

# On convergence of affine thin plate bending element

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*Abstract*—In the present paper the displacement-based non-conforming quadrilateral affine thin plate bending finite element ARPQ4 is presented, derived directly from non-conforming quadrilateral thin plate bending finite element RPQ4 proposed by Wanji and Cheung [19]. It is found, however, that element RPQ4 is only conditionally unisolvent. The new element is shown to be inherently unisolvent. This convenient property results in the element ARPQ4 being more robust and thus better suited for computations than its predecessor. The convergence is proved and the rate of convergence estimated. The mathematically rigorous proof of convergence presented in the paper is based on Stummel's generalized patch test and the consideration of the element approximability condition, which are both necessary and sufficient for convergence.

*Keywords*—quadrilateral thin plate bending element, convergence, generalized patch test

## I. INTRODUCTION

**D**ISPLACEMENT-based thin plate bending finite elements are often employed in practical structural design to model plate-like structures. Yet their theoretical formulation is difficult because of the  $C^1$ -continuity requirement across finite element boundaries, which is hard to achieve. A possible solution of this difficulty is the introduction of weaker  $C^1$ -continuity requirements, resulting in a non-conforming plate bending element. Unfortunately, such a non-conforming element need not be automatically convergent and its convergence should be theoretically proved.

An example of an innovative non-conforming quadrilateral thin plate bending finite element is element RPQ4, proposed by Wanji and Cheung [19]. They employed a refined non-conforming displacement field improved by the introduction of the averaged constraint conditions of the inter-element continuity, termed the 'weak continuity conditions' [19]. By performing extensive numerical comparisons for a number of different plate elements, the authors [19] draw the conclusion that the element possesses high accuracy. They also checked that element RPQ4 passes Irons' numerical patch test [3], [16], [18], [20], [22]. It is well known [12]–[15], however, that Irons' numerical patch test is neither necessary nor sufficient condition for convergence.

According to Ciarlet [7] each finite element should satisfy the unisolvence condition. The essential drawback of element RPQ4 is that the element may not satisfy this condition. This may endanger the applicability of element RPQ4 for randomly designed and/or very dense element meshes. The unisolvence problem of element RPQ4 is in the present paper solved with a proposed new affine non-conforming thin plate element termed 'ARPQ4', which is derived directly from the original element

RPQ4 by introducing an affine transformation as described in [10]. Such an element appears to be inherently unisolvent.

One of the objectives of the present paper is thus to present a rigorous mathematical proof that necessary and sufficient conditions for convergence of ARPQ4 element are satisfied indeed. Our derivation of the proof is based on Stummel's generalized patch test [14] and the approximability condition [7], [14], but is somewhat unusual [11] in order to incorporate the specific type of the weak continuity conditions employed in [19]. As also discussed by Wanji [20], the generalized patch test is difficult to apply to a broad class of elements and should normally be performed on each particular element. A rare example of the convergence analysis according to Stummel's generalized patch test of a whole class of non-conforming simplex elements is presented by Wang [18]. His findings cannot be directly used for element ARPQ4, however, which is due to a different element geometry and specific type of the weak continuity conditions. See [21] for the approach in this direction. In addition to the convergence proof, the error estimates are also derived using partially the methodology of Shi [11], Flajs et al. [9] and the inequalities derived by Brenner and Scott [4].

Both convergence and error estimates as derived for element ARPQ4 also hold true for its predecessor RPQ4, provided that it is unisolvent. The modified element ARPQ4 is found not only to be unconditionally unisolvent but, consequently, also more robust and stable when compared to the original element RPQ4.

The outline of the paper is as follows. In Sec. II-A, element RPQ4 is briefly presented and Ciarlet's mathematical definition [7] of finite element is set up. In Sec. III the affine finite element ARPQ4 is defined with the derived unisolvence condition proof. The boundary problem to be solved is defined in Sec. IV. The error is estimated in Sec. V, for each, the consistency and the approximability terms. Numerical examples are presented and discussed in Sec. VI. The paper ends with Conclusions.

## II. THIN PLATE FINITE ELEMENT RPQ4

Finite element RPQ4 is a non-conforming thin plate bending quadrilateral element (Fig. 1), developed directly in the Cartesian coordinates and characterized by the satisfaction of the so called 'weak continuity of displacements on the interelement boundaries'. The ideas behind the formulation and the technical derivation of the stiffness matrix are fully described in Wanji and Cheung [19] and will, thus, not be repeated here. In what follows we somewhat generalize the geometry of the element and assume that the shape of the element should be convex.

In order to prove convergence and estimate the error of finite element RPQ4, we have to recast the original equations

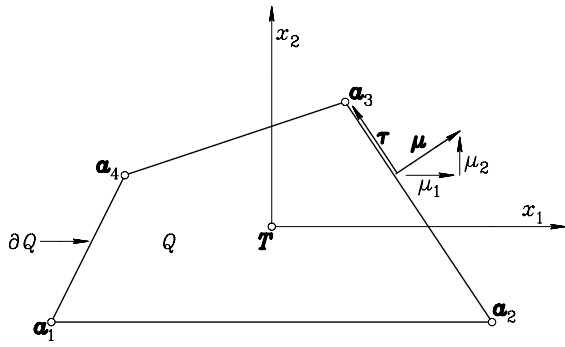


Fig. 1. Quadrilateral thin plate bending finite element RPQ4 [19].

of Wanji and Cheung [19] into the form appropriate for our convergence analysis. For this purpose the following notations are introduced:

$$\begin{aligned} \partial_{1\bullet} &:= \frac{\partial \bullet}{\partial x}, \quad \partial_{2\bullet} := \frac{\partial \bullet}{\partial y}, \quad \partial_{ij\bullet} := \frac{\partial^2 \bullet}{\partial x_i \partial x_j}, \quad 1 \leq i, j \leq 2, \\ \partial_{\mu\bullet} &:= \frac{\partial \bullet}{\partial \mu}, \quad \partial_{\tau\bullet} := \frac{\partial \bullet}{\partial \tau}, \quad \partial^{\alpha\bullet} \equiv \partial^{(\alpha_1, \alpha_2)\bullet} := \frac{\partial^{(\alpha_1 + \alpha_2)\bullet}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \\ \mathbf{X} &:= [1 \ x \ y \ x^2 \ xy \ y^2 \ x^3 \ x^2y \ xy^2 \ y^3 \ x^3y \ xy^3]^T, \\ \mathbf{q} &:= [\dots w_h(\mathbf{a}_i) \ \partial_1 w_h(\mathbf{a}_i) \ \partial_2 w_h(\mathbf{a}_i) \dots]^T, \quad 1 \leq i \leq 4 \end{aligned}$$

$$A := \begin{bmatrix} A_n(\mathbf{a}_1) \\ \vdots \\ A_n(\mathbf{a}_4) \end{bmatrix}, \quad A_n((x, y)) := \begin{bmatrix} \mathbf{X}^T \\ \partial_1 \mathbf{X}^T \\ \partial_2 \mathbf{X}^T \end{bmatrix}. \quad (1)$$

The derivatives in the above equations are understood in the generalized sense, see [1].  $\mathbf{X}$  is the non-conforming interpolation basis of the element, and  $\mathbf{q}$  its vector of nodal degrees of freedom  $q_i, i = 1 \dots 12$ ;  $\mathbf{a}_1, \dots, \mathbf{a}_4$  are the position vectors of the nodal points. Let the origin of the Cartesian coordinate system  $(x, y)$ , in Figs. 1, 2, 3 denoted by  $(x_1, x_2)$ , be the geometric center  $T$  of the quadrilateral  $Q$ . The components of the outer normal of its border  $\partial Q$  in  $x$  and  $y$  directions are denoted by  $\mu_1$  and  $\mu_2$ , respectively (Fig. 1). The area of the quadrilateral is denoted by  $|Q|$ . Let  $w_h|_Q$  denote the non-conforming displacement approximation described by basis  $\mathbf{X}$ , and  $v_h|_Q$  the Wanji and Cheung refined non-conforming displacement approximation on  $Q$  given by [19]

$$\begin{aligned} v_h|_Q &:= w_h|_Q + \lambda_1 \frac{x^2}{2} + \lambda_2 \frac{y^2}{2} + \lambda_3 \frac{xy}{2} = w_h|_Q + \Lambda_Q \quad (2) \\ &= \mathbf{X}^T A^{-1} \mathbf{q} + \Lambda_Q. \end{aligned}$$

