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Application of the Central-Difference with Half-Sweep Gauss-Seidel Method for Solving First Order Linear Fredholm Integro-Differential Equations

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***Abstract*—**The objective of this paper is to analyse the application of the Half-Sweep Gauss-Seidel (HSGS) method by using the Half-sweep approximation equation based on central difference (CD) and repeated trapezoidal (RT) formulas to solve linear fredholm integro-differential equations of first order. The formulation and implementation of the Full-Sweep Gauss-Seidel (FSGS) and Half-Sweep Gauss-Seidel (HSGS) methods are also presented. The HSGS method has been shown to rapid compared to the FSGS methods. Some numerical tests were illustrated to show that the HSGS method is superior to the FSGS method.

***Keywords*—**Integro-differential equations, Linear fredholm equations, Finite difference, Quadrature formulas, Half-Sweep iteration.

I. Introduction

INTEGRO-DIFFERENTIAL equations (IDEs) arise from many branches of science, for example in control theory and financial mathematics [1]. Especially in physics, it arises naturally such as scattering theory, colloidal dispersions, heat transfer in the presence memory effects, quark dynamic [2], etc. IDE is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. Commonly, it can be classified into Fredholm equations or Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. However, in this paper we focus on Fredholm integro-differential. Generally, first-order linear Fredholm integro-differential equations can be defined as follows

$$y'(x) = p(x)y(x) + f(x) + \int_a^b K(x,t)y(t)\,dt, \quad a \le x \le b \quad (1)$$
$$y(a) = y_a$$

where the functions , and the kernel are known and is the solution to be determined. In the engineering field, numerical methods for solution of linear Fredholm integro-differential equations (LFIDEs) have been studied by many authors such as Lagrange interpolation method [3], Tau method [4], quadrature-difference method [5], variational method [6], collocation method [7], homotopy perturbation method [8], Euler-Chebyshev method [9] and GMRES method [10].

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LFIDEs are usually difficult to solve analytically so numerical approaches are practiced to obtain an approximation solution for the problem (1). To solve a LFIDE equation numerically, discretization of differential and integral parts to the solution of system of linear algebraic equations is the basic concept used by researchers to solve LFIDE problems. By considering numerical techniques, there are many schemes that can be used to discretize problem (1) independently for linear differential and integral terms. Many researchers have implemented discretization schemes for linear differential term such as finite difference scheme [11]-[12]), Taylor polynomial scheme [13], Chebyshev polynomial method [14], Runge-Kutta scheme [15] and Euler implicit schemes [16] whilst to discretize linear integral term numerically, many discretization schemes can be used for approximation such as quadrature [17]-[20], projection method [21]-[22]) and least squares [23]. The concept of Half-sweep iterative method was introduced by[24] by the employ of Explicit Decoupled Group (EDG) to solve two-dimensional Poisson equations. Then this concept has been discussed in [25]-[30]. This concept is essential to reduce the computational complexities during the iterative process, whereas the implementation of the half-sweep iterations will only consider nearly half of all node points in a solution domain. In this paper, we carried out the application of the half-sweep iteration technique with Gauss-Seidel (GS) iterative methods by using approximation equation based on finite difference and quadrature schemes for solving problem (1). The standard GS iterative method also called as the Full-Sweep Gauss-Seidel iterative method was implemented with half-sweep iterations process whereas it can be indicated as Half-Sweep Gauss-Seidel (HSGS). The organization of the paper is as follows. In section 2, the formulation of the finite difference and quadrature approximation equations for full- and half-sweep cases will be elaborated. In section 3, formulation of the FSGS and HSGS methods will be demonstrated. In section 4, some numerical results will be illustrated to emphasize effectiveness of the methods. Conclusion is in section 5.

II. Formulation of Half-Sweep Approximation Equation

Based on Fig. 1, the full- and half-sweep iterative methods will compute approximate values onto only solid node points until the convergence criterion is reached. It seems that the implementation of the half-sweep iterative method just involves by nearly one-half of whole inner points as shown in Figure 1(b) compared with the full-sweep iterative method.

Then the other approximation solutions for the remaining points are calculated by using direct methods. [1, 33]

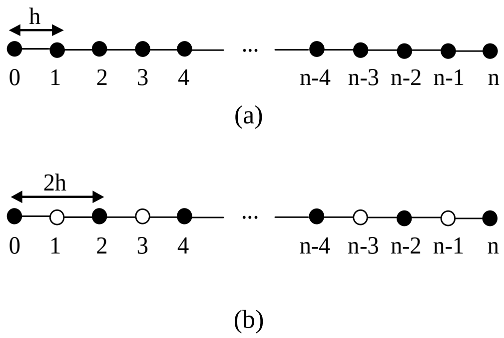

Fig. 1 (a) and (b) show distribution of uniform node points for the full and half-sweep cases respectively.

