RBF-based Meshless Method for Free Vibration Analysis of Laminated Composite Plates

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Abstract—The governing differential equations of laminated plate utilizing trigonometric shear deformation theory are derived using energy approach. The governing differential equations discretized by different radial basis functions are used to predict the free vibration behavior of symmetric laminated composite plates. Effect of orthotropy and span to thickness ratio on frequency parameter of simply supported laminated plate is presented. Numerical results show the accuracy and good convergence of radial basis functions.

Keywords—Composite plates, Meshfree method, free vibration, Shear deformation, RBFs

I. INTRODUCTION

COMPOSITE laminates have found many applications in variety of engineering structures because of their high strength-to-weight, stiffness-to-weight ratios, good energy and sound absorption, and often also low production cost. The applicability of laminated composites ranges from deep ocean to high in the sky, where weight saving is crucial. The flexure, vibration and buckling analysis of the laminated composite plates subjected to mechanical loading has been the subject of research interest of many investigators. The significant increase of the industrial use of laminated composites calls for the development of new numerical tools/methods for the analysis of such high performance structures. Most popular techniques are based on finite differences and finite elements. In the last decade, meshless methods have received much attention because of their flexibility in the construction of finite dimensional subspaces. When the dominant domain is extremely complex, traditional numerical method for partial differential equations (PDEs), such as the finite element method, finite difference method etc, is somewhat difficult to carry out. Most of the work till date on meshless methods using RBFs relates to scattered data approximation, there has recently been an increased interest in their use for solving partial differential equations (PDEs). This approach, which approximates the whole solution of the PDE directly using RBFs, is very attractive due to the fact that this is truly a meshfree technique. A considerable work has been done on meshfree analysis of composites in the last few years. Kansa\textsuperscript{[1]} introduced the concept of solving PDEs using RBFs. Kansa’s method is an unsymmetrical RBF collocation method based upon the MQ interpolation functions. Radial basis functions (RBFs) is one of the best recently developed methods that has attracted several researchers in recent years especially in the area of computational mechanics. A review of meshless methods for laminated plated has been presented by Liew et al\textsuperscript{[2]}. Radial basis function was applied by Xiang et al\textsuperscript{[3-4]} for linear flexural and free vibration analysis of the laminated composite and sandwich plates. Castro et al\textsuperscript{[5]} used wavelet collocations for static analysis of sandwich plates using layer wise theory. Recently Ferreira et al\textsuperscript{[6-7]} used wavelets and Wendland radial basis function for buckling analysis of laminated composite plates. Liew and Huang\textsuperscript{[8]} used moving least-squares differential quadrature for bending and buckling; Liew et al\textsuperscript{[8-10]} used reproducing kernel approximations and meshfree method for buckling analysis of isotropic circular and skew plates. In the present study, the free vibration analysis of laminated composite plates using different radial basis function and trigonometric shear deformation theory is presented. The simply supported immovable rectangular plates are analyzed. Results have been compared with the results obtained by other numerical and analytical methods.

II. MATHEMATICAL FORMULATION

The mathematical formulation of the actual physical problem of the laminated composite plate is presented. The governing differential equations are derived using energy approach and utilizing trigonometric shear deformation theory and linear kinematics. The displacement field at any point in the laminated composite plate made up of perfectly bonded layers of uniform thickness is expressed as Xiang et al\textsuperscript{[3]}:

\begin{align}
u'(x,y,z) &= u(x,y) - z \frac{\partial w}{\partial x} + f(z) \phi_x \\
v'(x,y,z) &= v(x,y) - z \frac{\partial w}{\partial y} + f(z) \phi_y \\
w'(x,y,z) &= w(x,y)
\end{align}

Where, \( f(z) = \frac{h}{\pi} \sin(\frac{\pi z}{h}) \) and the parameters \( \nu' \), \( \phi' \) and \( w' \) are the in-plane and transverse displacements of the plate at any point \((x, y, z)\) in \(x\), \(y\) and \(z\) directions, respectively. \( u\), \( v\) and \( w\) are the displacements at mid plane of the plate at any point \((x, y)\) in \(x\) and \(y\) directions, respectively. The functions \( \phi_x \) and \( \phi_y \) are the higher order rotations of the normal to the mid plane due to shear deformation about \(y\) and \(x\) axes, respectively.
Strain displacements relations are given by

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}, \\
\varepsilon_{yy} &= \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}, \\
\gamma_{xy} &= \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}, \\
\gamma_{yz} &= \frac{\partial w'}{\partial z} + \frac{\partial u'}{\partial x}, \\
\gamma_{zx} &= \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}
\end{align*}
\]

