Some Separations in Covering Approximation Spaces

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I. INTRODUCTION

Rough set theory, which was proposed by Z. Pawlak in [8], is a useful tool in researches and applications of pattern recognition and artificial intelligence [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. In rough set theory, Pawlak approximation spaces are based on equivalence class partitioning of sets. However, in an equivalence class partitioning of a set, we can usually not separate points by equivalence classes. In other words, Pawlak approximation spaces do not satisfy “granularity-wise separations” in general. This leads us to explore the richer rough set theory. In the past years, covering approximation spaces arouse our extensive interest and their usefulness has been demonstrated by many successful applications in pattern recognition and artificial intelligence [10], [11], [12], [13], [14], [15], [16]. Naturally, the following question is worthy to be considered.

Question 1.1: (1) How to characterize granularity-wise separations in covering approximation spaces?

(2) What relations are there among granularity-wise separations in covering approximation spaces?

As some investigations of the above question, we adopt Zakowski’s upper approximation operator \( \overline{\mathcal{C}} \) and lower approximation operator \( \underline{\mathcal{C}} \) to characterize granularity-wise separations in a covering approximation space \((U, \mathcal{C})\) and establish some relations among these separations. Results of this paper give further applications of Pawlak rough set theory in pattern recognition and artificial intelligence.

II. PRELIMINARIES

Definition 2.1 ([20]): Let \( U \), the universe of discourse, be a finite set and \( \mathcal{C} \) be a family of nonempty subsets of \( U \).

(1) \( \mathcal{C} \) is called a cover of \( U \) if \( \bigcup \{ K : K \in \mathcal{C} \} = U \).

(2) The pair \((U; \mathcal{C})\) is called a covering approximation space if \( \mathcal{C} \) is a cover of \( U \).

Definition 2.2 ([8]): Let \((U; \mathcal{C})\) be a covering approximation space.

(1) \( \mathcal{C} \) is called a partition on \( U \) if \( K \cap K' = \emptyset \) for all \( K, K' \in \mathcal{C} \) where \( K \neq K' \).

(2) \((U; \mathcal{C})\) is called a Pawlak approximation space if \( \mathcal{C} \) is a partition on \( U \).

Notation 2.3: Let \((U; \mathcal{C})\) be a covering approximation space. Throughout this paper, we use the following notations, where \( x \in U \), \( X \subseteq U \) and \( \mathcal{F} \subseteq 2^U \).

\[
\begin{align*}
(1) & \quad \mathcal{F} = \bigcap \{ F : F \subseteq \mathcal{F} \} \\
(2) & \quad \mathcal{F} = \bigcup \{ F : F \subseteq \mathcal{F} \} \\
(3) & \quad \mathcal{C}_x = \{ K : x \in K \in \mathcal{C} \} \\
(4) & \quad N(x) = \bigcap \{ K : K \in \mathcal{C}_x \} = \bigcap \mathcal{C}_x \\
(5) & \quad D(X) = U - \bigcup \{ K : K \in \mathcal{C} \land K \cap X = \emptyset \} \\
(6) & \quad D(x) = D(\{ x \}) = U - \bigcup \{ \mathcal{C} - \mathcal{C}_x \}.
\end{align*}
\]

Now we give the following granularity-wise separations in covering approximation spaces. Ideas of these separations come from topology (see [2], for example).

Definition 2.4: Let \((U; \mathcal{C})\) be a covering approximation space. \((U; \mathcal{C})\) is called a \( G_0 \) (resp. \( G_{1}\), \( G_2\), \( G_3\), \( G_4\)) covering approximation space if \((U; \mathcal{C})\) satisfies the following \( G_0 \) (resp. \( G_{1}\), \( G_2\), \( G_3\), \( G_4\)) separation axiom.

