New Classes of Salagean type Meromorphic Harmonic Functions

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Abstract—In this paper, a necessary and sufficient coefficient are given for functions in a class of complex valued meromorphic harmonic univalent functions of the form \( f = h + \bar{g} \) using Salagean operator. Furthermore, distortion theorems, extreme points, convolution condition and convex combinations for this family of meromorphic harmonic functions are obtained.

Keywords—Harmonic mappings, Meromorphic functions, Salagean operator.

I. INTRODUCTION

A continuous function \( f = u + iv \) is a complex valued harmonic function in a complex domain \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). In any simply connected domain \( \mathbb{D} \subset \mathbb{C} \) we can write \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic in \( D \). A necessary and sufficient condition for \( f \) to be locally univalent and sense preserving in \( D \) is that \( |h'(z)| > |g'(z)| \) in \( D \) (see [2]). In [3], Hengartner and Schober investigated functions harmonic in the exterior of the unit disc \( U = \{ z : |z| > 1 \} \). They showed that complex valued, harmonic, sense preserving, univalent mapping \( f \) must admits the representation

\[
f(z) = h(z) + \bar{g}(z) + A \log |z|,
\]

where

\[
h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k}
\]

and

\[
g(z) = \beta \bar{z} + \sum_{k=1}^{\infty} b_k z^{-k}
\]

for \( 0 \leq |\beta| < |\alpha|, \ A \in \mathbb{C} \).

Let \( MH \) denote the class of functions

\[
f(z) = h(z) + g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{-k} + \sum_{k=1}^{\infty} b_k z^{-k}
\]

which are harmonic in the punctured unit disk \( U \setminus \{ 0 \} \). \( h(z) \) and \( g(z) \) are analytic in \( U \setminus \{ 0 \} \) and \( U \), respectively, and \( h(z) \) has a simple pole at the origin with residue 1 here.

For \( f = h + \bar{g} \) given by (1), Jahangiri [4] defined the modified Salagean operator of \( f \) as

\[
D^n f(z) = D^n h(z) + (-1)^n D^n g(z); \quad n = 0, 1, 2, \ldots,
\]

where

\[
D^n h(z) = \left( \frac{-1}{z} \right)^n + \sum_{k=1}^{\infty} k^n a_k z^{-k}
\]

and

\[
D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^{-k}.
\]

A function \( f(z) \in MH \) is said to be in the subclass \( MHS^* \) of meromorphically harmonic starlike in \( U \setminus \{ 0 \} \) if it satisfies the condition

\[
Re \left\{ - \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} > 0, \quad z \in U \setminus \{ 0 \}.
\]

Now we define a new class \( MHS^*_S(n, \alpha) \) (see [1]).

Definition 1.1: For \( 0 \leq \alpha < 1 \), we let \( MHS^*_S(n, \alpha) \) denote the class of meromorphic harmonic functions \( f \) of the form (1) such that

\[
Re \left\{ - \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\} > \alpha, \quad z \in U \setminus \{ 0 \}.
\]

We let the subclass \( MHS^*_S(n, \alpha) \) consist of meromorphic harmonic functions \( f_n = h_n + g_n \in MHS^*_S(n, \alpha) \) so that \( h_n \) and \( g_n \) are of the form

\[
h_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and}
\]

\[
g_n(z) = \frac{(-1)^n}{z} \sum_{k=1}^{\infty} b_k z^{-k},
\]

where \( a_k \geq 0, b_k \geq 0 \).

In this paper, we have obtained the coefficient conditions for the classes \( MHS^*_S(n, \alpha) \) and \( MHS^*_S(n, \alpha) \). Further a representation theorem, inclusion properties and distortion bound for the class \( MHS^*_S(n, \alpha) \) are established.

