World Academy of Science, Engineering and Technology
International Journal of Mechanical and Mechatronics Engineering
Vol:5, No:7, 2011

RBF- based Meshless Method for Free Vibration Analysis of Laminated Composite Plates

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***Abstract*—**The governing differential equations of laminated plate utilizing trigonometric shear deformation theory are derived using energy approach. The governing differential equations discretized by different radial basis functions are used to predict the free vibration behavior of symmetric laminated composite plates. Effect of orthotropy and span to thickness ratio on frequency parameter of simply supported laminated plate is presented. Numerical results show the accuracy and good convergence of radial basis functions.

***Keywords*—**Composite plates, Meshfree method, free vibration, Shear deformation, RBFs

I. INTRODUCTION

COMPOSITE laminates have found many applications in variety of engineering structures because of their high strength-to-weight, stiffness-to-weight ratios, good energy and sound absorption, and often also low production cost. The applicability of laminated composites ranges from deep ocean to high in the sky, where weight saving is crucial. The flexure, vibration and buckling analysis of the laminated composite plates subjected to mechanical loading has been the subject of research interest of many investigators. The significant increase of the industrial use of laminated composites calls for the development of new numerical tools/methods for the analysis of such high performance structures. Most popular techniques are based on finite differences and finite elements. In the last decade, meshless methods have received much attention because of their flexibility in the construction of finite dimensional subspaces. When the dominant domain is extremely complex, traditional numerical method for partial differential equations (PDEs), such as the finite element method, finite difference method etc, is somewhat difficult to carry out. Most of the work till date on meshless methods using RBFs relates to scattered data approximation, there has recently been an increased interest in their use for solving partial differential equations (PDEs). This approach, which approximates the whole solution of the PDE directly using RBFs, is very attractive due to the fact that this is truly a mesh-free technique. A considerable work has been done on meshfree analysis of composites in the last few years. Kansa [1] introduced the concept of solving PDEs using RBFs.

Kansa's method is an unsymmetrical RBF collocation method based upon the MQ interpolation functions. Radial basis functions (RBFs) is one of the best rescently developed methods that has attracted several researchers in recent years especially in the area of computational mechanics. A review of meshless methods for laminated plated has been presented by Liew *et al* [2]. Radial basis function was applied by Xiang *et al* [3-4] for linear flexural and free vibration analysis of the laminated composite and sandwich plates. Castro *et al* [5] used wavelet collocations for static analysis of sandwich plates using layer wise theory. Recently Ferreira *et al* [6-7] used wavelets and Wendland radial basis function for buckling analysis of laminated composite plates. Liew and Huang [8] used moving least-squares differential quadrature for bending and buckling; Liew et al [8-10] used reproducing kernel approximations and meshfree method for buckling analysis of isotropic circular and skew plates. In the present study, the free vibration analysis of laminated composite plates using different radial basis function and trigonometric shear deformation theory is presented. The simply supported immovable rectangular plates are analyzed. Results have been compared with the results obtained by other numerical and analytical methods.

II. MATHEMATICAL FORMULATION

The mathematical formulation of the actual physical problem of the laminated composite plate is presented. The governing differential equations are derived using energy approach and utilizing trigonometric shear deformation theory and linear kinematics. The displacement field at any point in the laminated composite plate made up of perfectly bonded layers of uniform thickness is expressed as Xiang *et al* [3]:

$$u'(x,y,z) = u(x,y) - z\frac{\partial w}{\partial x} + f(z)\phi_x \tag{1.1}$$

$$v'(x,y,z) = v(x,y) - z\frac{\partial w}{\partial y} + f(z)\phi_y \tag{1.2}$$

$$w'(x,y) = w(x,y) \tag{1.3}$$

Where, $f(z) = \frac{h}{\pi}\sin(\frac{\pi z}{h})$ and the parameters u', v' and w' are the in-plane and transverse displacements of the plate at any point (x, y, z) in x, y and z directions, respectively. u, v and w are the displacements at mid plane of the plate at any point (x, y) in x, y and z directions, respectively. The functions ϕ_x and ϕ_y are the higher order rotations of the normal to the mid plane due to shear deformation about y and x axes, respectively.

