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Hyers-Ulam Stability of Functional Equation

\[ f(3x) = 4f(3x - 3) + f(3x - 6) \]

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Abstract—The functional equation \( f(3x) = 4f(3x - 3) + f(3x - 6) \) will be solved and its Hyers-Ulam stability will be also investigated in the class of functions \( f : \mathbb{R} \to X \), where \( X \) is a real Banach space.

Keywords—Functional equation, Lucas sequence of the first kind, Hyers-Ulam stability.

I. INTRODUCTION

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [20]). Among those was the question concerning the stability of homomorphisms:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with a metric \( d(\cdot, \cdot) \). Given any \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \)?

In the following year, Hyers affirmatively answered in his paper [8] the question of Ulam for the case where \( G_1 \) and \( G_2 \) are Banach spaces. Later, the result of Hyers has been generalized by Rassias (ref. [16]).

Let \( (G_1, \cdot) \) be a groupoid and let \( (G_2, +) \) be a groupoid with the metric \( d \). The equation of homomorphism

\[ f(x \cdot y) = f(x) + f(y) \]

is stable in the Hyers-Ulam sense (or has the Hyers-Ulam stability) if for every \( \delta > 0 \) there exists an \( \varepsilon > 0 \) such that for every function \( h : G_1 \to G_2 \) satisfying

\[ d(h(x \cdot y), h(x) + h(y)) \leq \varepsilon \]

for all \( x, y \in G_1 \) there exists a solution \( g : G_1 \to G_2 \) of the equation of homomorphism with

\[ d(h(x), g(x)) \leq \delta \]

for any \( x \in G_1 \) (see [15, Definition 1]).

This terminology is also applied to the case of other functional equations. It should be remarked that a lot of references concerning the stability of functional equations can be found in the books [3], [9], [12] (see also [1], [4], [5], [6], [7], [10], [11], [17], [18], [19]).

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The \( n \)th Fibonacci number will be denoted by \( F_n \) for \( n \in \mathbb{N} \). It is well known that the Fibonacci numbers satisfy the equality \( F_{3n} = 4F_{3n-3} + F_{3n-6} \) for all \( n \geq 2 \) (see [14, p. 89]). From this famous formula, the following functional equation

\[ f(3x) = 4f(3x - 3) + f(3x - 6) \]

may be derived.

In this paper, using the idea from [13], the functional equation (1) will be solved and its Hyers-Ulam stability will be investigated in the class of functions \( f : \mathbb{R} \to X \), where \( X \) is a real Banach space.

Throughout this paper, the positive and the negative root of the equation \( x^2 - 4x - 1 = 0 \) will be denoted by \( a \) and \( b \), respectively, i.e.,

\[ a = 2 + \sqrt{5} \quad \text{and} \quad b = 2 - \sqrt{5}. \]

Moreover, the Lucas sequence of the first kind will be denoted by \( \{U_n(4, -1)\} \) and an abbreviation \( U_n \) will be used instead of \( U_n(4, -1) \), i.e., \( U_n \) is defined by

\[ U_n = U_n(4, -1) = \frac{a^n - b^n}{a - b} \]

for all integers \( n \). It is not difficult to see that

\[ U_{n+2} = 4U_{n+1} + U_n \]

for any integer \( n \). For any \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) stands for the largest integer that does not exceed \( x \).

II. GENERAL SOLUTION TO EQ. (1)

Throughout this section, let \( X \) be a real vector space. The general solution of the functional equation (1) will be investigated.

Theorem 2.1. Let \( X \) be a real vector space. A function \( f : \mathbb{R} \to X \) is a solution of the functional equation (1) if and only if there exists a function \( h : [-3, 3] \to X \) such that

\[ f(x) = U_{\lfloor x/3 \rfloor + 1} h(x - 3 \lfloor x/3 \rfloor) + U_{\lfloor x/3 \rfloor} h(x - 3 \lfloor x/3 \rfloor - 3). \]