A. Formulation of Half-Sweep Finite Difference Schemes

As mentioned in section 1, CD scheme based on finite difference method was used to form an approximation equation for differential term. In this paper CD scheme was used to discretize the first order LFIDE. In general, first order derivative of second order error central difference formula can be derived from the Taylor series expansion as follows

for $i = 1, 2, n-1$,

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + O(h^2) \qquad (2)$$

for $i = n$,

$$y'(x_i) = \frac{3y(x_i) - 4y(x_{i-1}) + y(x_{i-2})}{2h} + O(h^2) \qquad (3)$$

where $h = \dfrac{b-a}{n}$ is size interval between nodes.

while $O(h^2)$ is truncations error which, is will not be considered in this paper. The size of the truncation error is mostly under our control because we can choose the mesh size.

In order to obtain the finite grid work network for formulation of the full- and half-sweep finite difference approximation equations over the problem as stated in Eq (1), further discussion will be restricted onto CD scheme which is as follows

$$y'(x_i) = \frac{y(x_{i+p}) - y(x_{i-p})}{2ph} \qquad (3)$$

and

$$y'(x_i) = \frac{3y(x_i) - 4y(x_{i-p}) + y(x_{i-2p})}{2ph} \qquad (4)$$

where the value of p, which corresponds to 1 and 2 , represents the full- and half -sweep respectively.

B. Formulation of Half-Sweep Quadrature Method

For the integral term, RT discretization scheme based on quadrature method was used to construct an approximation equation. In general quadrature formula can be defined as follows

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \qquad (4)$$

where $t_j \ (j = 0,1,\ldots,n)$ are the abscissas of the partition points of the integration interval $[a,b]$ or quadrature (interpolation) nodes, $A_j \ (j = 0,1,\ldots,n)$ are numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of Eq. (2). Based on RT rule, numerical coefficients A_j are satisfied following relation

$$A_j = \begin{cases} \frac{1}{2}ph, & j = 0, n \\ ph, & otherwise \end{cases} \qquad (5)$$

where the constant step size, h is defined

$$h = \frac{b-a}{n} \qquad (6)$$

n is the number of subintervals in the interval $[a, b]$. Meanwhile, the value of p, which corresponds to 1 and 2, represents the full- and half-sweep respectively.

Based on Eqs. (3), (4) and (5), by substitute into Eq. (1), a system of linear algebraic equations obtained for approximation values $y(x)$ at the nodes $x_1, x_2 \ldots, x_n$. The following linear system generated either by the full- or half-sweep approximation equation can be easily shown as

$$M \underset{\sim}{y} = \underset{\sim}{f} \qquad (7)$$

where

$$M = \begin{bmatrix} -2hP_p - 2hA_pK_{p,p} & -2hA_{2p}K_{p,2p} & \cdots & -2hA_nK_{p,n} \\ -2hA_pK_{2p,p} & -2hP_p - 2hA_{2p}K_{2p,2p} & \cdots & -hA_pK_{2p,n} \\ \vdots & \vdots & \ddots & \vdots \\ -2hA_pK_{n,p} & -2hA_{2p}K_{n,2p} & \cdots & -3 - 2hP_n - 2hA_nK_{n,n} \end{bmatrix}_{\left(\frac{n}{p} x \frac{n}{p}\right)}$$

$$\underset{\sim}{y} = \begin{bmatrix} y_p \\ y_{2p} \\ \vdots \\ y_{n-p} \\ y_n \end{bmatrix} \quad \text{and} \quad \underset{\sim}{f} = \begin{bmatrix} (2hA_0K_{p,0} + 1)y_0 + 2hf_p \\ (2hA_0K_{2p,0})y_0 + 2hf_{2p} \\ \vdots \\ (2hA_0K_{n-p,0})y_0 + 2hf_{n-p} \\ (2hA_0K_{n,0})y_0 + 2hf_n \end{bmatrix}$$

The value of p, which corresponds to 1 and 2, represents the full- and half-sweep cases respectively.

III. Formulation of the Full- and Half-Sweep Gauss-Seidel Methods

In this paper, FSGS and HSGS iterative methods will be applied to solve linear system generated from the discretization of the problem (1) as shown in Eq. (7). Let matrix M be articulated into

$$M = D - L - U \tag{8}$$

where D, L and U are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Thus, the general scheme for FSGS and HSGS iterative methods can be written as

$$\underset{\sim}{y}^{(k+1)} = (D-L)^{-1}\left(U\,\underset{\sim}{y}^{(k)} + \underset{\sim}{f}\right). \tag{9}$$

The iterative methods attempt to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector $\underset{\sim}{y}$. Iterations for FSGS and HSGS methods continue until the solution is within a predetermined acceptable bound on the error. The general algorithms for FSGS and HSGS iterative methods to solve problem (1) would be generally described in Algorithm 1.