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Strain displacements relations are given by

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u'}{\partial x} \\ \frac{\partial v'}{\partial y} \\ \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \\ \frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y} \\ \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \end{Bmatrix}$$

(2)

The governing differential equations of plate are obtained using Hamilton's principle and expressed as:

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial \tau^2} - I_1 \frac{\partial^3 w_0}{\partial x \partial \tau^2} + I_3 \frac{\partial^2 \phi_x}{\partial \tau^2} \tag{3.1}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial \tau^2} - I_1 \frac{\partial^3 w_0}{\partial y \partial \tau^2} + I_3 \frac{\partial^2 \phi_y}{\partial \tau^2} \tag{3.2}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2.\frac{\partial^2 M_{xy}}{\partial x \partial y} = I_0 \frac{\partial^2 w_0}{\partial t^2} \qquad (3.2)$$

$$+ I_1 \left(\frac{\partial^3 u}{\partial x \partial \tau^2} + \frac{\partial^3 v}{\partial y \partial \tau^2} \right) - I_2 \left(\frac{\partial^4 w}{\partial x^2 \partial \tau^2} + \frac{\partial^4 w}{\partial y^2 \partial \tau^2} \right) \qquad (3.3)$$

$$+ I_4 \left(\frac{\partial^3 \phi_x}{\partial x^2 \partial \tau^2} + \frac{\partial^3 \phi_y}{\partial y^2 \partial \tau^2} \right)$$

$$\frac{\partial M^f_{xx}}{\partial x} + \frac{\partial M^f_{xy}}{\partial y} - Q^f_x = I_3 \frac{\partial^2 u_0}{\partial \tau^2} - I_4 \frac{\partial^3 w_0}{\partial x \partial \tau^2} + I_5 \frac{\partial^2 \phi_x}{\partial \tau^2} \tag{3.4}$$

$$\frac{\partial M^f_{xy}}{\partial x} + \frac{\partial M^f_{yy}}{\partial y} - Q^f_y = I_3 \frac{\partial^2 v_0}{\partial \tau^2} - I_4 \frac{\partial^3 w_0}{\partial y \partial \tau^2} + I_5 \frac{\partial^2 \phi_y}{\partial \tau^2} \tag{3.5}$$

The force and moment resultants and their derivatives used in equations (3.1-3.5) are given in Appendix A.

The plate stiffness coefficients and inertia terms are expressed as:

$$A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} =$$

$$\sum_{k=1}^{n} \overline{Q}_{ij} \int_{z_k}^{z_{k+1}} \left(1, z, z^2, f(z), (z) \times f(z), f^2(z)\right) dz$$

Where i, j = 1, 2, 6: (4.1)

$$A_{ij} = \sum_{k=1}^{n} \overline{Q}_{ij} \int_{z_k}^{z_{k+1}} \left(\frac{\partial f(z)}{\partial z} \right)^2 dz \qquad \text{i, j= 4, 5} \tag{4.2}$$

$$I_0, I_1, I_2, I_3, I_4, I_5 = \rho \int_{h_1}^{h_2} \left(1, z, z^2, f(z), zf(z), f^2(z)\right) dz \tag{4.3}$$