Constants  $\lambda_1, \lambda_2$  and  $\lambda_3$  are determined from the weak continuity conditions [19] resulting in

$$\begin{aligned} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} &= \frac{1}{|Q|} \int_{\partial Q} \left( \widetilde{\partial_{\mu} w_h} \begin{bmatrix} \mu_1^2 \\ \mu_2^2 \\ 2\mu_1\mu_2 \end{bmatrix} + \widetilde{\partial_{\tau} w_h} \begin{bmatrix} -\mu_1\mu_2 \\ \mu_1\mu_2 \\ \mu_1^2 - \mu_2^2 \end{bmatrix} \right) ds \\ &\quad - \frac{1}{|Q|} \int_Q \begin{bmatrix} \partial_{11} w_h \\ \partial_{22} w_h \\ 2 \partial_{12} w_h \end{bmatrix} dx. \end{aligned} \quad (3)$$

Functions  $\widetilde{\partial_{\tau} w_h}$  and  $\widetilde{\partial_{\mu} w_h}$  denote a piecewise linear or parabolic interpolation of the tangential derivative and a piecewise linear interpolation of the normal derivative on the border  $\partial Q$ , respectively, interpolated solely by the nodal values  $q_i, i = 1, \dots, 12$ . With such a choice of constants  $\lambda_1, \lambda_2$  and  $\lambda_3$ , the approximation functions fulfill the weak continuity conditions as introduced in [19, Eq. (1)]:

$$\begin{aligned} \int_Q \partial_{11} v_h dx - \int_{\partial Q} \left( \widetilde{\partial_{\mu} w_h} \mu_1^2 - \widetilde{\partial_{\tau} w_h} \mu_1 \mu_2 \right) ds &= 0, \\ \int_Q \partial_{22} v_h dx - \int_{\partial Q} \left( \widetilde{\partial_{\mu} w_h} \mu_2^2 + \widetilde{\partial_{\tau} w_h} \mu_1 \mu_2 \right) ds &= 0, \\ \int_Q 2 \partial_{12} v_h dx - \int_{\partial Q} \left( 2 \widetilde{\partial_{\mu} w_h} \mu_1 \mu_2 + \widetilde{\partial_{\tau} w_h} (\mu_1^2 - \mu_2^2) \right) ds &= 0. \end{aligned} \quad (4)$$

The finite element was shown numerically to pass Irons' patch test [19].

*Remark 2.1:* Note that the refined displacement  $v_h$  is non-conforming both across the boundaries of the elements and in the nodal points.

The first step in proving convergence is the introduction of Ciarlet's mathematical definition of the finite element [7, p. 78].

#### A. Finite element $(Q, P_Q, \Phi_Q)$

Let  $V_h$  denote a finite element space,  $V_h|_Q := P_Q \subset \mathcal{P}_3(Q) \oplus \mathcal{L}\{x^3y, xy^3\}$ ,  $u_h^*$  a finite element approximation of the weak solution,  $u^*$  the weak solution,  $v$  an arbitrary function and  $W_2^m(Q) \equiv H^m(Q)$  the Sobolev spaces with norms  $\|\cdot\|_{m,2,Q} \equiv \|\cdot\|_{m,Q}$  and subnorms  $|\cdot|_{m,2,Q} \equiv |\cdot|_{m,Q}$  for  $0 \leq m \leq 4$ .

With the help of Eqs. (2) and (3) it is easy to show that

$$\widetilde{\partial_{\alpha} v_h} - \partial_{\alpha} v_h = \widetilde{\partial_{\alpha} w_h} - \partial_{\alpha} w_h, \quad \alpha \in \{\mu, \tau\}. \quad (5)$$

The above relations hold true for both linear and parabolic interpolation of  $\partial_{\alpha} w_h$  on the element border. Consequently, we have  $\lambda_i(w_h) = \lambda_i(v_h), 1 \leq i \leq 3$ . We introduce two sets of linear functionals,  $\Sigma_Q := \{\varphi_i \equiv \varphi_i^Q, i = 1, \dots, 12\}$  and  $\Phi_Q := \{\phi_i \equiv \phi_i^Q, i = 1, \dots, 12\}$ , as

$$\begin{aligned} \varphi_{3i-2}(w_h) &:= w_h(\mathbf{a}_i) = \mathbf{q}_{3i-2} = \\ &= v_h(\mathbf{a}_i) - \Lambda_Q(\mathbf{a}_i) =: \phi_{3i-2}(v_h), \quad 1 \leq i \leq 4, \\ \varphi_{3i-1}(w_h) &:= \partial_1 w_h(\mathbf{a}_i) = \mathbf{q}_{3i-1} = \\ &= \partial_1 v_h(\mathbf{a}_i) - \partial_1 \Lambda_Q(\mathbf{a}_i) =: \phi_{3i-1}(v_h), \quad 1 \leq i \leq 4, \\ \varphi_{3i}(w_h) &:= \partial_2 w_h(\mathbf{a}_i) = \mathbf{q}_{3i} = \\ &= \partial_2 v_h(\mathbf{a}_i) - \partial_2 \Lambda_Q(\mathbf{a}_i) =: \phi_{3i}(v_h), \quad 1 \leq i \leq 4. \end{aligned}$$

By Ciarlet's definition of a finite element [7], the set  $\Phi_Q$  must be  $P_Q$ -unisolvant in the following sense: given any real scalars  $\alpha_i, i = 1, \dots, 12$ , there exists a unique function  $p \in P_Q$  which satisfies the conditions

$$\phi_i(p) = \alpha_i, \quad 1 \leq i \leq 12 \quad (6)$$

[7, p. 78].

*Remark 2.2:* The proof of unisolvence becomes straightforward, if the following lemma is proved first. This lemma will convert the  $P_Q$ -unisolvence problem of the set  $\Phi_Q$  into the  $P_Q$ -unisolvence problem of the set  $\Sigma_Q$ , which is equivalent to requiring the regularity of the interpolation matrix  $A$ .

*Lemma 2.3:* Let the space  $P'_Q$  denote an algebraic dual of space  $P_Q$ . The set  $\Phi_Q$  is the base for  $P'_Q$ , if and only if the set  $\Sigma_Q$  is the base for  $P'_Q$ .

*Proof:* Let the set  $\Sigma_Q$  be the base for  $P'_Q$ . Since  $\varphi_i(w_h) = \phi_i(v_h) = 0$  for  $i = 1, \dots, 12$ , we have  $w_h = 0$ . Hence we can conclude from Eqs. (2) and (3) that  $v_h = 0$ . Lemma 3.1.4 in [4] then guarantees that the set  $\Phi_Q$  is the base. Similarly, invoking Eq. (5), we can prove that if the set  $\Phi_Q$  is the base, the set  $\Sigma_Q$  is also the base. ■

Element RPQ4 [19] is inherently prone to singularity of the interpolation matrix  $A$  if convex or non-convex. One such example of a convex quadrilateral having a singular interpolation matrix  $A$  is the rhombus. Note that in [19] the possibility of the singularity of the interpolation matrix was not explicitly mentioned.