Proof. Since \( a + b = 4 \) and \( ab = -1 \), it follows from (1) that

\[
\begin{cases}
  f(3x) = af(3x - 3) = b[f(3x - 3) - af(3x - 6)], \\
  f(3x) = bf(3x - 3) = a[f(3x - 3) - bf(3x - 6)].
\end{cases}
\]
If a function $g : \mathbb{R} \to X$ is defined by $g(x) = f(3x)$ for each $x \in \mathbb{R}$, then it follows from the above equalities that

$$\begin{align*}
g(x) - ag(x-1) &= b[g(x-1) - ag(x-2)], \\
g(x) - bg(x-1) &= a[g(x-1) - bg(x-2)].
\end{align*}$$

By the mathematical induction, it can be proved that

$$\begin{align*}
g(x) - ag(x-1) &= b^n [g(x-n) - ag(x-n-1)], \\
g(x) - bg(x-1) &= a^n [g(x-n) - bg(x-n-1)]
\end{align*}$$

for all $x \in \mathbb{R}$ and $n \in \{0, 1, 2, \ldots \}$. Substitute $x + n$ ($n \geq 0$) for $x$ in (5) and divide the resulting equations by $b^n$ resp. $a^n$, and then substitute $-m$ for $n$ in the resulting equations to obtain the equations in (5) with $m$ in place of $n$, where $m \in \{0, -1, -2, \ldots \}$. Therefore, the equations in (5) are true for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Multiply the first and the second equation of (5) by $b$ and $a$, respectively. And subtract the first resulting equation from the second one to obtain

$$g(x) = U_{n+1}g(x-n) + U_n g(x-n-1)$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Putting $n = [x]$ in (6) yields

$$g(x) = U_{[x]+1}g(x-[x]) + U_{[x]}g(x-[x]-1),$$

i.e., by the definition of $g$, it holds that

$$f(x) = U_{[x]+1}f(x-3[x/3]) + U_{[x]}f(x-3[x/3]-3)$$

for all $x \in \mathbb{R}$.

Since $0 \leq 3[x/3] < 3$ and $-3 \leq x-3[x/3] < 3$, if a function $h : [-3, 3) \to X$ is defined by $h := f_{[-3,3)}$, then $f$ is a function of the form (3).

Now, assume that $f$ is a function of the form (3), where $h : [-3, 3) \to X$ is an arbitrary function. Then, it follows from (3) that

$$\begin{align*}
f(3x) &= U_{[x]+1}h(3x-[x]) + U_{[x]}h(3x-3[x]-3), \\
f(3x-3) &= U_{[x]}h(3x-3[x]) + U_{[x]-1}h(3x-3[x]-3), \\
f(3x-6) &= U_{[x]-1}h(3x-3[x]) + U_{[x]-2}h(3x-3[x]-3)
\end{align*}$$

for any $x \in \mathbb{R}$. Thus, by (2), it holds that

$$\begin{align*}
f(3x) - 4f(3x-3) - f(3x-6) &= (U_{[x]+1} - 4U_{[x]} + U_{[x]-1})h(3x-3[x]) \\
&\quad + (U_{[x]} - 4U_{[x]-1} + U_{[x]-2})h(3x-3[x]-3) \\
&= 0,
\end{align*}$$

which completes the proof.

### III. Hyers-Ulam Stability of Eq. (1)

In this section, $a$ denotes the positive root of the equation $x^2 - 4x - 1 = 0$ and $b$ is its negative root. The Hyers-Ulam stability of the functional equation (1) will be proved in the following theorem.

**Theorem 3.1.** Let $(X, \| \cdot \|)$ be a real Banach space. If a function $f : \mathbb{R} \to X$ satisfies the inequality

$$\|f(3x) - 4f(3x-3) - f(3x-6)\| \leq \varepsilon$$

for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a unique solution function $F : \mathbb{R} \to X$ of (1) such that

$$\|f(x) - F(x)\| \leq \frac{5 + \sqrt{5}}{20} \varepsilon$$

for all $x \in \mathbb{R}$.