The boundary conditions for an arbitrary edge with simply supported conditions are as follows:

$$x = 0, a : v = 0; \phi_y = 0; w = 0; M_x = 0; N_x = 0 \tag{5.1}$$

$$y = 0, b : u = 0; \phi_x = 0; w = 0; M_y = 0; N_y = 0 \tag{5.2}$$

III. Solution Methodology

Consider a general differential equation of form

$$A\{u\} = f(x, y) \qquad \text{in} \quad \Omega \tag{6.1}$$

$$B\{u\} = g(x, y) \qquad \text{on} \quad \delta\Omega \tag{6.2}$$

Where, A is a linear differential operator, B is a linear boundary operator imposed on boundary conditions. u is unknown quantity to be determined. Let $\{P = (x, y)\}_{i=1}^{N}$ be *N* collocation points in domain Ω of which $\{(x_i, y_i)\}_{i=1}^{N_I}$ are interior points; $\{(x_i, y_i)\}_{i=N_I+1}^{N}$ are boundary points. In RBF method, the approximate solution for differential equation (3.1-3.5) and boundary conditions (6.1-6.2) can be expressed as:

$$u(x, y) = \sum_{j=1}^{N} \alpha_j \varphi_j(x, y) \tag{6.3}$$

Where, $\varphi_j(x, y)$ is radial basis function. Most commonly used radial basis functions are:

$g = r^{2c} \log r$	Logarithmic or thin plate spline
$g = e^{-c^2 \times r}$	Gaussian function
$g = r^c$	Polynomial function
$g = \left(r^2 + c^2\right)^{(1/2)}$	Multiquadric function
$g = \left(r^2 + c^2\right)^{-(1/2)}$	Inverse multiquadric function

Where, $r = \|X - X_j\| = \sqrt{(x - x_j)^2 + (y - y_j)^2}$ and c is the shape parameter.

Free Vibration Analysis

The discretized governing differential equation is expressed as eigenvalue for predicting the natural frequency.

$$\left([K] - \Omega^2 [M]\right)\{\ \} = \{\ \} \tag{6.4}$$

Where [K] is the stiffness matrix and [M] refers to mass matrix, and the parameter Ω refers to the frequency parameter

IV. Results and Discussions

A RBF based meshless code in MATLAB is developed following the analysis procedure as discussed above. In order to demonstrate the accuracy and applicability of present formulation, several examples have been analyzed and the computed results are compared with the published results. Based on convergence study, a 15×15 node is used throughout the study. All individuals' layers are taken to be of equal thickness. The material properties of laminates have been taken as follows:E_1/E_2 = open; $G_{12} = G_{13} = 0.6\times E_2$; $G_{23}=0.5\times E_2$; $\nu_{12}=0.25$;
The dimensionless natural frequency parameter is defined as:

$$\Omega = \left(\frac{\omega a^2}{h}\right)\left(\frac{\rho}{E_2}\right)^{(1/2)}$$

(7.1)

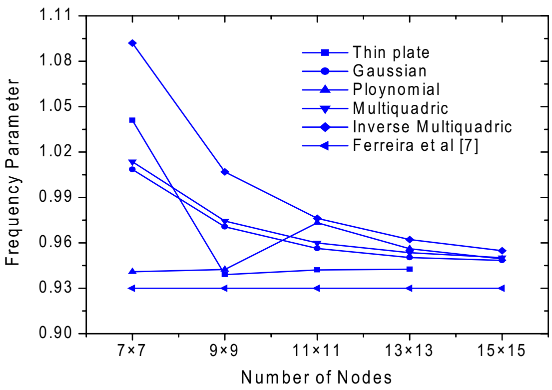

Fig. 1 Convergence study for free vibrations of simply supported isotropic plate (a/h = 10)

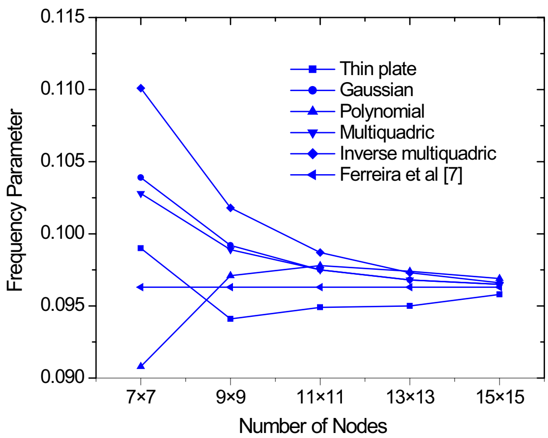

Fig. 2 Convergence study for free vibrations of simply supported isotropic plate (a/h = 100)

In order to show the accuracy and efficiency of the present solution methodology, detailed convergence studies for simply supported isotropic plate (a/h=10 and a/h=100) is carried out. The plate edge length and thickness are denoted by 'a' and 'h', respectively. The convergences of the frequency parameters for different radial basis functions are shown in Fig. 1&2. It can be seen that convergence of thin plate spline radial basis function is better as compared to others, however reasonably good convergence is achieved at 15×15 nodes.