In order to exclude the degenerate convex quadrilaterals, we require

*Condition 2.4:*

$$\frac{h_Q}{\ell_Q} \leq c_\ell, \\ |\cos \theta| \leq c_\theta < 1, \quad \text{for any angle } \theta \text{ from } Q$$

must hold for all  $Q \in \mathcal{Q}_h$  [6].

In the convergence proof we will need a constant  $\gamma$  introduced by

*Condition 2.5:* Let  $h_Q$  and  $\ell_Q$  denote the diameter of quadrilateral  $Q$  and the length of the shortest side on the border  $\partial Q$ , respectively. Suppose that  $Q$  is star-shaped with respect to the ball  $B_Q$  with radius  $\rho_Q := \sup\{\text{diam}(\text{ball } S), S \subset Q, \forall \mathbf{x} \in Q \forall \mathbf{y} \in S \forall \lambda \in [0, 1] \Rightarrow (1 - \lambda)\mathbf{x} + \lambda\mathbf{y} \in Q\}$ . Then we can define the chunkiness parameter  $c_\rho := \frac{h_Q}{\rho_Q}$  and parameter  $c_\ell := \frac{h_Q}{\ell_Q}$ . We assume that some constant  $\gamma$  exists for which the following inequality holds

$$\max(\cup_{Q \in \mathcal{Q}_h} \max(c_\rho, c_\ell)) \leq \gamma.$$

Let  $\mathcal{N}_h$ ,  $\mathcal{Q}_h$ ,  $\mathcal{Q}_h(\mathbf{a})$  and  $\mathcal{Q}_1(\mathbf{a})$  denote the set of all vertices, the set of all quadrilaterals, the set of quadrilaterals with common vertex  $\mathbf{a}$  and the first quadrilateral from the set  $\mathcal{Q}_h(\mathbf{a})$ , respectively. For a quadrilateral  $Q$  with nodes  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ , we rewrite the set of linear functionals as  $\Phi_Q = \{\phi_{\mathbf{a}_j, k}^Q, j = 1, \dots, 4, k = 1, \dots, 3\}$ . We can now define the finite element space

$$X_h := \left\{ v_h \in \prod_{Q \in \mathcal{Q}_h} P_Q, \forall \mathbf{a} \in \mathcal{N}_h, \forall Q_i, Q_j \in \mathcal{Q}_h(\mathbf{a}), \right. \\ \left. \forall k, \phi_{\mathbf{a}, k}^{Q_i}(v_h|_{Q_i}) = \phi_{\mathbf{a}, k}^{Q_j}(v_h|_{Q_j}) \right\}$$

and the related set of linear functionals

$$\Phi_h := \{\phi_{\mathbf{a}, k} = \phi_{\mathbf{a}, k}^{Q_1(\mathbf{a})}, \mathbf{a} \in \mathcal{N}_h, 1 \leq k \leq 3\}.$$

Next we employ the dual functions  $p_{\mathbf{a}, k}$  from  $V_h$  for functionals  $\phi_{\mathbf{a}, k}$ , on the open set  $\Omega_h = \Omega - \cup_{Q \in \mathcal{Q}_h} \partial Q$ , and define

the interpolation operator

$$I_h : v \mapsto \sum_{\mathbf{a} \in \mathcal{N}_h, 1 \leq k \leq 3} \phi_{\mathbf{a}, k}(v) p_{\mathbf{a}, k}.$$

### III. AFFINE THIN PLATE FINITE ELEMENT ARPQ4

A potential singularity of the interpolation matrix is a serious drawback of element RPQ4. In this section we derive an improved finite element ARPQ4, marked as ‘an affine modification of element RPQ4’, which is free of singularity for any non-degenerate convex quadrilateral. This is achieved by the introduction of the affine equivalent quadrilaterals [10], as discussed in the sequel.

#### A. Reference quadrilateral $\hat{Q}$

Quadrilateral  $Q$  from set  $\mathcal{Q}_h$  is called the affine equivalent quadrilateral with respect to its reference quadrilateral  $\hat{Q}$  [10], if it is obtained by the affine mapping

$$F : \hat{\mathbf{x}} \mapsto B \hat{\mathbf{x}} + \mathbf{b} = \mathbf{x},$$

where the matrix  $B$ , the shift vector  $\mathbf{b}$  and the node displacement vector  $\mathbf{d}$  (Fig. 2) take the forms

$$B = \frac{1}{4} [\mathbf{a}_1 - \mathbf{a}_3 - \mathbf{a}_2 + \mathbf{a}_4, \quad \mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_2 - \mathbf{a}_4], \\ \mathbf{b} = \frac{1}{4} (\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4), \\ \mathbf{d} = \frac{1}{4} B^{-1} (\mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_2 - \mathbf{a}_4).$$

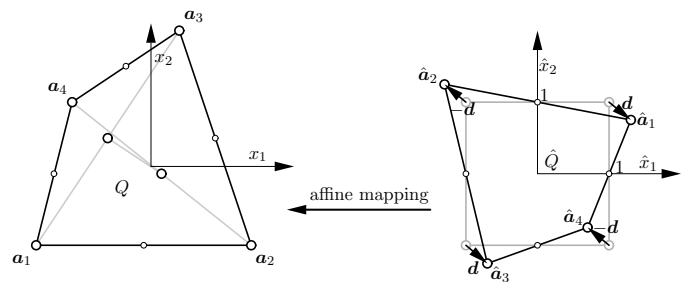


Fig. 2. Affine equivalent quadrilaterals.

*Lemma 3.1:* For the bisection scheme of mesh subdivisions [11], the norm  $|\mathbf{d}|$  is of order  $O(h)$ .

*Proof:* Let us denote  $A := 4B$  and  $\mathbf{b}_m := \mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_2 - \mathbf{a}_4$ . Then we can write  $\mathbf{d} = A^{-1} \mathbf{b}_m$ . It is easy to see that  $\frac{|A|}{4}$  is exactly the area  $|Q|$  of quadrilateral  $Q$ , and  $\frac{|\mathbf{b}_m|}{2}$  is the distance between the midpoints of diagonals. The distance between the midpoints of diagonals of  $Q \in \mathcal{Q}_h$  is of order  $O(h^2)$  uniformly for all quadrilaterals, if mesh is designed by the bisection scheme of the mesh subdivision [11]. For such a kind of a mesh subdivision, the area is of order  $O(h^2)$ , i.e.  $|Q| \geq c(\gamma) h^2$ . This implies that  $|\mathbf{d}|$  is of order  $O(h)$ . ■

Let us write the components of vector  $\mathbf{d}$  explicitly. We get (see Fig. 3)

$$d_1 = \frac{(x_1 - x_4)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_4)}{(x_4 - x_2)(y_1 - y_3) + (x_1 - x_3)(y_2 - y_4)} = \frac{2|T_{023}|}{2|Q|}, \\ d_2 = \frac{(x_4 - x_3)(y_1 - y_2) + (x_1 - x_2)(y_3 - y_4)}{(x_4 - x_2)(y_1 - y_3) + (x_1 - x_3)(y_2 - y_4)} = \frac{-2|T_{012}|}{2|Q|}. \quad (8)$$

If the quadrilateral is a trapezoid or a parallelogram, one or two components of vector  $\mathbf{d}$  are zero.

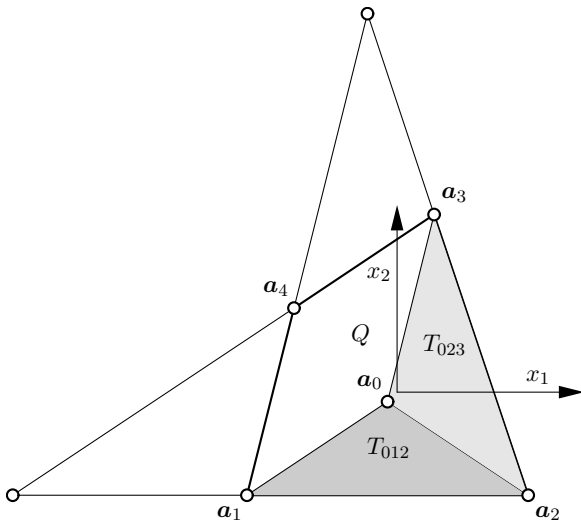


Fig. 3. Convex quadrilateral  $Q$ .