\[
\begin{align*}
(1) & \quad G_0\text{-separation axiom: } x, y \in U \land x \neq y \implies \exists K \in \mathcal{C}(K \cap \{x, y\} = \{x\} \lor K \cap \{x, y\} = \{y\}) \\
(2) & \quad G_{1}\text{-separation axiom: } x, y \in U \land x \neq y \implies \exists K_x, K_y \in \mathcal{C}(K_x \cap \{x, y\} = \{x\} \lor K_y \cap \{x, y\} = \{y\}) \\
(3) & \quad G_2\text{-separation axiom: } x, y \in U \land x \neq y \implies \exists K_x, K_y \in \mathcal{C}(x \in K_x \land y \in K_y \land K_x \cap K_y = \emptyset) \\
(4) & \quad G_3\text{-separation axiom: } x \in U \land x \notin X \subseteq U \implies \exists K \in \mathcal{C}(x \in K \land K \cap X = \emptyset). \\
(5) & \quad G_4\text{-separation axiom: } x \in U \implies \exists K \in \mathcal{C}(\{x\} = K \cap D(x)). \\
(6) & \quad G_r\text{-separation axiom: } x \in K \in \mathcal{C} \implies D(x) \subseteq K.
\end{align*}
\]

For short, \( G_i \)-covering approximation spaces are called \( G_i \)-spaces, \( i = 0, 1, 2, 3, d, r \).

In order to investigate the above separations in covering approximation spaces by means of Pawlak rough sets, we need the following definition (see [11], for example).

Definition 2.5: Let \((U; \mathcal{C})\) be a covering approximation space. For each \( X \subseteq U \), Put
\[
\mathcal{C}(X) = \bigcup \{ K : K \in \mathcal{C} \text{ and } K \subseteq X \};
\]
Theorem 3.5: Let $(U;C)$ be a covering approximation space. Then the following are equivalent.

(1) $(U;C)$ is a $G_0$-space.

(2) $x, y \in U \wedge x \neq y \implies \exists y^* \land y \not\in \{x^*\}$. 

(3) $x \in U \implies \{x\} \in N(x)$. 

Proof. $(1) \implies (2)$: Suppose that $(U;C)$ is a $G_0$-space. Let $x, y \in U$ and $x \neq y$. Without loss of generality, assume that there is $K \in C$ such that $K \cap \{x, y\} = \{x\}$. Then $x \not\in \{y^*\}$ by Lemma 3.2.

$(2) \implies (3)$: Suppose that (2) holds. Let $x, y \in U$ and $x \neq y$. Without loss of generality, assume that $x \not\in \{y^*\}$. Since $x \in \{x^*\}$ by Lemma 3.1(2), $\{x^*\} \not\in \{y^*\}$. 

$(3) \implies (1)$: Suppose that (3) holds. Let $x, y \in U$ and $x \neq y$. Then $\{x^*\} \not\in \{y^*\}$. Without loss of generality, assume that there is $z \in \{y^*\}$ and $z \in \{x^*\}$. By Lemma 1(6), $z \in \{y^*\} = U \setminus (U \setminus \{y\})$, and hence $z \in (U \setminus \{y\})$. So there is $K \in C$ such that $z \in K \subset U \setminus \{y\}$, i.e., $z \notin K \in C$. Since $z \in \{x^*\}$, $x \in K$ by Corollary 3.3. Thus $K \cap \{x, y\} = \{x\}$. It follows that $(U;C)$ is a $G_0$-space.
Remark 3.7: In Theorem 3.6(2), “$K \cap K_y = \emptyset$” can not be replaced by “$K \cap K_y = \emptyset$”.

Proof. Let $U = \{a, b, c, \ldots, e\}$ and $C = \{\{a, b\}, \{a, c\}, \ldots\}$. It is not difficult to check that $(U; C)$ is a $G_2$-space. If $K_a \subseteq C$, $K_b \subseteq C_b$ and $K_a \cap K_b = \emptyset$, then $K_a = \{a, c\} \equiv K_b = \{b, d\}$. Since $K_a^{*} = \{a\} \neq U - (U - \{a, c\}) = U - \{b, d\}$, $K_b^{*} = \{b\} \neq U - (U - \{b, d\}) = U - \{a, c\}$. Thus, for each $K \in C$, if $y \in C$, then $K \subseteq U - \{x\}$, and hence $x \in K$, i.e., $K \subseteq C_a$. So $y \not\subseteq K$ for each $K \in C - C_a$ and hence $y \not\subseteq \bigcup \{C - C_a\}$. Consequently, $y \not\subseteq U - \bigcup \{C - C_a\} = D(x)$. On the other hand, let $y \in D(x)$. Then we can obtain $y \in \{x^{*}\}$ by reversing the above proof. This proves that $\{x^{*}\} = D(x)$.