II. MAIN RESULTS

Theorem 2.1: Let \( f \) be of the form (1). If

\[
\sum_{k=1}^{\infty} (|a_{2k}| + |b_{2k}|) (2k)^{n+1} + (2k - 1 + \alpha) |a_{2k-1}| + (2k - \alpha) |b_{2k-1}| (2k - 1)^n \leq 1 - \alpha,
\]

then \( f \) is harmonic univalent, sense preserving in \( U \setminus \{ 0 \} \) and \( f \in MHS^*_S(n, \alpha) \).
Proof: For $0 < |z_1| \leq |z_2| < 1$ we have

\[
|f(z_1) - f(z_2)| \\
\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
\geq \frac{z_1 - z_2}{|z_1||z_2|} \\
- |z_1 - z_2| \sum_{k=1}^{\infty} (|a_k| + |b_k|) |z_1^{-k+1} + \cdots + z_2^{-k+1}| \\
> \frac{z_1 - z_2}{|z_1||z_2|} \left[ 1 - |z_2|^2 \sum_{k=1}^{\infty} k(|a_k| + |b_k|) \right] \\
= \frac{z_1 - z_2}{|z_1||z_2|} \left[ 1 - |z_2|^2 \left( \sum_{k=1}^{\infty} 2k(|a_{2k}| + |b_{2k}|) \\
+ \sum_{k=1}^{\infty} (2k - 1)(|a_{2k-1}| + |b_{2k-1}|) \right) \right] \\
> \frac{z_1 - z_2}{|z_1||z_2|} \left[ 1 - \sum_{k=1}^{\infty} (2k - 1)(|a_{2k-1}| + |b_{2k-1}|) \right] \\
+ \sum_{k=1}^{\infty} (2k - 1)^{n+1} |a_{2k}| |z^{k-1}| \\
- \sum_{k=1}^{\infty} (2k - 1)^n (2k - 1 + \alpha) |a_{2k-1}| \\
\geq \sum_{k=1}^{\infty} (2k - 1)^n |a_{2k}| |z^{k-1}| \\
+ \sum_{k=1}^{\infty} (2k - 1)^n (2k - 1 - \alpha) |b_{2k-1}| \\
\geq \sum_{k=1}^{\infty} 2k|b_{2k}| + \sum_{k=1}^{\infty} (2k - 1)|b_{2k-1}| \\
> \sum_{k=1}^{\infty} k|b_k||z^{k-1}| \geq |g(z)|.
\]

This last expression is non negative by (6) and so $f$ is univalent in $U \setminus \{0\}$. To show that $f$ is sense preserving in $U \setminus \{0\}$, we need to show that $|h'(z)| \geq |g'(z)|$ in $U \setminus \{0\}$. We have

\[
|h'(z)| \geq 1 - \sum_{k=1}^{\infty} k|a_k||z^{k-1}| \\
= 1 - \sum_{k=1}^{\infty} k|a_k| |z^{k-1}| - \sum_{k=1}^{\infty} k|a_k| \\
\geq 1 - \sum_{k=1}^{\infty} (2k)^{n+1} |a_{2k}| \\
- \sum_{k=1}^{\infty} (2k - 1)^n (2k - 1 + \alpha) |a_{2k-1}| \\
\geq \sum_{k=1}^{\infty} (2k - 1)^n |a_{2k}| |z^{k-1}| \\
\geq \sum_{k=1}^{\infty} (2k - 1)^n |b_{2k}| |z^{k-1}|.
\]

Now, we will show that $f \in MHS_{\mathbb{S}}(n, \alpha)$. According to (2) and (3), for $0 \leq \alpha < 1$, we have

\[
Re \left\{ \frac{-2D^{n+1}h(z) - 2(-1)^n D^{n+1} g(z)}{T^n(z)} \right\} \geq \alpha,
\]

where

\[
T^n(z) = D^n h(z) + (-1)^n D^n g(z) - D^n h(-z) - (-1)^n D^n g(-z).
\]

Using the fact that $Re\{w\} \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

\[
1 - \alpha - \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \geq \alpha + \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)}
\]

which is equivalent to

\[
|2D^{n+1}f(z) - (1 - \alpha)(D^n f(z) - D^n f(-z))| \\
- 2D^{n+1}f(z) + (1 + \alpha)(D^n f(z) - D^n f(-z)) \geq 0. \quad (7)
\]