### B. Function $v_h$ and its derivatives

With the notations  $\mathbf{e}_1 = [1 \ 0]^T$ ,  $\mathbf{e}_2 = [0 \ 1]^T$ ,  $\beta = [\beta_1, \dots, \beta_{12}]^T$ ,  $Dw_h(\mathbf{a}) = \left[ \frac{\partial w_h(\mathbf{a})}{\partial x} \quad \frac{\partial w_h(\mathbf{a})}{\partial y} \right]^T$ ,  $D\hat{w}_h(\hat{\mathbf{a}}) = \left[ \frac{\partial \hat{w}_h(\hat{\mathbf{a}})}{\partial \hat{x}} \quad \frac{\partial \hat{w}_h(\hat{\mathbf{a}})}{\partial \hat{y}} \right]^T$ ,  $\hat{\mathbf{X}}^T = [1, \dots, \hat{x}\hat{y}^3]$ ,

$$\hat{\mathbf{q}} = \left[ \hat{w}_h(\hat{\mathbf{a}}_1), \dots, \frac{\partial \hat{w}_h(\hat{\mathbf{a}}_4)}{\partial \hat{y}} \right]^T \text{ and } \hat{A} = \begin{bmatrix} \hat{A}_n(\hat{\mathbf{a}}_1) \\ \vdots \\ \hat{A}_n(\hat{\mathbf{a}}_1) \end{bmatrix}, \text{ where}$$

$$\hat{A}_n((\hat{x}, \hat{y})) := \begin{bmatrix} \hat{\mathbf{X}}^T \\ \partial_1 \hat{\mathbf{X}}^T \\ \partial_2 \hat{\mathbf{X}}^T \end{bmatrix},$$

we can define, as in [10], the non-conforming displacement approximation in the affine equivalent quadrilateral  $Q$  by

$$w_h(\mathbf{x}) := w_h(F(\hat{\mathbf{x}})) = (w_h \circ F)(\hat{\mathbf{x}}) = \hat{w}_h(\hat{\mathbf{x}}) = \hat{\mathbf{X}}^T \beta = \hat{\mathbf{X}}^T \hat{A}^{-1} \hat{\mathbf{q}}$$

and, additionally, the refined non-conforming displacement approximation [19]

$$v_h(\mathbf{x}) := w_h(\mathbf{x}) + \frac{1}{2} \begin{bmatrix} x^2 & y^2 & xy \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^T = \hat{\mathbf{X}}^T \hat{A}^{-1} \hat{\mathbf{q}} + \frac{1}{2} \begin{bmatrix} x^2 & y^2 & xy \end{bmatrix} (B_c \mathbf{q} - B_0 \mathbf{q}).$$

Employing the result of Ciarlet [7, p. 86] we can write

$$\begin{aligned} \partial_i w_h(\mathbf{x}) &= D\hat{w}_h^T(\hat{\mathbf{x}}) B^{-1} \mathbf{e}_i, \quad 1 \leq i \leq 2, \\ \partial_{ij} w_h(\mathbf{x}) &= \mathbf{e}_i^T (B^{-1})^T D^2 \hat{w}_h(\hat{\mathbf{x}}) B^{-1} \mathbf{e}_j, \quad 1 \leq i, j \leq 2. \end{aligned}$$

Inserting the above expressions into  $\hat{\mathbf{q}}$  gives

$$\begin{aligned} \hat{\mathbf{q}} &= \begin{bmatrix} \hat{w}_h(\hat{\mathbf{a}}_1) \\ D\hat{w}_h(\hat{\mathbf{a}}_1) \\ \vdots \\ \hat{w}_h(\hat{\mathbf{a}}_4) \\ D\hat{w}_h(\hat{\mathbf{a}}_4) \end{bmatrix} = \begin{bmatrix} w_h(\mathbf{a}_1) \\ B^T D w_h(\mathbf{a}_1) \\ \vdots \\ w_h(\mathbf{a}_4) \\ B^T D w_h(\mathbf{a}_4) \end{bmatrix} = \\ &= \begin{bmatrix} 1 & & & & \\ & B^T & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & B^T \end{bmatrix} \mathbf{q} =: B_A \mathbf{q}. \end{aligned}$$

Thus we have derived

$$\hat{w}_h(\hat{\mathbf{x}}) = \hat{\mathbf{X}}^T \hat{A}^{-1} B_A \mathbf{q}.$$

### C. Unisolvence proof

After a short calculation we have

$$|\hat{A}| = 2^{20} ((d_1 - d_2)^2 - 1) \cdot ((d_1 + d_2)^2 - 1) \cdot (d_1^2 + d_2^2 - 1) \cdot ((d_1^2 + d_2^2 - 1)^2 + 4d_1^2 d_2^2), \quad (9)$$

which shows that the bisection scheme of mesh subdivision preserves the regularity of the matrix  $\hat{A}$ . In fact, the matrix  $\hat{A}$  remains nonsingular for every non-degenerate convex quadrilateral  $Q$ , as clearly observed from Eq. (8) and Fig. 3.

1) Estimate of the determinant of the interpolation matrix  $\hat{A}$ : Let us denote the side lengths and the lengths of the diagonals of a convex quadrilateral with  $a, b, c, d, p$  and  $q$ , respectively. From Brethsneider's formula

$$|Q| = \frac{1}{4} \sqrt{4p^2 q^2 - (b^2 + d^2 - a^2 - c^2)^2},$$

Fig. 3, Eqs. (8) and inequality  $c_\ell \geq \sqrt{2}$  we have

$$|d_1| + |d_2| = 1 - \frac{|Q_{0341}|}{|Q|} \leq 1 - \frac{\ell_Q^2 \sqrt{1 - c_\theta^2}}{|Q|} \leq 1 - \frac{2\sqrt{1 - c_\theta^2}}{c_\ell^2},$$

$$\begin{aligned} d_1^2 + d_2^2 &\leq (|d_1| + |d_2|)^2 \\ &\leq 1 - 4 \frac{\sqrt{1 - c_\theta^2}}{c_\ell^2} + 4 \frac{1 - c_\theta^2}{c_\ell^4} \leq 1 - 4 \frac{1 - c_\theta^2}{c_\ell^4} \end{aligned}$$

and

$$\begin{aligned} 1 - \max((d_1 - d_2)^2, (d_1 + d_2)^2, d_1^2 + d_2^2) &\geq \\ &\geq 1 - (|d_1| + |d_2|)^2 \geq 4 \frac{1 - c_\theta^2}{c_\ell^4}. \end{aligned}$$

The above inequalities are inserted into Eq. (9) yielding

$$\text{abs}(|\hat{A}|) \geq 2^{10} \left( \frac{2}{c_\ell} \right)^{20} (1 - c_\theta^2)^5. \quad (10)$$

The lower bound is attained only in the case of a square mesh.

