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Periodic solutions for some strongly nonlinear oscillators by He's energy balance method

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Abstract—In this paper, applying He's energy balance method to determine frequency formulation relations of nonlinear oscillators with discontinuous term or fractional potential. By calculation and computer simulations, compared with the exact solutions show that the results obtained are of high accuracy.

Keywords—He's energy balance method; Periodic solution; Nonlinear oscillator; Discontinuous; Fractional potential.

I. Introduction

In this paper, we considers the following generalized nonlinear oscillators

$$u'' + f(u)u = 0,\ u(0) = A,\ u'(0) = 0, \tag{1}$$

where $f(u) > 0$ is a known function of u.

In case that there exists no small parameter in the equation, the traditional perturbation methods cannot be applied directly to (1). Hereby, we will give two examples to show how to solve the problems of nonlinear oscillators with discontinuous terms or fractional potential by a new variational approach proposed by He's energy balance method, which is an easy, effective and convenient mathematical tool for nonlinear oscillators.

II. He's energy balance method

Firstly, we consider the following nonlinear oscillator with discontinuous term

$$u'' + au^3 + bu + cu|u| = 0,\ u(0) = A,\ u'(0) = 0. \tag{2}$$

Using the semi-inverse method [1], the variational principle of equation (2) can be easily obtained:

$$\begin{aligned} J(u) &= \int_{-\frac{T}{2}}^{0} \Big[-\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 - \frac{1}{3}cu^3 \Big]\,\mathrm{d}t \\ &\quad + \int_{0}^{\frac{T}{2}} \Big[-\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 + \frac{1}{3}cu^3 \Big]\,\mathrm{d}t \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \Big[-\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 \\ &\quad + \mathrm{sgn}(u)\frac{1}{3}cu^3 \Big]\,\mathrm{d}t. \end{aligned} \tag{3}$$

The Hamiltonian of equation (2), therefore, can be written in the form

$$\begin{aligned} H &= \frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 + \mathrm{sgn}(u)\frac{1}{3}cu^3 \\ &= \frac{1}{4}aA^4 + \frac{1}{2}bA^2 + \mathrm{sgn}(A)\frac{1}{3}cA^3, \end{aligned}$$

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that is

$$\begin{aligned} &\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 + \mathrm{sgn}(u)\frac{1}{3}cu^3 - \frac{1}{4}aA^4 - \frac{1}{2}bA^2 \\ &-\mathrm{sgn}(A)\frac{1}{3}cA^3 = 0. \end{aligned} \tag{4}$$

We use the following trial function to determine the angular frequency ω:

$$u = A\cos\omega t. \tag{5}$$

Substituting (5) into (4), we obtain the following residual

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{4}aA^4\cos^4\omega t + \frac{1}{2}bA^2\cos^2\omega t \\ &\quad +\mathrm{sgn}(A\cos\omega t)\frac{1}{3}cA^3\cos^3\omega t \\ &\quad -\frac{1}{4}aA^4 - \frac{1}{2}bA^2 - \mathrm{sgn}(A)\frac{1}{3}cA^3. \end{aligned}$$

We set

$$\int_0^{\frac{T}{4}} R(t)\cos\omega t\,\mathrm{d}t = 0,\ T = \frac{2\pi}{\omega}$$

to determine the $\omega - A$ relationship, which reads

$$\omega^2 = \frac{13}{10}aA^2 + (\frac{16-3\pi}{8})cA + b,$$

then

$$\omega = \Big[\frac{13}{10}aA^2 + (\frac{16-3\pi}{8})cA + b\Big]^{\frac{1}{2}}. \tag{6}$$

Therefore, we can obtain the following periodic solution

$$u(t) = A\cos\Big[\Big(\frac{13}{10}aA^2 + (\frac{16-3\pi}{8})cA + b\Big)^{\frac{1}{2}} t\Big].$$

To illustrate the accuracy of the obtained results, we give an example as follows:

In case $a = 0, b = 0$, equation (1) becomes

$$u'' + cu|u| = 0,\ u(0) = A,\ u'(0) = 0,$$

its frequency reads $\omega = \left(\frac{16-3\pi}{8}\right)^{\frac{1}{2}} c^{\frac{1}{2}} A^{\frac{1}{2}}$, its exact frequency is $\omega_{ex} = 0.921318c^{\frac{1}{2}}A^{\frac{1}{2}}$. Therefore its accuracy reaches 0.0088. The above result is of high accuracy.