Substituting for $D^n f(z)$ and $D^{n+1} f(z)$ in (7) yields

\[
\left| \frac{2(-1)^n}{z} - 2 \sum_{k=1}^{\infty} k^n a_k z^k + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} b_k z^k \\
+ (1 - \alpha) \left[ \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k \\
+ (-1)^n \sum_{k=1}^{\infty} k^n b_k z^k \right] \right| \\
- \left| \frac{2(-1)^n}{z} - 2 \sum_{k=1}^{\infty} k^n a_k z^k \\
+ 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} b_k z^k - (1 + \alpha) \left[ \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n b_k z^k \right] \right| \\
- \left| \sum_{k=1}^{\infty} (1 - \alpha)^n k^n a_k z^k \right| \\
- \left| \sum_{k=1}^{\infty} (1 - \alpha)^n k^n b_k z^k \right| \right| \right| \\
- \left| \sum_{k=1}^{\infty} (1 - \alpha)^n k^n b_k z^k \right| \\
= \frac{2(2 - \alpha)(-1)^n}{z} \\
- \sum_{k=1}^{\infty} (2k - (1 - \alpha) + (1)^k (1 + \alpha)) k^n a_k z^k \\
+ (1 - \alpha) \left[ \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n b_k z^k \right] \\
- \left| \sum_{k=1}^{\infty} (2k + (1 + \alpha) - (1)^k (1 + \alpha)) k^n a_k z^k \\
- (1 - \alpha) \left[ \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n b_k z^k \right] \right| \\
- \left| \sum_{k=1}^{\infty} (2k - (1 - \alpha) + (1)^k (1 + \alpha)) k^n b_k z^k \right| \\
= \frac{2(2 - \alpha)(-1)^n}{z}.
\]
\[-2 \sum_{k=1}^{\infty} (2k - 2 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}\]
\[-2 \sum_{k=1}^{\infty} (2k)^{n+1}a_{2k}z^{2k} + 2(-1)^{n} \sum_{k=1}^{\infty} (2k)^{n+1}b_{2k}z^{2k}\]
\[+2(-1)^{n} \sum_{k=1}^{\infty} (2k - \alpha)(2k - 1)^{n}b_{2k-1}z^{2k-1}\]
\[-2(1-a)(-1)^{n}\]
\[\geq \frac{2(2 - \alpha)(-1)^{n}}{z}\]
\[-2 \sum_{k=1}^{\infty} (2k - 2 + \alpha)(2k - 1)^{n}|a_{2k-1}| |z|^{2k-1}\]
\[-2 \sum_{k=1}^{\infty} (2k)^{n+1}|a_{2k}| |z|^{2k} - 2 \sum_{k=1}^{\infty} (2k)^{n+1}|b_{2k}| |z|^{2k}\]
\[-2 \sum_{k=1}^{\infty} (2k - \alpha)(2k - 1)^{n}|b_{2k-1}| |z|^{2k-1}\]
\[-2(1-a)(-1)^{n}\]
\[\geq \frac{2\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}}{\phi(z)}\]
\[-2 \sum_{k=1}^{\infty} (2k - 2 + \alpha)(2k - 1)^{n}|a_{2k-1}| |z|^{2k-1}\]
\[-2 \sum_{k=1}^{\infty} (2k)^{n+1}|a_{2k}| |z|^{2k} - 2 \sum_{k=1}^{\infty} (2k)^{n+1}|b_{2k}| |z|^{2k}\]
\[-2 \sum_{k=1}^{\infty} (2k - \alpha)(2k - 1)^{n}|b_{2k-1}| |z|^{2k-1}\]
\[-2(1-a)(-1)^{n}\]
\[\geq \frac{2(2 - \alpha)(-1)^{n}}{\phi(z)}\]
\[-2 \sum_{k=1}^{\infty} (2k - 2 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}\]
\[-2 \sum_{k=1}^{\infty} (2k)^{n+1}a_{2k}z^{2k} + 2(-1)^{n} \sum_{k=1}^{\infty} (2k)^{n+1}b_{2k}z^{2k}\]
\[+2(-1)^{n} \sum_{k=1}^{\infty} (2k - \alpha)(2k - 1)^{n}b_{2k-1}z^{2k-1}\]
\[-2(1-a)(-1)^{n}\]
\[\geq \frac{2(2 - \alpha)(-1)^{n}}{\phi(z)}\]
\[-2 \sum_{k=1}^{\infty} (2k - 2 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}\]
\[-2 \sum_{k=1}^{\infty} (2k)^{n+1}a_{2k}z^{2k} + 2(-1)^{n} \sum_{k=1}^{\infty} (2k)^{n+1}b_{2k}z^{2k}\]
\[+2(-1)^{n} \sum_{k=1}^{\infty} (2k - \alpha)(2k - 1)^{n}b_{2k-1}z^{2k-1}\]
\[-2(1-a)(-1)^{n}\]
\[\geq \frac{2(2 - \alpha)(-1)^{n}}{\phi(z)}\]
\[-2 \sum_{k=1}^{\infty} (2k - 2 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}\]
\[-2 \sum_{k=1}^{\infty} (2k)^{n+1}a_{2k}z^{2k} + 2(-1)^{n} \sum_{k=1}^{\infty} (2k)^{n+1}b_{2k}z^{2k}\]
\[+2(-1)^{n} \sum_{k=1}^{\infty} (2k - \alpha)(2k - 1)^{n}b_{2k-1}z^{2k-1}\]
\[-2(1-a)(-1)^{n}\]
\[\geq \frac{2(2 - \alpha)(-1)^{n}}{\phi(z)}\]