The governing differential equations of plate are obtained using Hamilton’s principle and expressed as:

\[
\begin{align*}
\frac{\partial^2 N_{xx}}{\partial x^2} + \frac{\partial^2 N_{yy}}{\partial y^2} &= I_0 \frac{\partial^4 u}{\partial t^4} - I_1 \frac{\partial^4 v}{\partial y^2 \partial t^2} + I_3 \frac{\partial^4 \phi_x}{\partial t^2} \quad (3.1) \\
\frac{\partial^2 N_{yy}}{\partial x^2} + \frac{\partial^2 N_{xy}}{\partial y \partial x} &= I_0 \frac{\partial^4 w}{\partial t^4} - I_1 \frac{\partial^4 \phi_y}{\partial x^2 \partial y \partial t^2} + I_3 \frac{\partial^4 \phi_y}{\partial x \partial t^2} \quad (3.2) \\
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} &= I_0 \frac{\partial^4 \phi_x}{\partial x^4} + I_1 \frac{\partial^4 \phi_y}{\partial x^2 \partial y^2} + I_3 \frac{\partial^4 \phi_y}{\partial y^4} \quad (3.3) \\
\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yx}}{\partial y \partial x} &= -Q_{xy} = I_0 \frac{\partial^2 u}{\partial x^2} - I_1 \frac{\partial^2 v}{\partial y^2} + I_3 \frac{\partial^2 \phi_x}{\partial y^2} \quad (3.4) \\
\frac{\partial^2 M_{yx}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} &= -Q_{yx} = I_0 \frac{\partial^2 v}{\partial x^2} - I_1 \frac{\partial^2 w}{\partial y^2} + I_3 \frac{\partial^2 \phi_y}{\partial y^2} \quad (3.5)
\end{align*}
\]

The force and moment resultants and their derivatives used in equations (3.1-3.5) are given in Appendix A.

The plate stiffness coefficients and inertia terms are expressed as:

\[
A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} = \\
\sum_{k=1}^{n} \bar{Q}_{ij} \int_{z_k}^{z_{k+1}} \left( 1, z, z^2, f(x), (z) \times f(z), f^2(z) \right) dz
\]

Where, \( i, j = 1, 2, 6 \) (4.1)

The boundary conditions for an arbitrary edge with simply supported conditions are as follows:

\[
x = 0, a: u = 0; \phi_y = 0; w = 0; M_x = 0; N_x = 0 \quad (5.1)
\]

\[
y = 0, b: u = 0; \phi_x = 0; w = 0; M_y = 0; N_y = 0 \quad (5.2)
\]

III. SOLUTION METHODOLOGY

Consider a general differential equation of form

\[
A[u] = f(x, y) \quad \text{in} \quad \Omega \quad (6.1)
\]

\[
B[u] = g(x, y) \quad \text{on} \quad \partial \Omega \quad (6.2)
\]

Where, A is a linear differential operator, B is a linear boundary operator imposed on boundary conditions. u is an unknown quantity to be determined. Let \( \{P(x, y)\}_{j=1}^{N} \) be \( N \) collocation points in domain \( \Omega \) of which \( \{(x_j, y_j)\}_{j=1}^{N} \) are interior points; \( \{(x_j, y_j)\}_{j=N+1}^{N+S} \) are boundary points. In RBF method, the approximate solution for differential equation (3.1-3.5) and boundary conditions (6.1-6.2) can be expressed as:

\[
u(x, y) = \sum_{j=1}^{N} \alpha_j \varphi_j(x, y) \quad (6.3)
\]

Where, \( \varphi_j(x, y) \) is radial basis function. Most commonly used radial basis functions are:

- \( g = r^2 \log r \) Logarithmic or thin plate spline
- \( g = e^{-c^2 r^2} \) Gaussian function
- \( g = r^c \) Polynomial function
- \( g = (r^2 + c^2)^{(1/2)} \) Multiquadric function
- \( g = (r^2 + c^2)^{(-1/2)} \) Inverse multiquadric function

Where, \( r = \|X - X_j\| = \sqrt{(x - x_j)^2 + (y - y_j)^2} \) and \( c \) is the shape parameter.

Free Vibration Analysis

The discretized governing differential equation is expressed as eigenvalue for predicting the natural frequency:

\[
[K] - \Omega^2 [M] \{u\} = \{0\} \quad (6.4)
\]
Where \([K]\) is the stiffness matrix and \([M]\) refers to mass matrix, and the parameter \(\Omega\) refers to the frequency parameter.