Using the equations from Sec. V one can easily connect the constants  $\gamma$ ,  $c_\ell$  and  $c_\theta$  from Conditions 2.4 and 2.5 respectively by Eq.

$$\gamma = \max \left( c_\ell, \frac{1}{\sqrt{1 - c_\theta^2}} \right).$$

D. Derivation of matrices  $B_c$  and  $B_0$

The choice of the interpolation for  $w_h$  over the element does not affect the derivation of matrices  $B_c$  and  $B_0$ . Consequently, matrices  $B_c$  and  $B_0$  remain formally as in [19]

$$B_c \mathbf{q} = \frac{1}{|Q|} \int_{\partial Q} \left( \widetilde{\partial}_\mu w_h \begin{bmatrix} \mu_1^2 \\ \mu_2^2 \\ 2\mu_1\mu_2 \end{bmatrix} + \widetilde{\partial}_\tau w_h \begin{bmatrix} -\mu_1\mu_2 \\ \mu_1\mu_2 \\ \mu_1^2 - \mu_2^2 \end{bmatrix} \right) ds,$$

$$B_0 \mathbf{q} = \frac{1}{|Q|} \int_Q \begin{bmatrix} \partial_{11} w_h \\ \partial_{22} w_h \\ 2\partial_{12} w_h \end{bmatrix} dx.$$

Moreover, since matrix  $B_c$  depends on  $\mathbf{q}$  only, it is identical to the expression for  $B_c$  of element RPQ4 [19].

One can construct ARPQ4 finite element  $(Q, P_Q, \Phi_Q)$  following the same steps as in Sec. II-A.

IV. BOUNDARY VALUE PROBLEM

We seek the weak solution  $u^*$  for the deflection of the thin clamped plate subjected to a given surface load  $f$

$$a(u^*, v) = (f, v), \quad u^*, v \in V := H_0^2(\Omega),$$

where

$$a(u, v) = \int_\Omega \left( \nu \Delta u \Delta v + (1 - \nu) \sum_{i=1, j=1}^2 \partial_{ij} u \partial_{ij} v \right) dx,$$

$$(f, v) = \int_\Omega f v dx, \quad f \in L^2(\Omega).$$

Its non-conforming approximation  $u_h^* \in V_h := X_{00h} := \{v_h \in X_h, \forall \mathbf{a} \in \partial\Omega, \phi_{\mathbf{a},k}(v_h) = 0, 1 \leq k \leq 3\}$  is obtained by the solution of the variational equation

$$a_h(u_h^*, v_h) = (f, v_h), \quad u_h^*, v_h \in V_h,$$

where

$$a_h(u_h, v_h) = \sum_{Q \in \mathcal{Q}_h} \int_Q \left( \nu \Delta u_h \Delta v_h + (1 - \nu) \sum_{i=1, j=1}^2 \partial_{ij} u_h \partial_{ij} v_h \right) dx,$$

$$(f, v_h) = \int_\Omega f v_h dx, \quad f \in L^2(\Omega).$$

Here,  $\nu \in [0, \frac{1}{2}]$  is Poisson's coefficient of material.

V. CONVERGENCE AND ERROR ESTIMATE

According to the second Strang Lemma [5]

$$\|u^* - u_h^*\|_{2,h} \leq c \left( \inf_{v_h \in V_h} \|u^* - v_h\|_{2,h} + \sup_{v_h \in V_h} \frac{|(f, v_h) - a_h(u^*, v_h)|}{\|v_h\|_{2,h}} \right),$$

the error of the weak non-conforming solution consists of two parts, i.e. the error of approximation and the error of the consistency term. In what follows each part will be estimated separately [9]. The error of the consistency term will be estimated with the help of Stummel's generalized patch test [14].

Let  $h$  denote the largest diameter of all quadrilaterals in quasi-uniform quadriangulation  $\mathcal{Q}_h$  of polygonal domain  $\Omega$ . Let  $c$  denote a generic constant independent on  $h$ , which may have different values at different places. For each quadrilateral  $Q$  we introduce the quadrilateral  $\hat{Q}$  with the same shape, yet with the diameter  $h_{\hat{Q}}$  being equal to 1.

A. Error estimate of the consistency term

According to [14], [18] the sequence  $\{V_h\} \cup H_0^2(\Omega)$  passes the generalized patch test if and only if

$$\lim_{h \rightarrow 0} T_{\alpha,i}(\psi, v_h) := \lim_{h \rightarrow 0} \sum_{Q \in \mathcal{Q}_h} T_{\alpha,i}^Q(\psi, v_h) =$$

$$= \lim_{h \rightarrow 0} \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \psi \partial^\alpha v_h \mu_i ds = 0$$

for all  $i = 1, 2$ , all  $|\alpha| \leq 1$ , all bounded sequences  $\{V_h\}$  and all  $\psi \in C^\infty(\bar{\Omega})$ . We employ the operators  $P_0 : v \mapsto Q^1 v := \int_{B_Q} v(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}$ ,  $R_0 : v \mapsto v - Q^1 v$  and  $\hat{R}_0 : \hat{v} \mapsto \hat{v} - \hat{Q}^1 \hat{v}$ , where we have used the cut-off function  $\phi$  ( $\text{supp} \phi \in \bar{B_Q}$ ,  $\int_{\mathbf{R}^2} \phi(\mathbf{x}) d\mathbf{x} = 1$ ) [4, p. 97]. Introducing the affine mapping  $F_Q : \hat{\mathbf{x}} \mapsto h_Q \hat{\mathbf{x}}$  one can write  $v := \hat{v} \circ F_Q^{-1}$ ,  $Q^1 v := \hat{Q}^1 \hat{v} \circ F_Q^{-1}$ .

Let us define the border interpolation functions

$$\partial_1 \widetilde{w}_h := \mu_1 \widetilde{\partial}_\mu w_h - \mu_2 \widetilde{\partial}_\tau w_h, \quad \partial_2 \widetilde{w}_h := \mu_2 \widetilde{\partial}_\mu w_h + \mu_1 \widetilde{\partial}_\tau w_h.$$

We have to estimate the following terms:

$$T_{(0,0),i}(\psi, v_h) = T_i(\psi, v_h),$$

$$T_{(1,0),i}(\psi, v_h) = T_i(P_0 \psi, \partial_1 v_h - \partial_1 \widetilde{w}_h) + T_i(\psi, \partial_1 \widetilde{w}_h) + T_i(R_0 \psi, \partial_1 w_h - \partial_1 \widetilde{w}_h + \partial_1 \Lambda),$$

$$T_{(0,1),i}(\psi, v_h) = T_i(P_0 \psi, \partial_2 v_h - \partial_2 \widetilde{w}_h) + T_i(\psi, \partial_2 \widetilde{w}_h) + T_i(R_0 \psi, \partial_2 w_h - \partial_2 \widetilde{w}_h + \partial_2 \Lambda). \quad (11)$$

Using the relations  $dx = -\mu_2 ds$ ,  $dy = \mu_1 ds$ , the Green formula and Eq. (12),

$$\begin{bmatrix} \partial_\mu w_h \\ \partial_\tau w_h \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 \\ -\mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} \partial_1 w_h \\ \partial_2 w_h \end{bmatrix}, \quad (12)$$

$$\begin{bmatrix} \partial_1 w_h \\ \partial_2 w_h \end{bmatrix} = \begin{bmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} \partial_\mu w_h \\ \partial_\tau w_h \end{bmatrix},$$

we can rewrite Eqs. (4) as

$$T_1^Q(1, \partial_1 v_h - \partial_1 \widetilde{w}_h) := \int_{\partial Q} \partial_1 v_h \mu_1 ds - \int_{\partial Q} \partial_1 \widetilde{w}_h \mu_1 ds = 0,$$

$$T_2^Q(1, \partial_2 v_h - \partial_2 \widetilde{w}_h) := \int_{\partial Q} \partial_2 v_h \mu_2 ds - \int_{\partial Q} \partial_2 \widetilde{w}_h \mu_2 ds = 0,$$

$$\int_{\partial Q} \left( 2 \partial_1 v_h \mu_2 - (\partial_1 \widetilde{w}_h \mu_2 + \partial_2 \widetilde{w}_h \mu_1) \right) ds$$

$$=: 2 T_2^Q(1, \partial_1 v_h - \partial_1 \widetilde{w}_h) - \int_{\partial Q} \partial_\tau \widetilde{w}_h ds,$$

$$\int_{\partial Q} \left( 2 \partial_2 v_h \mu_1 - (\partial_1 \widetilde{w}_h \mu_2 + \partial_2 \widetilde{w}_h \mu_1) \right) ds$$

$$=: 2 T_1^Q(1, \partial_2 v_h - \partial_2 \widetilde{w}_h) + \int_{\partial Q} \partial_\tau \widetilde{w}_h ds.$$

