Persistence of Termination for Term Rewriting Systems with Ordered Sorts

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Abstract—A property $P$ is persistent if for any many-sorted term rewriting system $\mathcal{R}$, $\mathcal{R}$ has the property $P$ if and only if term rewriting system $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property $P$. Zantema showed that termination is persistent for term rewriting systems without collapsing or duplicating rules. In this paper, we show that the Zantema’s result can be extended to term rewriting systems on ordered sorts, i.e., termination is persistent for term rewriting systems on ordered sorts without collapsing, decreasing or duplicating rules. Furthermore we give the example as application of this result. Also we obtain that completeness is persistent for this class of term rewriting systems.

Keywords: Theory of computing, Model-based reasoning, term rewriting system, termination

I. INTRODUCTION

Term rewriting systems (TRSs) can offer both flexible computing and effective reasoning with equations and have been widely used as a model of functional and logic programming languages and as a basis of theorem provers, symbolic computation, algebraic specification and verification [4].

A rewrite system is called terminating (strongly normalizing) if there is no infinite rewrite sequence. The notion of termination for rewrite systems corresponds to the existence of answers of computations. So termination is the fundamental notion of term rewriting systems as computation models [7]. It is well-known that termination is undecidable for term rewriting systems in general. However, several sufficient approaches for proving this property have been successfully developed in particular cases.

Zantema [23] introduced the notion of persistence as follows. A property $P$ is persistent if for any many-sorted TRS $\mathcal{R}$, $\mathcal{R}$ has the property $P$ if and only if TRS $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property $P$. Usual many-sorted TRS was extended with ordered sorts by Aoto and Toyama [2]. And it was shown that the persistency of confluence [1] is preserved for this extension in [2]. Zantema [23] showed that termination is persistent for TRSS without collapsing or duplicating rules. Ohsaki and Middeldorp [20] studied the persistence of termination, acyclicity and non-loopingness on equational many-sorted TRSs. Aoto proved that the persistence of termination for TRSs in which all variables are of the same sort [3]. We showed that the persistence of termination for non-overlapping TRSs [11]. Also, we showed that the persistence of termination for locally confluent overlay TRSs [12]. And we showed that the persistence of termination for right-linear overlay TRSSs [13]. Furthermore we showed that the persistence of semi-completeness for TRSs [14].

In this paper, we show that the above Zantema’s result is preserved for Aoto and Toyama’s extension in the subclass of order sorted term rewriting systems.

This research was first appeared in [9] and studied in [10]. Furthermore, Ohsaki [21] studied the case of equational order-sorted TRSs. Their equational order-sorted TRSs [21] were based on ordered-sorted algebras in [8], [22]. However, our TRSs on ordered sorts are based on Aoto and Toyama [2]. For example, we consider the sorts $\text{Zero}$ and $\text{Nat}$. If $\text{Zero} \prec \text{Nat}$ then $A_{\text{Zero}} \subseteq A_{\text{Nat}}$ where $A_{\text{Zero}}$ and $A_{\text{Nat}}$ are order-sorted algebras in equational order-sorted TRSs [21]. However, in our TRSs on ordered sorts we do not consider order-sorted algebras. In our research, if $\text{Zero} \prec \text{Nat}$ then $\mathcal{T}_{\text{Zero}} \cap \mathcal{T}_{\text{Nat}} = \emptyset$ holds where $\mathcal{T}_{\text{Zero}}$ and $\mathcal{T}_{\text{Nat}}$ are set of terms with sort $\text{Zero}$ and $\text{Nat}$, respectively. So our research does not depend on order-sorted algebras.

In section 2, many-sorted TRS is formulated on ordered sorts. Then, the persistence of termination on ordered sorts is shown in section 3 and 4. The proof is a generalization of a simplified proof of modularity of termination [18]. Furthermore we give the example as application of this result. Also we obtain that completeness is persistent for term rewriting systems on ordered sorts.

II. PRELIMINARIES

We mainly follow basic definitions and basic lemmas in the literature [2].

A. Sorted Term Rewriting Systems

In this subsection, we introduce the basic notions of sorted term rewriting systems. Usual term rewriting systems [4] are considered as special cases of sorted term rewriting systems.