Secondly, we consider another nonlinear oscillator of the following form

$$u'' + au^{\frac{1}{m}} + bu^{2n+1} = 0,\ u(0) = A,\ u'(0) = 0. \tag{7}$$

Using the semi-inverse method [1], the variational principle of equation (7) can be easily obtained:

$$J(u) = \int_0^T \left[-\frac{1}{2}u'^2 + \frac{am}{m+1}u^{\frac{m+1}{m}} + \frac{b}{2n+2}u^{2n+2} \right] \mathrm{d}t.$$

The Hamiltonian of equation (7), therefore, can be written in the form

$$\begin{aligned} H &= \frac{1}{2}u'^2 + \frac{am}{m+1}u^{\frac{m+1}{m}} + \frac{b}{2n+2}u^{2n+2} \\ &= \frac{am}{m+1}A^{\frac{m+1}{m}} + \frac{b}{2n+2}A^{2n+2}, \end{aligned}$$

that is

$$\begin{aligned} &\frac{1}{2}u'^2 + \frac{am}{m+1}u^{\frac{m+1}{m}} + \frac{b}{2n+2}u^{2n+2} \\ &-\frac{am}{m+1}A^{\frac{m+1}{m}} - \frac{b}{2n+2}A^{2n+2} = 0. \end{aligned} \tag{8}$$

We use the following trial function to determine the angular frequency ω:

$$u = A\cos\omega t. \tag{9}$$

Substituting (9) into (8), we obtain the following residual

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{am}{m+1}A^{\frac{m+1}{m}}\cos^{\frac{m+1}{m}}\omega t \\ &+\frac{b}{2n+2}A^{2n+2}\cos^{2n+2}\omega t \\ &-\frac{am}{m+1}A^{\frac{m+1}{m}} - \frac{b}{2n+2}A^{2n+2}, \end{aligned}$$

that is

$$\begin{aligned} R(t) &= \frac{1}{2}\left[A^2\omega^2\sin^2\omega t + \frac{2am}{m+1}A^{\frac{m+1}{m}}\cos^{\frac{m+1}{m}}\omega t \right.\\ &+\frac{b}{n+1}A^{2n+2}\cos^{2n+2}\omega t \\ &\left.-\frac{2am}{m+1}A^{\frac{m+1}{m}} - \frac{b}{n+1}A^{2n+2}\right]. \end{aligned}$$

We set

$$\int_0^{\frac{T}{4}} R(t)\cos\omega t\,\mathrm{d}t = 0,\ T = \frac{2\pi}{\omega}$$

to determine the $\omega - A$ relationship, which reads

$$\begin{aligned} &\frac{1}{3}A^2\omega^2 - \frac{2am}{m+1}A^{\frac{m+1}{m}}\left[1 - \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma(\frac{3}{2}+\frac{1}{2m})}{\Gamma(2+\frac{1}{2m})}\right] \\ &-\frac{bA^{2n+2}}{n+1}\left[1 - \frac{(2n+2)!!}{(2n+3)!!}\right] = 0, \end{aligned}$$

where $\Gamma(\cdot)$ is Γ function, and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$. Then

$$\begin{aligned} \omega &= \frac{\sqrt{3}}{A}\left[\frac{2am}{m+1}A^{\frac{m+1}{m}}\left(1 - \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma(\frac{3}{2}+\frac{1}{2m})}{\Gamma(2+\frac{1}{2m})}\right)\right. \\ &\left.+\frac{bA^{2n+2}}{n+1}\left(1 - \frac{(2n+2)!!}{(2n+3)!!}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{10}$$

Therefore, we can obtain the following periodic solution

$$\begin{aligned} u(t) &= A\cos\left[\frac{\sqrt{3}}{A}\left[\frac{2am}{m+1}A^{\frac{m+1}{m}}\left(1 - \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma(\frac{3}{2}+\frac{1}{2m})}{\Gamma(2+\frac{1}{2m})}\right)\right.\right. \\ &\left.\left.+\frac{bA^{2n+2}}{n+1}\left(1 - \frac{(2n+2)!!}{(2n+3)!!}\right)\right]^{\frac{1}{2}} \cdot t\right]. \end{aligned}$$

To illustrate the accuracy of the obtained results, we give two examples as follows:

In case $m = 3, b = 0$, equation (7) becomes

$$u'' + au^{\frac{1}{3}} = 0,$$

its frequency reads $\omega = 1.0834b^{\frac{1}{2}}A^{-\frac{1}{3}}$, its exact frequency[2] is $\omega_{ex} = 1.0705a^{\frac{1}{2}}A^{-\frac{1}{3}}$. Therefore its accuracy reaches 0.0121.