In order to estimate the first and the second terms in the second and in the third equation, we have to estimate the term  $\int_{\partial Q} \partial_\tau \widetilde{w}_h ds$ . A short derivation shows that the sum  $\sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \psi \partial_\tau \widetilde{w}_h ds$  vanishes for both linear (LITD) and parabolic (PITD) interpolations of the tangential derivative. The difference function  $\partial_\tau \widetilde{w}_{hd}$  between the parabolic and linear interpolations of the tangential derivative gives

$$\sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \psi \partial_\tau \widetilde{w}_h ds = \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \psi \partial_\tau \widetilde{w}_{hd} ds = 0.$$

*Remark 5.1:* From relations  $T_i(1, \partial_j v_h) = 0$ ,  $1 \leq i, j \leq 2$ , which hold for all  $v_h \in V_{P0} := \{v_h \in V_h, \forall \mathbf{a} \in \partial(\cup_{Q \in \mathcal{Q}_P} Q), \phi_{\mathbf{a},k}(v_h) = 0, 1 \leq k \leq 3\}$ , it immediately follows

$$\sum_{Q \in \mathcal{Q}_P} \int_Q \partial^\alpha v_h dx = 0 \quad \forall v_h \in V_{P0}, \quad |\alpha| = 2.$$

So, according to [18, Lemma 4.1], the element passes Irons' patch test.

Now we can estimate the first term in both the second and the third equation of Eqs. (11). We write

$$\begin{aligned} T_2(P_0 \psi, \partial_1 v_h - \partial_1 \widetilde{w}_h) &= \sum_{Q \in \mathcal{Q}_h} T_2^Q(P_0 \psi, \partial_1 v_h - \partial_1 \widetilde{w}_h) \\ &= \frac{1}{2} \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} P_0 \psi \partial_\tau \widetilde{w}_h ds \\ &= \frac{1}{2} \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} (\psi - R_0 \psi) \partial_\tau \widetilde{w}_{hd} ds \\ &= -\frac{1}{2} \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} R_0 \psi \partial_\tau \widetilde{w}_{hd} ds \end{aligned}$$

and derive the estimate:

$$\begin{aligned} |T_2(P_0 \psi, \partial_1 v_h - \partial_1 \widetilde{w}_h)| &\leq \sum_{Q \in \mathcal{Q}_h} |R_0 \psi|_{0,2,\partial Q} |\partial_\tau \widetilde{w}_{hd}|_{0,2,\partial Q} \\ &\leq \sum_{Q \in \mathcal{Q}_h} c(\gamma) |\psi|_{1,2,Q} h_Q^{\frac{1}{2}} |w_h|_{3,\infty,Q} h_Q^{\frac{5}{2}} \\ &\leq \sum_{Q \in \mathcal{Q}_h} c(\gamma) |\psi|_{1,2,Q} \|v_h\|_{2,2,Q} h_Q. \end{aligned}$$

In order to estimate the remaining terms, we first introduce the node numbering function,  $N : \mathbf{a}_i \mapsto N\mathbf{a}_i$ , and function  $\widetilde{w}_h$ :

$$\widetilde{w}_h|_{\overline{\mathbf{a}_i \mathbf{a}_j}} := \begin{cases} w_{h_i} + w_{h'_i} \tau + \frac{\tau^2}{2L} (w_{h'_j} - w_{h'_i}), & N\mathbf{a}_i < N\mathbf{a}_j, \\ w_{h_j} - \frac{L(w_{h'_i} + w_{h'_j})}{2} + & N\mathbf{a}_i > N\mathbf{a}_j, \\ + w_{h'_i} \tau + \frac{\tau^2}{2L} (w_{h'_j} - w_{h'_i}), & \text{(LITD)}, \\ w_{h_i} + w_{h'_i} \tau - & \\ - \frac{\tau^2}{L^2} (2w_{h'_i} L + w_{h'_j} L + 3w_{h_i} - 3w_{h_j}) & \text{(PITD)}, \\ - \frac{\tau^3}{L^3} (-w_{h'_i} L - w_{h'_j} L - 2w_{h_i} + 2w_{h_j}), & \end{cases} \leq \begin{cases} \sum_{Q \in \mathcal{Q}_h} |R_0 \psi|_{0,2,\partial Q} (|\partial_j w_h - \partial_j \widetilde{w}_h|_{0,2,\partial Q} + |\partial_j \Lambda_Q|_{0,2,\partial Q}) \\ \sum_{Q \in \mathcal{Q}_h} c(\gamma) |\psi|_{1,2,Q} h_Q^{\frac{1}{2}} (|w_h|_{3,\infty,Q} h_Q^{\frac{5}{2}} + \sum_k |\lambda_k|_{0,\infty,Q} h_Q^{\frac{5}{2}}) \\ c(\gamma) \sum_{Q \in \mathcal{Q}_h} |\psi|_{1,2,Q} \|v_h\|_{2,2,Q} h_Q. \end{cases} \quad (15)$$

where  $L$  denotes the length of the side  $\overline{\mathbf{a}_i \mathbf{a}_j}$ ,  $w_{h'_j} := \partial_\tau w_h(\mathbf{a}_j)$  and  $w_{h_j} := w_h(\mathbf{a}_j)$  for  $1 \leq i, j \leq 4$ . The function  $\widetilde{w}_h$ , defined above, has the nice property  $\partial_\tau \widetilde{w}_h = \partial_\tau w_h$ . With

the help of the function  $\widetilde{w}_h$  we split the first term into the sum of three terms:

$$\begin{aligned} T_i(\psi, v_h) &= T_i(\psi, \widetilde{w}_h) + T_i(P_0 \psi, w_h - \widetilde{w}_h + \Lambda) \\ &\quad + T_i(R_0 \psi, w_h - \widetilde{w}_h + \Lambda). \end{aligned}$$

Because of conformity of the function  $\widetilde{w}_h$ , the first term vanishes; thus the remaining two terms only need be elaborated upon:

$$\begin{aligned} T_i(P_0 \psi, w_h - \widetilde{w}_h + \Lambda) &= \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} P_0 \psi (w_h - \widetilde{w}_h + \Lambda_Q) \mu_i ds \\ &\leq \sum_{Q \in \mathcal{Q}_h} P_0 \psi h_Q^{\frac{1}{2}} (|w_h - \widetilde{w}_h|_{0,2,\partial Q} + |\Lambda_Q|_{0,2,\partial Q}) \\ &\leq \sum_{Q \in \mathcal{Q}_h} c(\gamma) |\psi|_{0,2,Q} h_Q^{\frac{1}{2}} \left( |w_h|_{3,\infty,Q} h_Q^{\frac{7}{2}} + \sum_k |\lambda_k|_{0,\infty,Q} h_Q^{\frac{5}{2}} \right) \\ &\leq c(\gamma) \sum_{Q \in \mathcal{Q}_h} |\psi|_{0,2,Q} h_Q^{\frac{1}{2}} |w_h|_{3,\infty,Q} h_Q^{\frac{7}{2}} \\ &= c(\gamma) \sum_{Q \in \mathcal{Q}_h} |\psi|_{0,2,Q} |v_h|_{3,\infty,Q} h_Q^3 \\ &\leq c(\gamma) \sum_{Q \in \mathcal{Q}_h} |\psi|_{0,2,Q} \|v_h\|_{2,2,Q} h_Q, \end{aligned} \quad (13)$$