This last expression is non-negative by (7), and so the proof is complete.

**Theorem 2.2:** Let \(f_n = h_n + g_n\) where \(h_n\) and \(g_n\) are of the form (4) and (5). Then \(f_n \in MHS^2_z(n, \alpha)\), if and only if
\[
\sum_{k=1}^{\infty} [(a_{2k} + b_{2k})(2k)^{n+1} + ((2k - 1 + \alpha)a_{2k-1} - 1)\leq 1 - \alpha.
\]

Proof: Since \(MHS^2_z(n, \alpha) \subset MHS^2_z(n, \alpha)\), we only need to prove the (only if) part of the theorem. To this end, for functions \(f_n = h_n + g_n\), we notice that condition
\[
\text{Re} \left\{ \frac{-2D^{n+1} f(z)}{D^n f(z) - D^n f(-z)} \right\} \geq \alpha, \quad z \in U \setminus \{0\},
\]
is equivalent to
\[
\text{Re} \left\{ \frac{2(1-a)}{z} \sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}}{\phi(z)}\right\} \geq 0,
\]
which implies
\[
\text{Re} \left\{ \frac{2(1-a)}{z} \sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}z^{2k-1}}{\phi(z)}\right\} \geq 0,
\]

where
\[
\phi(z) = \frac{2}{z} + 2 \sum_{k=1}^{\infty} (2k - 1)^{n}a_{2k-1}z^{2k-1}\]

The condition (9) must hold for all \(z \in U \setminus \{0\}\). By choosing \(0 < r < 1\), from the left hand (9), we have
\[
1 - \alpha = \sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}r^{2k}
\]
\[
\sum_{k=1}^{\infty} (2k)^{n+1}a_{2k}r^{2k+1} + (1-\alpha)\sum_{k=1}^{\infty} (2k)^{n+1}b_{2k}r^{2k+1}
\]
\[
\sum_{k=1}^{\infty} (2k - 1)^{n}a_{2k-1}r^{2k-1}
\]
\[
\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}r^{2k}
\]
\[
\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}r^{2k}
\]
\[
\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}r^{2k}
\]
\[
\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}r^{2k}
\]
\[
\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^{n}a_{2k-1}r^{2k}
\]

If the condition (8) does not hold, then the number in (10) is negative for \(r\) sufficiently close to 1. Hence there exist
\(z_0 = r_0 \) in \((0, 1)\), for which the eqnarray in (10) is negative. This contradicts the required condition for and so the proof is complete. It is easily seen that \(f_n(z) \in MHS_S(n, \alpha)\). Thus we complete the of the Theorem 2.2. \[ \]

**Theorem 2.3:** If \(f_n = h_n + \bar{g}_n \in MHS_S(n, \alpha)\) for \(0 < |z| = r < 1\), then
\[
\frac{1}{r} - 1 - \frac{\alpha}{2n+1}r \leq |f_n(z)| \leq \frac{1}{r} + 1 - \frac{\alpha}{2n+1}r.
\]