**Proof.** First, define a function $g : \mathbb{R} \to X$ by $g(x) = f(3x)$ for all $x \in \mathbb{R}$. Analogously to the first equation of (4), it follows from (7) that

$$\|g(x) - ag(x-1) - bg(x-1) - ag(x-2)\| \leq \varepsilon$$

for each $x \in \mathbb{R}$. Replacing $x$ with $x-k$ in the last inequality yields

$$\|g(x-k) - ag(x-k-1) - bg(x-k-1) - ag(x-k-2)\| \leq \varepsilon$$

and further

$$\|\frac{b^k}{b^{k+1}}[g(x-k) - ag(x-k-1)] - \frac{b^k}{b^{k+1}}[g(x-k-1) - ag(x-k-2)]\| \leq \frac{|b|^k \varepsilon}{b^{k+1}}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (9), it obviously holds that

$$\|g(x) - ag(x-1) - b^n[g(x-n) - ag(x-n-1)]\| \leq \sum_{k=0}^{n-1} \frac{|b|^k \varepsilon}{b^{k+1}}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (9) implies that the sequence $\{b^n[g(x-n) - ag(x-n-1)]\}$ is a Cauchy sequence. (Note that $|b| < 1$). Therefore, a function $G_1 : \mathbb{R} \to X$ can be defined by

$$G_1(x) = \lim_{n \to \infty} b^n[g(x-n) - ag(x-n-1)],$$
since X is complete. It follows from the definition of $G_1$ that

$$4G_1(x - 1) + G_1(x - 2)$$

for all $x \in \mathbb{R}$, since $b^2 = 4b + 1$. If $n$ goes to infinity, then (10) yields that

$$\|g(x) - ag(x - 1) - G_1(x)\| \leq \frac{3 + \sqrt{5}}{4} \varepsilon$$

for every $x \in \mathbb{R}$.

On the other hand, it also follows from (7) that

$$\|g(x) - bG(x - 1) - a[g(x - 1) - b(g(x) - 2)]\| \leq \varepsilon$$

see the second equation in (4). Analogously to (9), replacing $x$ by $x + k$ in the above inequality and then dividing by $a^k$ both sides of the resulting inequality yield

$$\|a^{-k}[g(x + k) - b(g(x + k - 1)) - a^{-k+1}[g(x + k - 1) - b(g(x + k - 2))]\| \leq a^{-k}\varepsilon$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. It further follows from (13) that

$$\|a^{-n}[g(x + n) - b(g(x + n - 1)) - g(x) - b(g(x) - 1)]\| \leq \frac{4}{a} G_2(x) \leq \frac{4}{a} \frac{x}{3}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

On account of (13), the sequence $\{a^{-n}[g(x + n) - b(g(x + n - 1)) - g(x) - b(g(x) - 1)]\}$ is a Cauchy sequence for any fixed $x \in \mathbb{R}$. Hence, a function $G_2 : \mathbb{R} \to X$ can be defined by

$$G_2(x) = \lim_{n \to \infty} a^{-n}[g(x + n) - b(g(x + n - 1))].$$

It follows from the definition of $G_2$ that

$$4G_2(x - 1) + G_2(x - 2) = 4a^{-1} \lim_{n \to \infty} a^{-n-1}[g(x + n - 1) - b(g(x + n - 1) - 1)]$$

$$+ a^{-2} \lim_{n \to \infty} a^{-n-2}[g(x + n - 2) - b(g(x + n - 2) - 1)]$$

$$= 4a^{-1} G_2(x) + a^{-2} G_2(x)$$

$$= G_2(x)$$

for any $x \in \mathbb{R}$. By letting $n$ go to infinity, (14) yields

$$\|G_2(x) - g(x) + b(g(x) - 1)\| \leq \frac{\sqrt{5} - 1}{4} \varepsilon$$

for $x \in \mathbb{R}$.

From (12) and (16), it follows that

$$\|g(x) - \frac{b}{b - a} G_1(x) - \frac{a}{b - a} G_2(x)\|$$

$$= \frac{1}{|b - a|} \|[(b - a)g(x) - bG_1(x) - aG_2(x)]\|$$

$$\leq \frac{1}{|b - a|} \|bg(x) - abg(x - 1) - bG_1(x)\|$$

$$+ \frac{1}{|b - a|} \|ag_2(x) - ax + abg(x - 1)\|$$

$$\leq \frac{5 + \sqrt{5}}{20} \varepsilon$$

for all $x \in \mathbb{R}$. Now define a function $F : \mathbb{R} \to X$ by

$$F(x) = b\frac{G_1(x) - a}{b - a} G_2(x)$$

$$= F(3x)$$

for each $x \in \mathbb{R}$, i.e., $F$ is a solution of (1). Moreover, the inequality (8) follows from (17).