V.Numerical Examples

Frequency parameter of simply supported isotopic plate for first eight modes (a/h = 100) are analyzed using different Radial Basis Functions and the results obtained from the present analysis are compared with published results of Liew *et al* [10] and Ferreira *et al* [7] and are shown in Table-I The results are well compared however accuracy of the Polynomial Radial Basis function is relatively lesser as compared to others.

Table I
Frequency parameter of simply supported isotopic for first eight modes (a/h = 100)

Mode	TPS	GRF	PLOY	MQ	IMQ	Ferreira et al [7]	Liew et al [10]
1	0.096	0.097	0.097	0.097	0.097	0.096	0.096
2	0.238	0.242	0.242	0.242	0.244	0.240	0.242
3	0.241	0.242	0.242	0.242	0.244	0.240	0.242
4	0.386	0.388	0.393	0.389	0.391	0.385	0.386
5	0.481	0.484	0.482	0.484	0.488	0.480	0.490
6	0.483	0.484	0.484	0.484	0.488	0.481	0.490
7	0.627	0.629	0.643	0.632	0.637	0.625	0.632
8	0.672	0.629	0.670	0.632	0.637	0.625	0.632

Table II shows the effect of orthotropy and span to thickness ratio on frequency parameter of simply supported [0/90/90/0] cross ply plate The results are also depicted graphically in Fig. 3.. It is observed that as plate becomes thinner, the frequency parameter increases, however this nature decreases with decrease in orthotropy ratio and the frequency is almost constant for thin plate having higher orthotropy ratio.

V.Conclusions

The present study shows that the proposed RBFs are capable to accurately predict the natural frequencies of laminated composite plates, However, choice of shape parameter for better and faster convergence is still matter of further investigations. The numerical examples show that all radial basis functions used in present study can be successfully used to analyze the free vibration of simply supported laminated composite plates.

TABLE II
EFFECT OF ORTHOTROPY AND SPAN TO THICKNESS RATIO ON FREQUENCY PARAMETER OF SIMPLY SUPPORTED (0/90/90/0) CROSS PLY PLATE WITH MQRBF

a/h	Nodes	E_1/E_2				
		3	10	20	30	40
5	11×11	6.954	8.724	10.023	10.804	10.524
	13×13	6.940	8.707	10.003	10.781	11.196
	15×15	6.933	8.698	9.9920	10.767	11.310
	17×17	6.930	8.693	9.9840	10.758	11.298
10	11×11	7.420	10.031	12.446	14.116	15.377
	13×13	7.392	10.010	12.420	14.087	15.346
	15×15	7.378	10.000	12.408	14.073	15.330
	17×17	7.370	9.995	12.401	14.065	15.320
20	11×11	7.551	10.557	13.610	15.954	17.883
	13×13	7.523	10.517	13.559	15.894	17.814
	15×15	7.508	10.497	13.532	15.862	17.778
	17×17	7.500	10.486	13.517	15.844	17.757
50	11×11	7.590	10.708	13.986	16.605	18.841
	13×13	7.561	10.666	13.932	16.541	18.769
	15×15	7.546	10.645	13.905	16.508	18.732
	17×17	7.538	10.634	13.890	16.491	18.712
100	11×11	7.596	10.730	14.043	16.706	18.994
	13×13	7.567	10.688	13.989	16.641	18.920
	15×15	7.552	10.667	13.961	16.608	18.882
	17×17	7.544	10.656	13.946	16.590	18.862