**Proof:** Let \(f_n = h_n + \bar{g}_n \in MHS_S(n, \alpha)\). Taking the absolute value of \(f\) we obtain
\[
|f_n(z)| = \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k z^k \right|
\leq \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r^k
\leq \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r
\leq \frac{1}{r} + \frac{\alpha}{2n+1} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1 - \alpha} (|a_k| + |b_k|) r,
\]
and
\[
|f_n(z)| = \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k z^k \right|
\geq \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r^k
\geq \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r
\geq \frac{1}{r} - \frac{\alpha}{2n+1} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1 - \alpha} (|a_k| + |b_k|) r,
\]
where for \(k = 1, 2, \ldots\)
\[
\begin{align*}
h_{n0}(z) &= \left| g_{n0}(z) = \frac{1}{z} \right| + \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 + \alpha) z^{2k-1}} \\
h_{n2k-1}(z) &= \left| \frac{1}{z} + \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 + \alpha) z^{2k-1}} \right| \\
h_{n2k}(z) &= \left| \frac{1}{z} + \frac{1 - \alpha}{(2k)^\alpha(2k) z^{2k}} \right| \\
g_{n2k-1}(z) &= \left| \frac{1}{z} + \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 - \alpha) z^{2k-1}} \right| \\
g_{n2k}(z) &= \left| \frac{1}{z} + \frac{1 - \alpha}{(2k)^\alpha(2k) z^{2k}} \right|
\end{align*}
\]
and
\[
\sum_{k=0}^{\infty} (x_k + y_k) = 1, \quad x_k \geq 0 \quad \text{and} \quad y_k \geq 0.
\]
In particular, the extreme point of \(MHS_S^n(n, \alpha)\) are \(\{h_n\}\) and \(\{g_n\}\).

**Proof:** For functions \(f_n = h_n + \bar{g}_n\), where \(h_n\) and \(g_n\) of the form (4) and (5), we have
\[
f_n(z) = \sum_{k=0}^{\infty} (x_k h_{nk} + y_k g_{nk})
\]
\[
= x_0 h_{n0} + y_0 g_{n0} + \sum_{k=1}^{\infty} (x_k + y_k) \frac{1}{z} + \sum_{k=1}^{\infty} x_{2k-1} \left( \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 + \alpha) z^{2k-1}} \right)
\]
\[
+ \sum_{k=1}^{\infty} x_{2k} \left( \frac{1 - \alpha}{(2k)^\alpha(2k) z^{2k}} \right)
\]
\[
+ \sum_{k=1}^{\infty} y_{2k-1} \left( \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 - \alpha) z^{2k-1}} \right)
\]
\[
+ \sum_{k=1}^{\infty} y_{2k} \left( \frac{1 - \alpha}{(2k)^\alpha(2k) z^{2k}} \right)
\]
\[
= \sum_{k=0}^{\infty} (x_k + y_k) \frac{1}{z} + \sum_{k=0}^{\infty} \left( \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 + \alpha) z^{2k-1}} \right) x_{2k-1} z^{2k-1}
\]
\[
+ \sum_{k=0}^{\infty} \left( \frac{1 - \alpha}{(2k)^\alpha(2k) z^{2k}} \right) x_{2k} z^{2k}
\]
\[
+ \sum_{k=0}^{\infty} \left( \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 - \alpha) z^{2k-1}} \right) y_{2k-1} z^{2k-1}
\]
\[
+ \sum_{k=0}^{\infty} \left( \frac{1 - \alpha}{(2k)^\alpha(2k) z^{2k}} \right) y_{2k} z^{2k}.
\]

**Corollary 2.4:** Let \(A = \{ w : |w| < \frac{2^{n+1} - 1 + n}{2^{n+1}} \} \).
If \(f_n = h_n + \bar{g}_n \in MHS_S^+(n, \alpha)\), then
\[
f_n(U) \subset A^*.
\]