$$\begin{aligned} T_i(R_0 \psi, w_h - \widetilde{w}_h + \Lambda) &= \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} R_0 \psi (w_h - \widetilde{w}_h + \Lambda_Q) \mu_i ds \\ &\leq \sum_{Q \in \mathcal{Q}_h} |R_0 \psi|_{0,2,\partial Q} (|w_h - \widetilde{w}_h|_{0,2,\partial Q} + |\Lambda_Q|_{0,2,\partial Q}) \\ &\leq \sum_{Q \in \mathcal{Q}_h} c(\gamma) |\psi|_{1,2,Q} h_Q^{\frac{1}{2}} \left( |w_h|_{3,\infty,Q} h_Q^{\frac{7}{2}} + \sum_k |\lambda_k|_{0,\infty,Q} h_Q^{\frac{5}{2}} \right) \\ &\leq c(\gamma) \sum_{Q \in \mathcal{Q}_h} |\psi|_{1,2,Q} \|v_h\|_{2,2,Q} h_Q^2. \end{aligned} \quad (14)$$

The last terms of the second and the third equations of Eq. (11) are estimated in a similar way:

$$\begin{aligned} T_i(R_0 \psi, \partial_j w_h - \partial_j \widetilde{w}_h + \partial_j \Lambda) &= \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} R_0 \psi (\partial_j w_h - \partial_j \widetilde{w}_h + \partial_j \Lambda_Q) \mu_i ds \\ &\leq \sum_{Q \in \mathcal{Q}_h} |R_0 \psi|_{0,2,\partial Q} (|\partial_j w_h - \partial_j \widetilde{w}_h|_{0,2,\partial Q} + |\partial_j \Lambda_Q|_{0,2,\partial Q}) \\ &\leq \sum_{Q \in \mathcal{Q}_h} c(\gamma) |\psi|_{1,2,Q} h_Q^{\frac{1}{2}} \left( |w_h|_{3,\infty,Q} h_Q^{\frac{5}{2}} + \sum_k |\lambda_k|_{0,\infty,Q} h_Q^{\frac{3}{2}} \right) \\ &\leq c(\gamma) \sum_{Q \in \mathcal{Q}_h} |\psi|_{1,2,Q} \|v_h\|_{2,2,Q} h_Q. \end{aligned} \quad (15)$$

In the second and the third inequalities of Eq. (15), we have used the Sobolev Imbedding Theorem [1], Friedrichs' inequality [4], [17] and the inverse inequality [4, Lemma

4.5.3]. Employing the integration by parts or the equivalent procedure from [5, p. 376] we can write the error functional  $E_h$  in the form

$$\begin{aligned} E_h(u^*, v_h) &:= (f, v_h) - a_h(u^*, v_h) \\ &= \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \partial_\nu \Delta u^* v_h + (1 - \sigma) \partial_{\nu\tau} u^* \partial_\tau v_h \, ds \\ &\quad - \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} (\Delta u^* - (1 - \sigma) \partial_{\tau\tau} u^*) \partial_\nu v_h \, ds. \end{aligned}$$

With the help of the inequalities (13), (14) and (15) we finally derive the estimate of the error of the consistency term

$$|E_h(u^*, v_h)| \leq ch (\|u^*\|_{3,h} + h \|u^*\|_{4,h}) \|v_h\|_{2,h} \quad (16)$$

valid for all  $u^* \in H^4(\Omega)$ .

### B. Estimate of the approximability term

Let us first adapt Definition 4.4.2 and Theorem 4.4.4 from [4] to obtain the form suitable for the present purposes.

*Lemma 5.2:* Let  $(Q, P, \Phi)$  be a finite element satisfying

- (i)  $Q$  is star-shaped with respect to some ball, such that

$$|\hat{A}| \geq c(\gamma), \quad (17)$$

- (ii)  $\mathcal{P}_2 \subset P \subset W_\infty^3(Q)$  and
- (iii)  $\Phi \subset (C^1(\hat{Q}))'$ .

Assume  $p = 2$ . Then for  $0 \leq i \leq 2$  and  $v \in W_2^3(Q)$  we have

$$|v - I_h v|_{i,2,Q} \leq C_{\gamma,\sigma(\hat{Q})} h_Q^{3-i} |v|_{3,2,Q},$$

where  $\hat{Q} := \{\frac{x}{h_Q}, x \in Q\}$ , a constant  $C_{\gamma,\sigma(\hat{Q})}$  is dependent on parameter  $\gamma$ , introduced in Condition 2.5, and  $\sigma(\hat{Q})$  is the operator norm of  $\hat{I}_h : W_2^3(\hat{Q}) \rightarrow C^1(\hat{Q})$ .

Let us point out that the interpolation operator  $\hat{I}_h$  is well defined on  $W_2^3(\hat{Q})$ . This follows from the Sobolev Imbedding Theorems. Our aim is to estimate the norm  $\sigma(\hat{Q})$  of operator  $\hat{I}_h : W_2^3(\hat{Q}) \rightarrow C^1(\hat{Q})$ . For the sake of simplification of notation, we skip the hat over symbols. Employing the inequalities

$$\begin{aligned} \|I_h u\|_{3,2,Q} &\leq \sum_{i=1}^{12} |\phi_i(u)| \|p_i\|_{3,2,Q} \leq \\ &\leq \sum_{i=1}^{12} \|\phi_i\|_{W_2^3(Q)} \|p_i\|_{3,2,Q} \|u\|_{3,2,Q} \end{aligned}$$

gives

$$\sigma(Q) \leq \sum_{i=1}^{12} \|\phi_i\|_{W_2^3(Q)} \|p_i\|_{3,2,Q}. \quad (18)$$

Next we show that the norm  $\sigma(Q)$  is uniformly bounded for all quadrilaterals  $Q$ . As has been shown by Ciarlet [5] and Adams [1], the identity from  $W_2^3(Q)$  to  $C_1(Q)$  is uniformly continuous. First we estimate the norm  $\|\phi_i\|_{W_2^3(Q)}$ :

$$|\phi_i(v)| \leq c \|v\|_{1,\infty,Q} \leq c \|v\|_{3,2,Q}, \quad 1 \leq i \leq 12. \quad (19)$$

From Eq. (19) it immediately follows that the norms  $\|\phi_i\|_{W_2^3(Q)}$  are bounded from above by the constant  $c = c(\gamma)$ .

Eq. (17) assures that the base functions  $p_i$  are also bounded. Thus we have

*Lemma 5.3:* The base functions  $p_i$  are bounded:

$$\|p_i\|_{3,2,Q} \leq c(\gamma), \quad 1 \leq i \leq 12. \quad (20)$$

In order to achieve the ellipticity we must prove

*Lemma 5.4:* The seminorm  $v \mapsto \|v\|_{2,h} := (\sum_{Q \in \mathcal{Q}_h} |v|_{2,2,\Omega}^2)^{\frac{1}{2}}$  is a norm.

*Proof:* We will show that  $\|v_h\|_{2,h} = 0$  implies  $v_h = 0$ . From  $\|v_h\|_{2,h} = 0$  it follows that  $\partial_j(v_h|_Q)$  is constant for each  $Q \in \mathcal{Q}_h$  and  $1 \leq j \leq 2$ . Thus the function  $\Lambda_Q = \Lambda_Q(\partial_\alpha w_h - \partial_\alpha v_h) = \Lambda_Q(\partial_\alpha v_h - \partial_\alpha v_h)$  vanishes and, consequently,  $v_h|_Q = w_h|_Q$  for each  $Q \in \mathcal{Q}_h$ .