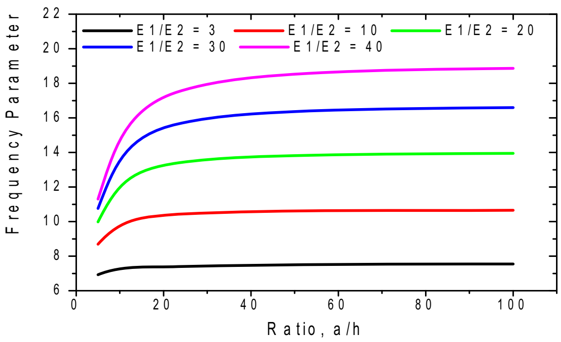

Fig. 3 Effect of orthotropy and span to thickness ratio on frequency parameter of simply supported (0/90/90/0) cross ply plate MQ

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APPENDIX- A

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_o}{\partial x^2} \\ \frac{\partial^2 w_o}{\partial y^2} \\ 2.\frac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} \tag{A.1}$$

$$+ \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{Bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_o}{\partial x^2} \\ \frac{\partial^2 w_o}{\partial y^2} \\ 2.\frac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} \tag{A.2}$$

$$+ \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} M^f{}_{xx} \\ M^f{}_{yy} \\ M^f{}_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{Bmatrix} - \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_o}{\partial x^2} \\ \frac{\partial^2 w_o}{\partial y^2} \\ 2.\frac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} \tag{A.3}$$

$$+ \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

$$Q_y^f = A_{44}\phi_y + A_{45}\phi_x \tag{A.4}$$

$$Q_x^f = A_{45}\phi_y + A_{55}\phi_x \tag{A.5}$$

(

$$\left\{N_{xx,x}\right\}=\left[A_{11}\quad A_{12}\quad A_{16}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial x^2}\\ \dfrac{\partial^2 v_o}{\partial x\partial y}\\ \dfrac{\partial^2 u_o}{\partial x\partial y}+\dfrac{\partial^2 v_o}{\partial x^2}\end{Bmatrix}-\left[B_{11}\quad B_{12}\quad B_{16}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^3}\\ \dfrac{\partial^3 w_o}{\partial x\partial y^2}\\ 2.\dfrac{\partial^3 w_o}{\partial x^2\partial y}\end{Bmatrix}+\left[E_{11}\quad E_{12}\quad E_{16}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x^2}\\ \dfrac{\partial^2 \phi_y}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_x}{\partial x\partial y}+\dfrac{\partial^2 \phi_y}{\partial x^2}\end{Bmatrix} \tag{A.6}$$

$$\left\{N_{yy,y}\right\}=\left[A_{12}\quad A_{22}\quad A_{26}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial y\partial x}\\ \dfrac{\partial^2 v_o}{\partial y^2}\\ \dfrac{\partial^2 u_o}{\partial y^2}+\dfrac{\partial^2 v_o}{\partial y\partial x}\end{Bmatrix}-\left[B_{12}\quad B_{22}\quad B_{26}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^2\partial y}\\ \dfrac{\partial^3 w_o}{\partial y^3}\\ 2.\dfrac{\partial^3 w_o}{\partial y^2\partial x}\end{Bmatrix}+\left[E_{12}\quad E_{22}\quad E_{26}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_y}{\partial y^2}\\ \dfrac{\partial^2 \phi_x}{\partial y^2}+\dfrac{\partial^2 \phi_y}{\partial x\partial y}\end{Bmatrix} \tag{A.7}$$