Let $\mathcal{S}$ be a set of sorts and $\mathcal{V}$ be a set of countably infinite sorted variables. We assume that $\mathcal{S}$ is equipped with a well-founded partial ordering $\succ$. We write $b \succeq b'$ if and only if $b \succ b'$ or $b = b'$. We assume there is a set $\mathcal{V}^0$ of countably infinite variables of sort $b$ for each sort $b \in \mathcal{S}$. Let $\mathcal{F}$ be a set of sorted function symbols. We assume that each sorted function symbol $f \in \mathcal{F}$ is given with the sorts of its arguments and the sort of its value, all of which are included in $\mathcal{S}$. We write $f b_1 \times \ldots \times b_n \rightarrow b'$ if and only if $f$ takes $n$ arguments of sorts

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This document discusses the theory of term rewriting systems (TRS) and introduces the concept of well-sortedness. It defines a term rewriting system (TRS) as a set of rewrite rules that modify terms. The document explains the notion of sort attachment and the concept of persistence in TRSs. It introduces the idea of strict well-sortedness and its importance in ensuring the termination of rewrite rules. The text also covers the definition of the underlying TRS of a TRS and the notion of persistence, which is crucial for proving termination of rewrite rules.

The document further elaborates on the concept of well-foundedness and how it relates to the termination of rewrite rules. It discusses the definition of a well-founded partial ordering and the notion of persistence with respect to such orderings. The text includes a section on the characterization of unsorted terms, where it introduces the concept of the top sort and discusses its role in determining the termination of rewrite rules.

Finally, the document provides an overview of the main concepts and definitions related to term rewriting systems, including the notions of strict well-sortedness, persistence, and the characterization of unsorted terms. It concludes with an analysis of the significance of these concepts in the theory of term rewriting systems.
Definition 3.2: Let $t = C[t_1, \ldots, t_n]$ $(n \geq 0)$ be an
unsorted term with $C[\ldots, \ldots] \neq \Box$. We write $t = C[t_1, \ldots, t_n]$ if and only if
1. $C:b_1 \times \ldots \times b_n \rightarrow b'$ is a context that is well-sorted under $\tau$.
2. $\text{top}(t_i) \neq b_i$ for all $i = 1, \ldots, n$.

The $t_1, \ldots, t_n$ are said to be the principal subterms of $t$. We denote $t = C[\langle t_1, \ldots, t_n \rangle]$ if either $t = C[t_1, \ldots, t_n]$ or $C = \Box$ and $t_i \notin \{t_1, \ldots, t_n\}$. Multiset $S(t)$ consists of all
principal subterms of $t = C[t_1, \ldots, t_n]$.

Definition 3.3: Let $t$ be an unsorted term. Rank of $t$ is defined as follows:

- $\text{rank}(t) = 1$ if $t$ is well-sorted term.
- $\text{rank}(t) = 1 + \max\{\text{rank}(t_1), \ldots, \text{rank}(t_n)\}$ if $t = C[t_1, \ldots, t_n]$.

Definition 3.4: Let $t$ be an unsorted term. Cap of $t$ is defined as follows:

- $\text{cap}(t) = t$ if $t$ is well-sorted term.
- $\text{cap}(t) = C[\ldots, \ldots]$ if $t = C[t_1, \ldots, t_n]$.

Definition 3.5: A rewrite step $s \rightarrow_{\mathcal{R}} t$ is said to be inner (written as $s \rightarrow_{\mathcal{R}} t$) if and only if $s = C[s_1, \ldots, C'[\theta], \ldots, s_n] \rightarrow_{\mathcal{R}} C'[\theta][s_1, \ldots, s_n] = t$ for some $s_1, \ldots, s_n, l \rightarrow r \in \mathcal{R}$, $\theta$ and $C'$, otherwise outer (written as $s \rightarrow t$).

Definition 3.6: A rewrite step $s \rightarrow_{\mathcal{R}} t$ is said to be vanishing if and only if $s = C[s_1, \ldots, \theta(x) \rightarrow C'[\theta, \ldots, s_n]] \rightarrow_{\mathcal{R}} C'[\theta, \ldots, s_n] = t$ for some $s_1, \ldots, s_n, C$ and $\theta$ such that $C' = C[s_1, \ldots, s_n]$ and $C'[x] \rightarrow x \in \mathcal{R}$.