Theorem 3.8: Let $(U; C)$ be a covering approximation space. Then the following are equivalent.

1. $(U; C)$ is a $G_3$-space.
2. $x \in U \setminus \{x\} \Rightarrow x \not\subseteq X^*.$
3. $X \subseteq U \Rightarrow X = X^*.$
4. $X \subseteq U \Rightarrow X = X^*.$
5. $x \in U \Rightarrow \{x\} = \{x\}^*.$
6. $(U; C)$ is a covering approximation space.

Proof. (1) $(\Rightarrow)$ (2): Suppose that $(U; C)$ is a $G_3$-space. Let $x \in U$ and $x \not\subseteq X^*$. Then there is $K \in C$ such that $x \in K$ and $K \cap X = \emptyset$. So $X \subseteq U - K$, and hence $X^* \subseteq (U - K)^* = U - K = K$. It follows that $x \not\subseteq X^*.$

(2) $(\Rightarrow)$ (3): Suppose that (2) holds. Then $x \subseteq X^*$. Then $x \not\subseteq X^*$. Since $x \not\subseteq X^*$. Thus $x \not\subseteq X^*$. On the other hand, $x \subseteq X^*$ by Lemma 3.1(2). Consequently, $X = X^*.$

(3) $(\Rightarrow)$ (4): Suppose that (3) holds. Then $X \subseteq U$. Then $U - X = (U - X^*) = U - X$, and so $X = U - (U - X) = U - (U - X) = X.$

(4) $(\Rightarrow)$ (5): It is clear.

(5) $(\Rightarrow)$ (6): Suppose that (5) holds. Let $x \in U$. Then $x \subseteq \{x\}^{*}$, so there is $K \in C$ such that $x \in K \subseteq \{x\}$. It follows that $\{x\} = K \subseteq C.$

(6) $(\Rightarrow)$ (1): Suppose that (6) holds. Let $x \in U \setminus X \subseteq U$. Then $x \subseteq C$ and $\{x\} \cap X = \emptyset$. So $(U; C)$ is a $G_3$-space.

Lemma 3.9: Let $(U; C)$ be a covering approximation space, $x \in U$ and $X \subseteq U$. The following are equivalent.

1. $x \not\subseteq X^*.$
2. $\exists K \in C \exists x (K \subseteq X \cap X^* = \emptyset).$
3. $\exists K \in C \exists x (K \subseteq X \cap X^* = \emptyset).$

Proof. (1) $(\Rightarrow)$ (2): Let $x \not\subseteq X^*$, i.e., $x \not\subseteq U - (U - X)$. Then $x \in (U - X)$, so there is $K \in C$ such that $x \in K \subseteq (U - X)^* = U - X^*$. Consequently, $K \subseteq X^* = \emptyset$.$$

(2) \Rightarrow (3):$ It holds by Lemma 3.1(2).

(3) $(\Rightarrow)$ (1): If there is $K \in C$ such that $x \in K$ and $K \cap X = \emptyset$, then $K \subseteq U - X$, and hence $x \in K \subseteq (U - X)^* = \emptyset$.

Corollary 3.10: Let $(U; C)$ be a covering approximation space, $x \in U$ and $X \subseteq U$. Then the following are equivalent.

1. $x \not\subseteq X^*.$
2. $\forall K \in C \exists x (K \subseteq X \cap X^* = \emptyset).$
3. $\forall K \in C \exists x (K \subseteq X \cap X^* = \emptyset).$

The following lemma can be obtained immediately from Definition 2.5.

Lemma 3.11: Let $(U; C)$ be a covering approximation space, $x \in U$ and $X \subseteq U$. Then the following are equivalent.

1. $x \not\subseteq X^*.$
2. $\exists K \in C (x \in K \subseteq X).$
3. $\exists K \in C (x \in K \subseteq X).$
Lemma 3.1(1). So \( \{ \{ x \} \} \cap U = \{ x \} ^* \cap U^* \). By Theorem 3.13, \((U;C)\) is a \( G_0\)-space.