IV. RESULTS AND DISCUSSIONS

A RBF based meshless code in MATLAB is developed following the analysis procedure as discussed above. In order to demonstrate the accuracy and applicability of present formulation, several examples have been analyzed and the computed results are compared with the published results. Based on convergence study, a 15\times15 node is used throughout the study. All individuals' layers are taken to be of equal thickness. The material properties of laminates have been taken as follows: \(E_1/E_2 = \text{open};\ G_{12} = G_{13} = 0.6 \times E_2;\ G_{23} = 0.5 \times E_2;\ \nu_{12} = 0.25;\)

The dimensionless natural frequency parameter is defined as:

\[
\Omega = \left( \frac{a^2}{h} \right) \left( \frac{\rho}{E_2} \right)^{1/2}
\]

In order to show the accuracy and efficiency of the present solution methodology, detailed convergence studies for simply supported isotropic plate (\(a/h=10\) and \(a/h=100\)) is carried out. The plate edge length and thickness are denoted by ‘\(a\)’ and ‘\(h\)’, respectively. The convergences of the frequency parameters for different radial basis functions are shown in Fig. 1&2. It can be seen that convergence of thin plate spline radial basis function is better as compared to others, however reasonably good convergence is achieved at 15\times15 nodes.

V. NUMERICAL EXAMPLES

Frequency parameter of simply supported isotropic plate for first eight modes (\(a/h = 100\)) are analyzed using different Radial Basis Functions and the results obtained from the present analysis are compared with published results of Liew et al [10] and Ferreira et al [7] and are shown in Table-I. The results are well compared however accuracy of the Polynomial Radial Basis function is relatively lesser as compared to others.

<table>
<thead>
<tr>
<th>Mode</th>
<th>TPS</th>
<th>GRF</th>
<th>POLY</th>
<th>MQ</th>
<th>IMQ</th>
<th>Ferreira et al [7]</th>
<th>Liew et al [10]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.096</td>
<td>0.097</td>
<td>0.097</td>
<td>0.097</td>
<td>0.097</td>
<td>0.096</td>
<td>0.096</td>
</tr>
<tr>
<td>2</td>
<td>0.238</td>
<td>0.242</td>
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<td>0.242</td>
<td>0.244</td>
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<td>0.242</td>
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<td>3</td>
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<td>0.242</td>
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<tr>
<td>4</td>
<td>0.386</td>
<td>0.388</td>
<td>0.393</td>
<td>0.389</td>
<td>0.391</td>
<td>0.385</td>
<td>0.386</td>
</tr>
<tr>
<td>5</td>
<td>0.481</td>
<td>0.484</td>
<td>0.482</td>
<td>0.484</td>
<td>0.488</td>
<td>0.480</td>
<td>0.490</td>
</tr>
<tr>
<td>6</td>
<td>0.483</td>
<td>0.484</td>
<td>0.484</td>
<td>0.484</td>
<td>0.488</td>
<td>0.481</td>
<td>0.490</td>
</tr>
<tr>
<td>7</td>
<td>0.627</td>
<td>0.629</td>
<td>0.643</td>
<td>0.632</td>
<td>0.637</td>
<td>0.625</td>
<td>0.632</td>
</tr>
<tr>
<td>8</td>
<td>0.672</td>
<td>0.629</td>
<td>0.670</td>
<td>0.632</td>
<td>0.637</td>
<td>0.625</td>
<td>0.632</td>
</tr>
</tbody>
</table>

Table II shows the effect of orthotropy and span to thickness ratio on frequency parameter of simply supported [0/90/90/0] cross ply plate. The results are also depicted graphically in Fig. 3. It is observed that as plate becomes thinner, the frequency parameter increases, however this nature decreases with decrease in orthotropy ratio and the frequency is almost constant for thin plate having higher orthotropy ratio.

V. CONCLUSIONS

The present study shows that the proposed RBFs are capable to accurately predict the natural frequencies of laminated composite plates. However, choice of shape parameter for better and faster convergence is still matter of further investigations. The numerical examples show that all radial basis functions used in present study can be successfully used to analyze the free vibration of simply supported laminated composite plates.
TABLE II
EFFECT OF ORTHOTROPY AND SPAN TO THICKNESS RATIO ON FREQUENCY PARAMETER OF SIMPLY SUPPORTED (0°/90/0/) CROSS PLY PLATE WITH MQRBF