Definition 3.7: A rewrite rule $l \rightarrow r$ in $\mathcal{R}$ is said to be decreasing if and only if $\text{top}(l) > \text{top}(r)$. A rewrite step $s \rightarrow_{\mathcal{R}} t$ is said to be decreasing if and only if $\text{top}(s) > \text{top}(t)$.

Definition 3.8: A rewrite step $s \rightarrow_{\mathcal{R}} t$ is said to be destructive at level $l$ if and only if $s \rightarrow_{\mathcal{R}} t$ is either vanishing or decreasing.

The rewrite step $s \rightarrow_{\mathcal{R}} t$ is said to be destructive at level $k+1$ if and only if $s = C[s_1, \ldots, s_j, \ldots, s_n] \rightarrow_{\mathcal{R}} C[s_1, \ldots, s_j, \ldots, s_n] = t$ with $s_j \rightarrow_{\mathcal{R}} t_j$ destructive at level $k$.

Lemma 3.9: If a rewrite step $s \rightarrow_{\mathcal{R}} t$ is not vanishing then $\text{top}(s) \geq \text{top}(t)$. If a rewrite step $s \rightarrow_{\mathcal{R}} t$ is not destructive at level 1 then $\text{top}(s) = \text{top}(t)$.

Lemma 3.10: If $s \rightarrow_{\mathcal{R}} t$ then $\text{rank}(s) \geq \text{rank}(t)$. If a rewrite step $s \rightarrow_{\mathcal{R}} t$ is vanishing then $\text{rank}(s) > \text{rank}(t)$.

Definition 3.11: The grade $(\text{grade}(t))$ of a term $t$ is defined by $\text{grade}(t) = \langle \text{rank}(t), \text{top}(t) \rangle$ where $N$ is the set of all natural numbers.

Let $>$ be the lexicographic ordering on $N \times S$ induced from $\succ$ on $N$ and $\succ$ on $S$. The lexicographic ordering $\succ$ on $N \times S$ is well-founded since orderings $\succ$ on $N$ and $\succ$ on $S$ are well-founded.

Lemma 3.12: If $s \rightarrow_{\mathcal{R}} t$ then $\text{grade}(s) \geq \text{grade}(t)$. If a rewrite step $s \rightarrow_{\mathcal{R}} t$ is destructive at level 1 then $\text{grade}(s) > \text{grade}(t)$.

IV. PERSISTENCE OF TERMINATION

In this section we discuss the persistence of termination for TRSs on ordered sorts and give the example as application
2. \( \text{cap}(s) = \text{cap}(t) \). By Lemma 4.4, \( S(t) = S(s) \setminus \{ t_j \} \setminus \{ t_j \} \cup \{ t_j \} \).

- \( s_j \to t_j \) is vanishing. Since \( \text{rank}(s_j) > \text{rank}(t_j) \), \( \text{grade}(s_j) > \text{grade}(t_j) \) holds.

- \( s_j \to t_j \) is decreasing. Since \( \text{rank}(s_j) \geq \text{rank}(t_j) \) and \( \text{top}(s_j) \geq \text{top}(t_j) \), \( \text{grade}(s_j) \geq \text{grade}(t_j) \) holds.

Therefore, \( \text{g} \text{p} \text{a} \) \( \geq \text{m} \text{a} \text{t} \) \( \text{t} \) \( g \) \( \text{h} \) \( \text{t} \) \( s \).

Lemma 4.9: Let \( \mathcal{R} = \) be a terminating STRS. Let \( D : s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \cdots \) be an infinite rewrite sequence of minimal grade with respect to \( \mathcal{R} \). Then the following statements hold.

1. There are infinitely many outer rewrite steps in \( D \).
2. There are infinitely many inner rewrite steps in \( D \) which are destructive at level 2.
3. There are infinitely many duplicating outer rewrite steps in \( D \).

Proof. Since \( \text{grade}(s_j) = \text{grade}(D) \) for any \( j \in N \), there is no rewrite step which is destructive at level 1.

1. Suppose that there are only finitely many outer rewrite steps in \( D \). Then we can assume that there is no outer rewrite step in \( D \) which is destructive at level 2 by Lemma 4.5. If \( s \to_{\mathcal{R}} t \) in \( D \) then \( \text{cap}(s) \to_{\mathcal{R}} \text{cap}(t) \) and if \( s \to_{\mathcal{R}} t \) in \( D \) then \( \text{cap}(s) = \text{cap}(t) \). By the case 1, \( \mathcal{R} \) is not terminating. This is contradiction by the assumption.