In case $n = 1, a = 0$, equation (7) becomes

$$u'' + bu^3 = 0,$$

its frequency reads $\omega = 0.8367Ab^{\frac{1}{2}}$, its exact frequency[2] is $\omega_{ex} = 0.8472Ab^{\frac{1}{2}}$. Therefore its accuracy reaches 0.0124. Compared with the results in [3], our result is higher accuracy.

III. Numerical Simulations

In this section, we will present some numerical results at different values.

One can see Figure 1 and Figure 2 on next page. Dashed line: exact solution, continuous line: approximate solution.

IV. Conclusions

In this work, the nonlinear equations are efficiently handled by He's frequency formulation which determined by He's energy balance method. It has been proved to be a powerful mathematical tool for searching exact solutions for nonlinear equations without requirement of perturbation or nonlinearities. The analytical approximation obtained by this new method is valid for the whole solution domain with high accuracy.

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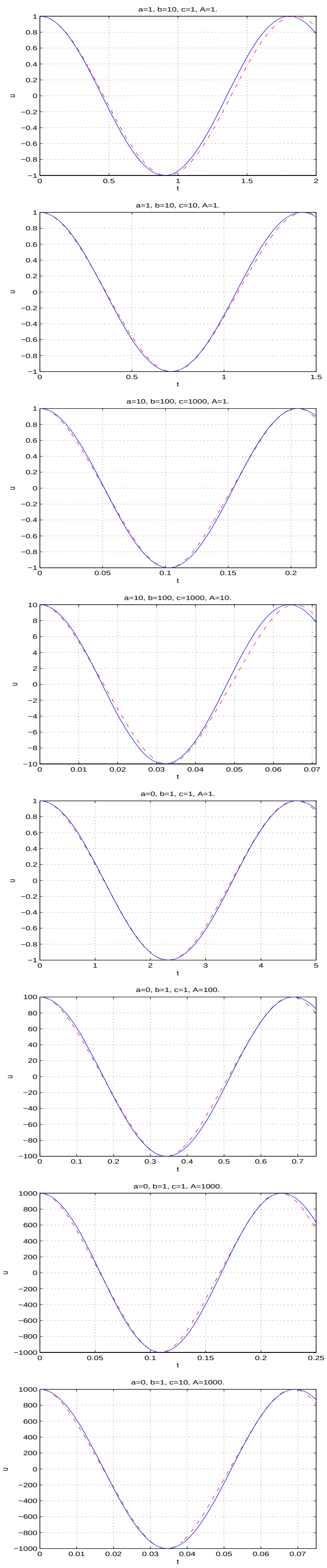

Fig.1. Comparison of exact solution of equation (2) with approximate solution $u = A\cos\omega t$ at different values of a, b, c and A, where ω is defined by equation (6).

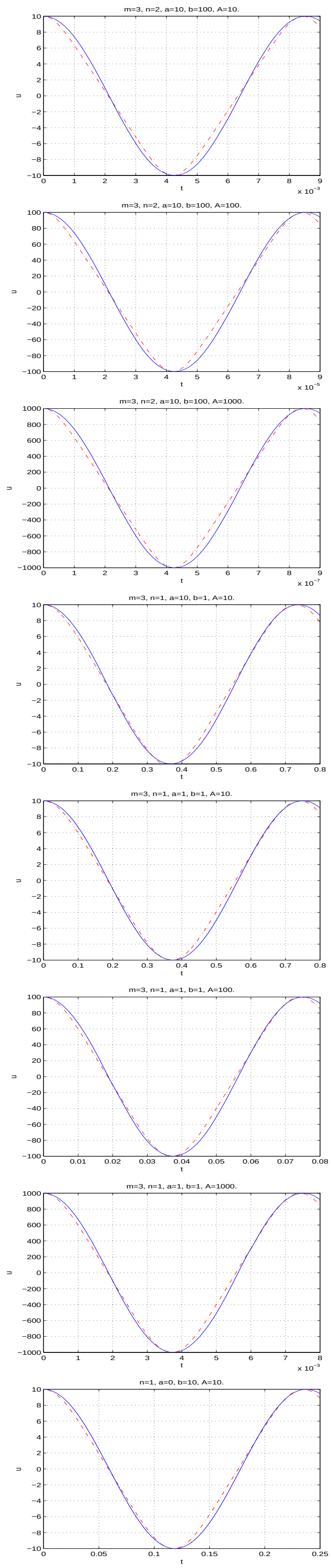

Fig.2. Comparison of exact solution of equation (7) with approximate solution $u = A\cos\omega t$ at different values of a, b, m, n and A, where ω is defined by equation (10).