Algorithm: FSGS and HSGS methods

i) Initializing all the parameters. Set k = 0.
ii) $i = p, 2p, \cdots, n-2p, n-p, n$
   Calculate

$$y_i^{(k+1)} = \frac{1}{M_{i,i}}\left(f_i - h\sum_{j=p,2p,}^{i-p} M_{i,j}\,y_j^{(k+1)} - h\sum_{j=i+p,i+2p}^{n} M_{i,j}\,y_j^{(k)}\right)$$

iii) Convergence test
iv) If the error of tolerance $\left|y_i^{(k+1)} - y_i^{(k)}\right| < \varepsilon = 10^{-10}$ is satisfied, the value option at that time is $y_i^{(k+1)}$ and the algorithm end.
v) Else, set $k = k+1$ and go to step (ii).

IV. Illustrative Examples

In this section, 3 numerical examples are illustrated to show the accuracy and effectiveness of the proposed methods and all of them were performed by using C language. Three criteria will be considered in comparison for FSGS and HSGS such as number of iterations, execution time and maximum absolute error.

Example 1 [31]

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)\,dt \quad 0 \le x \le 1$$

with the condition

$$y(0) = 0$$

and exact solution of the problem is

$$y(x) = x.$$

Example 2 [31]

$$y'(x) = xe^x + e^x - x + \int_0^1 y(t)\,dt \quad 0 \le x \le 1$$

with the condition

$$y(0) = 0$$

and exact solution of the problem is

$$y(x) = xe^x.$$

Example 3 [32]

$$y'(x) = \sinh x + \frac{1}{8}(1-e^{-1})x - \frac{1}{8}\int_0^1 xty(t)\,dt \quad 0 \le x \le 1$$

with the condition

$$y(0) = 1$$

and exact solution of the problem is

$$y(x) = \cosh x.$$

Throughout the experiments, the convergence test considered the tolerance error of $\varepsilon = 10^{-10}$. The experiments were carried out in different mesh sizes such as 60, 120, 240, 480 and 960. Results of numerical simulations which were obtained from implementations of the FSGS and HSGS iterative methods for Examples 1, 2 and 3 have been recorded in Tables 1, 2 and 3 respectively.

V. Conclusion

In this paper, the HSGS iterative method was employed to solve LFIDE for first-order. Based on numerical results observed in Tables 1, 2 and 3, it manifestly shows that the application of the half-sweep iterative concept significantly reduces computational time (refer table 4) with the tolerable precision. In the other hand, the number of iterations also reduced extensively corresponding to the mesh sizes. In all purpose, HSGS iterative method is faster for the computational works compared to FSGS iterative method. This is due to the computational complexity of the HSGS is reduced approximately 50% compared to FSGS method. In future works this concept can also can be used for high order IDEs problems.

TABLE I
COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS FOR EXAMPLE 1

Number of iteration					
Methods	Mesh size				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	33174	107988	375982	1394346	5487814
HSGS+CD+RT	10952	33174	107988	375982	1394346
Execution time (seconds)					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	512.36	17122.44	60347.03	143653.12	5434556.95
HSGS+CD+RT	47.87	563.54	19656.32	61202.98	153655.84
Maximum Absolute Error					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	2.623E-5	5.853E-6	3.506E-6	1.359E-7	9.858E-7
HSGS+CD+RT	1.057E-4	2.623E-5	5.853E-6	3.506E-6	1.359E-7

TABLE II
COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS FOR EXAMPLE 2

Number of iteration					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	43268	137637	459828	1653228	6136092
HSGS+CD+RT	14595	43268	137637	459828	1653228
Execution time (seconds)					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	421.65	5324.21	55324.20	155159.78	1073214.21
HSGS+CD+RT	20.66	795.54	16845.02	64324.17	579548.36
Maximum Absolute Error					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	2.9883E-4	6.2354E-4	2.3785E-5	4.3312E-5	1.2032E-6
HSGS+CD+RT	1.2228E-3	2.9883E-4	6.2354E-4	2.3785E-5	4.3312E-5

TABLE III
COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS FOR EXAMPLE 2

Number of iteration					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	27766	92736	331899	1229548	5988642
HSGS+CD+RT	8737	27766	92736	331899	1229548
Execution time (seconds)					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	256.65	4651.23	48898.78	145694.01	1002365.64
HSGS+CD+RT	11.86	257.84	4856.35	49584.13	149653.47
Maximum Absolute Error					
Methods	**Mesh size**				
	60	**120**	**240**	**480**	**960**
FSGS+CD+RT	2.572E-5	3.265E-6	4.397E-6	6.320E-7	1.254E-8
HSGS+CD+RT	1.335E-4	2.572E-5	3.265E-6	4.397E-6	6.320E-7

TABLE IV
PERCENTAGES OF REDUCTION FOR EXECUTION TIME FOR HSGS ITERATIVE METHODS COMPARED WITH FSGS METHOD

Methods	HSGS+CD+RT
	Execution time
Example 1	57.39%-97.17%
Example 2	45.99%-95.10%
Example 3	65.96%-95.37%

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