2. Suppose that there are only finitely many inner rewrite steps in \( D \) which are destructive at level 2. Then we can assume that there is no inner rewrite step in \( D \) which is destructive at level 2. By Lemma 4.5, if \( s \to_{\mathcal{R}} t \in D \) then \( \text{cap}(s) \to_{\mathcal{R}} \text{cap}(t) \) and if \( s \to_{\mathcal{R}} t \in D \) then \( \text{cap}(s) = \text{cap}(t) \). By the case 1, \( \mathcal{R} \) is not terminating. This is contradiction by the assumption.

Example 4.12: We show that the following TRS \( \mathcal{R} \) is terminating using theorem 4.11. To show the termination of the following TRS \( \mathcal{R} \) directly seems difficult form known results (E.g., recursive path ordering [7]). Also, we can not use the modularity results for composable TRSs [17], [19] and hierarchical combinations and hierarchical combinations with common subsystem of TRSs [16], [19]. Furthermore, we can not use the Zantema’s result [23] for proving termination of the following TRS. However, we can show the termination of next TRS using our results in this paper.

\[
\mathcal{R} = \begin{cases} 
\{ g(x, B) \to g(x, A) 
\}, & (r1) \\
g(x, B) \to d(x, A) & (r2) \\
g(x, d(z, B)) \to x & (r3) \\
I(A, g(x, d(y, C))) \to I(B, g(x, d(y, C))) & (r4) \\
I(x, g(x, d(z, z))) \to d(z, z) & (r5) \\
d(z, A) \to e(z, C) & (r6)
\end{cases}
\]

Let \( S = \{ 0, 1, 2 \} \), \( 1 \geq 0 \) and \( 2 \geq 0 \).

Any well-sorted term in \( T^0, T^1 \) and \( T^2 \) is terminating, i.e. any well-sorted term in \( T^s \) is terminating. We consider the following cases:

- \( t \in T^0 \). Then \( r6 \) is the only applicable rule. A TRS \( \{ r6 \} \) is terminating using recursive path ordering. Hence, \( t \) is terminating.
- \( t \in T^1 \). Then \( r1 \), \( r2 \), \( r3 \) and \( r6 \) are the only applicable rules. A TRS \( \{ r1, r2, r3, r6 \} \) is terminating using recursive path ordering. Hence, \( t \) is terminating.
- \( t \in T^2 \). Then \( r1 \), \( r2 \), \( r3 \), \( r4 \), \( r5 \) and \( r6 \) are the applicable rules. For any proper subterm \( s \) of \( t \), \( \text{top}(s) = 0 \) or \( \text{top}(s) = 1 \). Since the above two cases, \( s \) is terminating. Since \( \text{top}(t) = 2 \), \( r4 \) and \( r5 \) are the only applicable rules to root position of term \( t \). Hence, \( t \) is terminating.

Then STRS \( \mathcal{R} = \) is terminating. Since \( \mathcal{R} \) has no duplicating rules and theorem 4.11, \( \mathcal{R} \) is terminating.

Furthermore we obtain the persistence of completeness for TRSs on ordered sorts. The following theorem was given by Aoto and Toyama [2].

Theorem 4.13: ([2]) Confluence is a persistent property of TRSs on ordered sorts.

Since a complete TRS is confluent and terminating, we obtain the following corollary from theorem 4.11 and theorem 4.13.

Corollary 4.14: The following statements hold.

1. Completeness is a persistent property of TRSs on ordered sorts without collapsing and decreasing rules.
2. Completeness is a persistent property of TRSs on ordered sorts without duplicating rules.
V. CONCLUSION AND FUTURE WORK

In this paper, we have shown that the Zantema’s result [23] is preserved for Aoto and Toyama’s extension [2] in the subclass of order sorted term rewriting systems. That is, we have shown that termination is persistent for TRSs on ordered sorts without collapsing, decreasing or duplicating rules. Furthermore, we have given the example as application of our results. Also we obtain the persistence of completeness for TRSs on ordered sorts.

However, the difference between our work and Ohsaki’s work [21] is still unclear. So we consider this point for the future.

REFERENCES