<table>
<thead>
<tr>
<th>a/h</th>
<th>Nodes</th>
<th>E1/E2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>11x11</td>
<td>6.954</td>
</tr>
<tr>
<td></td>
<td>13x13</td>
<td>6.940</td>
</tr>
<tr>
<td></td>
<td>15x15</td>
<td>6.933</td>
</tr>
<tr>
<td></td>
<td>17x17</td>
<td>6.930</td>
</tr>
<tr>
<td>10</td>
<td>11x11</td>
<td>7.420</td>
</tr>
<tr>
<td></td>
<td>13x13</td>
<td>7.392</td>
</tr>
<tr>
<td></td>
<td>15x15</td>
<td>7.378</td>
</tr>
<tr>
<td></td>
<td>17x17</td>
<td>7.370</td>
</tr>
<tr>
<td>20</td>
<td>11x11</td>
<td>7.551</td>
</tr>
<tr>
<td></td>
<td>13x13</td>
<td>7.523</td>
</tr>
<tr>
<td></td>
<td>15x15</td>
<td>7.508</td>
</tr>
<tr>
<td></td>
<td>17x17</td>
<td>7.500</td>
</tr>
<tr>
<td>50</td>
<td>11x11</td>
<td>7.590</td>
</tr>
<tr>
<td></td>
<td>17x17</td>
<td>7.538</td>
</tr>
<tr>
<td>100</td>
<td>11x11</td>
<td>7.596</td>
</tr>
<tr>
<td></td>
<td>15x15</td>
<td>7.552</td>
</tr>
<tr>
<td></td>
<td>17x17</td>
<td>7.544</td>
</tr>
</tbody>
</table>

Fig. 3 Effect of orthotropy and span to thickness ratio on frequency parameter of simply supported (0°/90/0/) cross ply plate MQ

REFERENCES
\[
\begin{align*}
\{M_{xx,xy}\} &= [B_{12} \quad B_{22} \quad B_{23}] \\
&= \begin{bmatrix}
\frac{\partial^2 u_x}{\partial x^2} & \frac{\partial^2 v_x}{\partial x \partial y} & \frac{\partial^2 w_x}{\partial x \partial y} \\
\frac{\partial^2 u_y}{\partial x \partial y} & \frac{\partial^2 v_y}{\partial y^2} & \frac{\partial^2 w_y}{\partial y^2} \\
\frac{\partial^2 u_z}{\partial x \partial y} & \frac{\partial^2 v_z}{\partial y^2} & \frac{\partial^2 w_z}{\partial y^2}
\end{bmatrix} \\
&= \begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{bmatrix}
\end{align*}
\]
(A.14)

\[
\begin{align*}
\{F_0 \quad F_2 \quad F_3\} &= \begin{bmatrix}
\frac{\partial^2 \theta_x}{\partial x^2} \\
\frac{\partial^2 \theta_y}{\partial y^2} \\
\frac{\partial^2 \theta_z}{\partial z^2}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\{M_{yy,xy}\} &= [B_{22} \quad B_{23} \quad B_{26}] \\
&= \begin{bmatrix}
\frac{\partial^2 u_y}{\partial y^2} & \frac{\partial^2 v_y}{\partial y \partial x} & \frac{\partial^2 w_y}{\partial y \partial x} \\
\frac{\partial^2 u_x}{\partial y \partial x} & \frac{\partial^2 v_x}{\partial x^2} & \frac{\partial^2 w_x}{\partial x^2} \\
\frac{\partial^2 u_z}{\partial y \partial x} & \frac{\partial^2 v_z}{\partial x^2} & \frac{\partial^2 w_z}{\partial x^2}
\end{bmatrix} \\
&= \begin{bmatrix}
D_{12} & D_{22} & D_{23} \\
D_{12} & D_{22} & D_{26} \\
D_{12} & D_{22} & D_{26}
\end{bmatrix}
\end{align*}
\]
(A.15)

\[
\begin{align*}
\{F_2 \quad F_3 \quad F_3\} &= \begin{bmatrix}
\frac{\partial^2 \theta_y}{\partial x \partial y} \\
\frac{\partial^2 \theta_z}{\partial x \partial y} \\
\frac{\partial^2 \theta_x}{\partial y \partial z}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\{M_{yy,yy}\} &= [B_{26} \quad B_{26} \quad B_{26}] \\
&= \begin{bmatrix}
\frac{\partial^2 u_y}{\partial y^2} & \frac{\partial^2 v_y}{\partial y^2} & \frac{\partial^2 w_y}{\partial y^2} \\
\frac{\partial^2 u_x}{\partial y \partial y} & \frac{\partial^2 v_x}{\partial x^2} & \frac{\partial^2 w_x}{\partial x^2} \\
\frac{\partial^2 u_z}{\partial y \partial y} & \frac{\partial^2 v_z}{\partial x^2} & \frac{\partial^2 w_z}{\partial x^2}
\end{bmatrix} \\
&= \begin{bmatrix}
D_{66} & D_{66} & D_{66} \\
D_{66} & D_{66} & D_{66} \\
D_{66} & D_{66} & D_{66}
\end{bmatrix}
\end{align*}
\]
(A.16)

\[
\begin{align*}
\{F_6 \quad F_6 \quad F_6\} &= \begin{bmatrix}
\frac{\partial^2 \theta_y}{\partial z \partial y} \\
\frac{\partial^2 \theta_x}{\partial z \partial y} \\
\frac{\partial^2 \theta_z}{\partial z \partial y}
\end{bmatrix}
\end{align*}
\]