**Theorem 2.5:** \(f_n = h_n + \bar{g}_n \in MHS_S^+(n, \alpha)\) if and only if \(f_n\) can be expressed as
\[
f_n(z) = \sum_{k=0}^{\infty} (x_k h_{nk} + y_k g_{nk}), \quad (11)
\]
Then,
\[
\sum_{k=1}^{\infty} (2k - 1 + \alpha)(2k - 1)^n
\times \left\{ \frac{1 - \alpha}{(2k - 1)^\alpha(2k - 1 + \alpha) z^{2k-1}} \right\}
\]

\[ \]
Theorem 2.6: If \( f \in MHS_{S}(n, \alpha) \), then the diameter \( D_f \) of \( C \setminus f(U) \) satisfies
\[
D_f \geq 2|1 + b_1|.
\]

Proof: Let \( D_f(R) \) be the diameter of \( f([|z| = R], 0 < R < 1 \), and let \( D_f(R) = \max_{|z| = R} |f(z) - f(-z)| \). Then \( D_f(R) \to D_f(R) \to D_f(R) \). Since
\[
[D_f(R)]^2 \geq \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta}) - f(-Re^{i\theta})|^2 d\theta
\]
\[
= 4 \left[ \frac{1}{R^2} + b_1 + b_1 + \sum_{k=1}^{\infty} (|a_{2k-1}|^2 + |b_{2k-1}|^2) R^{2(2n-1)} \right]
\]
\[
\geq 4(1 + 2\text{Re}b_1)
\]
\[
\geq 4(1 + 2\text{Re}b_1 + \sum_{k=1}^{\infty} (|a_{2k-1}|^2 + |b_{2k-1}|^2) \right)
\]
we conclude that \( D_f \geq 2\sqrt{1 + b_1^2} \).

Note that if \( f \) and \( F \) are
\[
f(z) = h(z) + g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k
\]
and
\[
F(z) = H(z) + G(z) = \frac{1}{z} + \sum_{k=1}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,
\]
then the convolution (or Hadamard product) of \( f \) and \( F \) if defined to be the function
\[
(f * F)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k z^k.
\]
\(+ (2k - 1 - \alpha) b_{i2k-1}(2k - 1)^n \leq 1 - \alpha.\)

For \(\sum_{i=1}^{\infty} t_i = 1, 0 < t_i \leq 1,\)

the convex combinations of \(f_{n_i}\) may be written as

\[
\sum_{i=1}^{\infty} t_i f_{n_i}(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i a_{ik} \right) z^{-n} + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i b_{ik} \right) z^{-n}.
\]

Then by (13),

\[
\sum_{k=1}^{\infty} \left[ \frac{(2k)^n + 1}{1 - \alpha} \sum_{i=1}^{\infty} t_i (a_{ik} + b_{2k}) + \frac{(2k - 1)^n}{1 - \alpha} \left( \frac{(2k - 1 + \alpha)}{1 - \alpha} \sum_{i=1}^{\infty} t_i a_{2k-1} + \frac{(2k - 1 - \alpha)}{1 - \alpha} \sum_{i=1}^{\infty} t_i b_{2k-1} \right) \right] \leq \sum_{i=1}^{\infty} t_i = 1.
\]

Thus

\[
\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in MHS_G(n, \alpha).
\]

**Theorem 2.9:** If \(f_n \in MHS_G(n, \alpha),\) then

\[
\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}.
\]

Equality occurs if and only if \(C\setminus f(U)\) has area zero.

**Proof:** The area of the omitted set is

\[
\lim_{R \to 1} \frac{1}{2i} \int_{0<|z|=R<1} \bar{f} df = \lim_{R \to 1} \left[ \frac{1}{2i} \int_{0<|z|=R<1} \bar{h}h' dz + \frac{1}{2i} \int_{0<|z|=R<1} \overline{g\overline{g}} d\overline{z} + \frac{1}{2i} \int_{0<|z|=R<1} \bar{g}g' dz \right] = \pi \left( \sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) R^{2k} - \frac{1}{R^2} - 2Re\{b_1\} \right).
\]

For \(0 < r < 1\) the curve \(\Gamma_r = f(C_r)\) is a simple closed curve oriented clockwise. Hence, for \(R \to 1\) we obtain

\[
\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) - 1 - 2Re\{b_1\} \leq 0
\]

and the result follows.