The uniqueness of $F$ will be proved. Assume that $F_1, F_2 : \mathbb{R} \to X$ are solutions of (1) and that there exist positive constants $C_1$ and $C_2$ with

$$\|f(x) - F_1(x)\| \leq C_1 \quad \text{and} \quad \|f(x) - F_2(x)\| \leq C_2$$

for all $x \in \mathbb{R}$. According to Theorem 2.1, there exist functions $h_1, h_2 : [-3, 3] \to X$ such that

$$F_1(x) = U_{\mathbb{R} / \mathbb{Z} + 1} [h_1(x - 3|x/3)]$$

$$+ U_{\mathbb{R} / \mathbb{Z}} h_1(x - 3|x/3) - 3$$

$$F_2(x) = U_{\mathbb{R} / \mathbb{Z} + 1} [h_2(x - 3|x/3)]$$

$$+ U_{\mathbb{R} / \mathbb{Z}} h_2(x - 3|x/3)$$

for any $x \in \mathbb{R}$, since $F_1$ and $F_2$ are solutions of (1).

Fix a $t$ with $0 \leq t < 3$. It then follows from (18) and (19) that

$$\|h_1(t) - h_2(t)\|$$

$$\leq \|U_{\mathbb{R} / \mathbb{Z} + 1} [h_1(t - 3 - 3) - h_2(t - 3)]\|$$

$$\leq \|U_{\mathbb{R} / \mathbb{Z}} h_1(t - 3) - h_2(t - 3)\|$$

$$\leq \|F_1(3n + t) - F_2(3n + t)\|$$

$$\leq \|F_1(3n + t) - f(3n + t)\| + \|f(3n + t) - F_2(3n + t)\|$$

$$\leq C_1 + C_2$$
for each \( n \in \mathbb{Z} \), i.e.,
\[
\left\| \frac{a^{n+1} - b^{n+1}}{a - b} [h_1(t) - h_2(t)] + \frac{a^n - b^n}{a - b} [h_1(t - 3) - h_2(t - 3)] \right\| \leq C_1 + C_2
\]
for every \( n \in \mathbb{Z} \). Dividing both sides by \( a^n \) yields that
\[
\left\| \frac{a - (b/a)^n b}{a - b} [h_1(t) - h_2(t)] + \frac{1 - (b/a)^3}{a - b} [h_1(t - 3) - h_2(t - 3)] \right\| \leq C_1 + C_2 \cdot \frac{1}{a^n}.
\]
Let \( n \to \infty \) to get
\[
a[h_1(t) - h_2(t)] + [h_1(t - 3) - h_2(t - 3)] = 0. \tag{21}
\]
Analogously, divide both sides of (20) by \( |b|^n \) and let \( n \to -\infty \) to get
\[
b[h_1(t) - h_2(t)] + [h_1(t - 3) - h_2(t - 3)] = 0. \tag{22}
\]
From (21) and (22), it follows that
\[
\begin{pmatrix}
  a & 1 \\
  b & 1
\end{pmatrix}
\begin{pmatrix}
  h_1(t) - h_2(t) \\
  h_1(t - 3) - h_2(t - 3)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Because \( a - b \neq 0 \), it should hold that
\[
h_1(t) - h_2(t) = h_1(t - 3) - h_2(t - 3) = 0
\]
for any \( t \in [0, 3] \), i.e., \( h_1(t) = h_2(t) \) for all \( t \in [-3, 3] \).
Therefore, it is true that \( F_1(x) = F_2(x) \) for any \( x \in \mathbb{R} \).

**Remark 1.** The presented proof of uniqueness of \( F \) is due to an idea of Professor Changsung Choi. It should be remarked that the uniqueness of \( F \) can be obtained directly from [2, Proposition 1].

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