\((4) \implies (5)\): Suppose that \((U;C)\) is a \( G_0\)-space. Let \( x, y \in U \) and \( x \neq y \). Then there is \( K \in C \) such that \( \{ x \} \cap K = \emptyset \). By Theorem 3.9, there is \( K' \in C \) such that \( y \notin K' \) and \( \{ x \} \cap K' = y \). If \( y \notin \{ x \}^* \), then \( y \notin K' \) and hence \( \{ x \} \cap \{ x, y \} = \{ x \} \).

Consequently, \((U;C)\) is a \( G_0\)-space.

\((3) \implies (6)\): Suppose that \((U;C)\) is a \( G_1\)-space. For each \( x \in U \), if \( x \in K \in C \), then \( \{ x \} \cap \{ x, y \} = \{ x \} \).

Proof. (1) \implies (2): It holds by Theorem 4.1.

\((2) \implies (1)\): Suppose that \((U;C)\) is a \( G_0\)- and \( G_1\)-space. Let \( x, y \in U \) and \( x \neq y \). By Theorem 3.4(2), \( x \notin \{ y \}^* \) or \( y \notin \{ x \}^* \). Without loss of generality, assume that \( x \notin \{ y \}^* \).

Then \( y \notin \{ x \}^* \) by Theorem 3.14(2). So \((U;C)\) is a \( G_1\)-space by Theorem 3.5(2).

Although none of implications in Theorem 4.1 can be reversed, we have the following equivalences on separations in Pawlak approximation spaces.

Theorem 4.10: Let \((U;C)\) be a Pawlak approximation space. Then the following are equivalent.

\((1)\): \((U;C)\) is a \( G_3\)-space.

\((2)\): \((U;C)\) is a \( G_2\)-space.

\((3)\): \((U;C)\) is a \( G_1\)-space.

\((4)\): \((U;C)\) is a \( G_0\)-space.

\((6)\): \(C = \{ \{ x \} : x \in U \} \).

Proof. (1) \implies (2) \implies (3) \implies (4) \implies (5): They are hold by Theorem 4.1.

\((5) \implies (6)\): Suppose \((U;C)\) is a \( G_0\)-space. Let \( x \in U \).

Then there is \( K_x \in C \) such that \( x \in K_x \). If \( \{ x \} \notin C \), then \( K_x \neq \{ x \} \), and hence there is \( y \in K_x \) and \( y \neq x \). Thus \( K_x \cap \{ x, y \} = \{ x, y \} \). On the other hand, since \((U;C)\) is a Pawlak approximation space, \( C \) is a partition of \( U \), and hence \( K \cap \{ x, y \} = \emptyset \) for each \( K \in C \). This contradicts that \((U;C)\) is a \( G_0\)-space. So \( \{ x \} \in C \).

\((6) \implies (1)\): Suppose that (6) holds. Then \( \{ x \} \in C \) for each \( x \in U \). By Theorem 3.8(6), \((U;C)\) is a \( G_3\)-space.

Remark 4.11: By Theorem 4.10, \( G_0\)-Pawlak approximation spaces are \( G_1\)-spaces. But \( G_1\)-Pawlak approximation space \( \not\equiv G_0\)-space by Example 4.7. So the condition “\((U;C)\) is a \( G_1\)-space” is not equivalent to conditions in Theorem 4.10.

V. CONCLUSION

This paper explore a new property in covering approximation spaces: granularity-wise separation. Adopting Zakowski’s covering approximation operators \( \overline{C} \) and \( \underline{C} \), this paper give some characterization of covering approximation spaces with some granularity-wise separation and establish some relations among these spaces. Results of this paper deepen and enrich rough set theory, which is helpful to understand inherent property of covering approximation spaces completely.

In this paper, our investigations are based on Zakowski’s covering approximation operators \( \overline{C} \) and \( \underline{C} \). Because there are also other useful covering approximation operators \([10], [11], [14], [15], [19], [20]\), it is a natural question how to investigate separations in corresponding covering approximation spaces with these covering approximation operators. This is still worthy to be considered in subsequent research.

ACKNOWLEDGMENT

This project is supported by the National Natural Science Foundation of China (No. 10971185 and 10971186).
REFERENCES