Considering the zero boundary conditions and the continuity of the directional derivatives  $\partial_j w_h$  yields  $\partial_j(w_h|_Q) = 0$  for each  $Q \in \mathcal{Q}_h$  and  $1 \leq j \leq 2$ . Thus the function  $w_h|_Q$  is constant for each  $Q \in \mathcal{Q}_h$ .

Similarly, considering the zero boundary conditions and the continuity of the functions  $w_h$  yields  $w_h|_Q = 0$  for each  $Q \in \mathcal{Q}_h$ . This implies that the function  $v_h|_Q$  is zero for each  $Q \in \mathcal{Q}_h$ . ■

Taking into account the second Strang Lemma [5], the error functional estimate (16), the estimate of the approximability term (Lemma 5.2), and Eqs. (18), (19) and (20), one can finally derive the estimate of the error in the energy norm:

$$\|u^* - u_h^*\|_{2,h} \leq ch (\|u^*\|_{3,h} + h \|u^*\|_{4,h}).$$

Thus, the error in the energy norm decreases at least linearly with  $h$  for all  $u^* \in H^4(\Omega)$ .

*Remark 5.5:* Combining the steps of the derivations above with the ideas from [11] we can derive the same error estimate for weak solutions of other fourth-order  $V$ -elliptic boundary value problems.

## VI. NUMERICAL EXAMPLES

The theoretically derived error estimate is also verified numerically. We study the convergence behaviour of a thin clamped plate, subjected to the variable surface load  $f : (x, y) \mapsto 8(10 - 18y^2 + 3(x^4 + y^4 + 6x^2(-1 + 2y^2)))$ , defined on  $\Omega = [-1, 1] \times [-1, 1]$ . Poisson's ratio is taken to be  $\nu = \frac{1}{3}$ . The related analytical solution is  $u^* = (x^2 - 1)^2 (y^2 - 1)^2$ . In Sec. VI-A we show that the numerical results confirm the theoretically predicted linear convergence. In Sec. VI-B we show that the newly derived element ARPQ4 is more robust and thus more convenient for practical computations.

### A. Rate of convergence

Two different series of meshes (a) and (b) (Fig. 4) and elements RPQ4 and ARPQ4 with the linear variation of displacement derivative along the sides of the element have been employed in the convergence analysis. The refined meshes have been constructed by the bisection dividing scheme [11]. Ten meshes with  $h \approx 1.9, \dots, 0.004$  have been applied with the related number of linear equations ranging from  $3(2^2 - 1)^2 = 27$  to  $3(2^{10} - 1)^2 = 3\,139\,587$ . The decrease of the actual error of the solution in the energy norm with

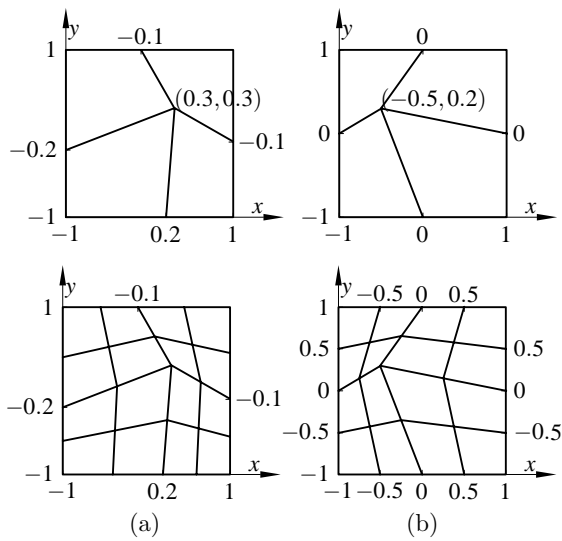


Fig. 4. Initial and the first refined meshes constructed by the bisection dividing scheme [11].

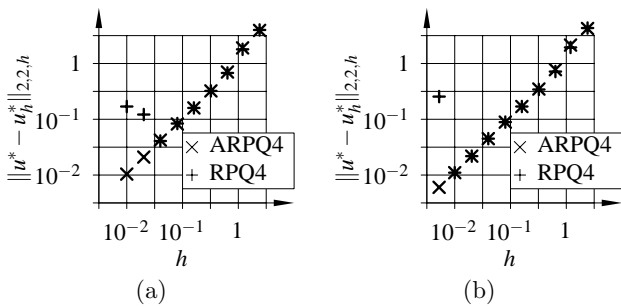


Fig. 5. Error in energy norm.

the decrease of  $h$  for elements RPQ4 and ARPQ4 is depicted in Fig. 5 for the range of  $h$ 's from  $h \approx 1.9$  to  $h \approx 0.004$ . There RPQ4 denotes the original RPQ4 element by Wanji and Cheung [19] and the ARPQ4 denotes its affine version derived in the present paper. Observe that the actual error in the energy norm decreases linearly with  $h$  for element ARPQ4 and for both meshes, exactly as predicted theoretically. The results of the RPQ4 element for small values of  $h$ , which do not fall on the straight line, are discussed in the next section.

### B. Condition number of the structure stiffness matrix

Both theoretical and numerical results for finite element ARPQ4 show that the accuracy of the solution monotonically increases, if the number of finite elements grows.

Clearly, with the increasing number of equations the condition number of the structure stiffness matrix increases, too. The analysis of the present numerical examples has shown that the condition numbers of the stiffness matrices range

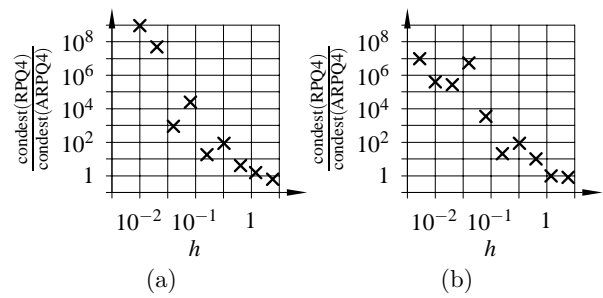


Fig. 6. Ratios of stiffness matrix condition numbers.

from, roughly,  $10^2$  to  $10^{12}$ , if applying the newly proposed element ARPQ4. The condition numbers of the related RPQ4 stiffness matrices increase much faster, see Fig. 6, where the ratios of the stiffness matrix condition numbers of elements ARPQ4 and RPQ4 are presented for each  $h$ . There the condition numbers of elements ARPQ4 and RPQ4 are denoted by  $\text{cond}(\text{ARPQ4})$  and  $\text{cond}(\text{RPQ4})$ , respectively. The condition numbers were estimated by the MATLAB function `cond`. As clearly seen from Fig. 6, the condition number ratios grow from, approx., 1 to  $10^8$ , which indicates that the condition numbers of element RPQ4 grow from, approx.,  $10^2$  to  $10^{20}$ , where the complete lost of the accuracy of solution is observed (see Figs. 5). Hence in almost every step of the bisection dividing algorithm, the condition number of RPQ4 structure stiffness matrix increases roughly by factor 7 or more regarding to the condition number of the ARPQ4 structure stiffness matrix. This indicates the important computational advantage of element ARPQ4 over element RPQ4. Note, however, that in case of square element meshes, the condition numbers of structure matrices do not differ.

## VII. CONCLUSION

In the present paper we have proved convergence and estimated the rate of convergence of new non-conforming affine quadrilateral thin plate bending finite element ARPQ4 derived directly from the finite element RPQ4 proposed by Wanji and Cheung [19].

Our mathematically rigorous proof of convergence is based on Stummel's generalized patch test [14] and the consideration of the element approximability condition [7], which are both necessary and sufficient for convergence.

This new element has theoretically the same convergence characteristics as its predecessor, RPQ4, only that it is unconditionally unisolvent.

This convenient property of the new element helps to reduce the condition number of the structure stiffness matrices and consequently results in the element being more robust and thus better suited for highly refined finite element meshes.

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