$$\left\{N_{xy,x}\right\}=\left[A_{16}\quad A_{26}\quad A_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial x^2}\\ \dfrac{\partial^2 v_o}{\partial x\partial y}\\ \dfrac{\partial^2 u_o}{\partial x\partial y}+\dfrac{\partial^2 v_o}{\partial x^2}\end{Bmatrix}-\left[B_{16}\quad B_{26}\quad B_{66}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^3}\\ \dfrac{\partial^3 w_o}{\partial x\partial y^2}\\ 2.\dfrac{\partial^3 w_o}{\partial x^2\partial y}\end{Bmatrix}+\left[E_{16}\quad E_{26}\quad E_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x^2}\\ \dfrac{\partial^2 \phi_y}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_x}{\partial x\partial y}+\dfrac{\partial^2 \phi_y}{\partial x^2}\end{Bmatrix} \tag{A.8}$$

$$\left\{N_{xy,y}\right\}=\left[A_{16}\quad A_{26}\quad A_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial y\partial x}\\ \dfrac{\partial^2 v_o}{\partial y^2}\\ \dfrac{\partial^2 u_o}{\partial y^2}+\dfrac{\partial^2 v_o}{\partial y\partial x}\end{Bmatrix}-\left[B_{16}\quad B_{26}\quad B_{66}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^2\partial y}\\ \dfrac{\partial^3 w_o}{\partial y^3}\\ 2.\dfrac{\partial^3 w_o}{\partial y^2\partial x}\end{Bmatrix}+\left[E_{16}\quad E_{26}\quad E_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_y}{\partial y^2}\\ \dfrac{\partial^2 \phi_x}{\partial y^2}+\dfrac{\partial^2 \phi_y}{\partial x\partial y}\end{Bmatrix} \tag{A.9}$$

$$\left\{M^f{}_{xx,x}\right\}=\left[E_{11}\quad E_{12}\quad E_{16}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial x^2}\\ \dfrac{\partial^2 v_o}{\partial x\partial y}\\ \dfrac{\partial^2 u_o}{\partial x\partial y}+\dfrac{\partial^2 v_o}{\partial x^2}\end{Bmatrix}-\left[F_{11}\quad F_{12}\quad F_{16}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^3}\\ \dfrac{\partial^3 w_o}{\partial x\partial y^2}\\ 2.\dfrac{\partial^3 w_o}{\partial x^2\partial y}\end{Bmatrix}+\left[H_{11}\quad H_{12}\quad H_{16}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x^2}\\ \dfrac{\partial^2 \phi_y}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_x}{\partial x\partial y}+\dfrac{\partial^2 \phi_y}{\partial x^2}\end{Bmatrix} \tag{A.10}$$

$$\left\{M^f{}_{yy,y}\right\}=\left[E_{12}\quad E_{22}\quad E_{26}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial y\partial x}\\ \dfrac{\partial^2 v_o}{\partial y^2}\\ \dfrac{\partial^2 u_o}{\partial y^2}+\dfrac{\partial^2 v_o}{\partial y\partial x}\end{Bmatrix}-\left[F_{12}\quad F_{22}\quad F_{26}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^2\partial y}\\ \dfrac{\partial^3 w_o}{\partial y^3}\\ 2.\dfrac{\partial^3 w_o}{\partial y^2\partial x}\end{Bmatrix}+\left[H_{12}\quad H_{22}\quad H_{26}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_y}{\partial y^2}\\ \dfrac{\partial^2 \phi_x}{\partial y^2}+\dfrac{\partial^2 \phi_y}{\partial x\partial y}\end{Bmatrix} \tag{A.11}$$

$$\left\{M^f{}_{xy,x}\right\}=\left[E_{16}\quad E_{26}\quad E_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial x^2}\\ \dfrac{\partial^2 v_o}{\partial x\partial y}\\ \dfrac{\partial^2 u_o}{\partial x\partial y}+\dfrac{\partial^2 v_o}{\partial x^2}\end{Bmatrix}-\left[F_{16}\quad F_{26}\quad F_{66}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^3}\\ \dfrac{\partial^3 w_o}{\partial x\partial y^2}\\ 2.\dfrac{\partial^3 w_o}{\partial x^2\partial y}\end{Bmatrix}+\left[H_{16}\quad H_{26}\quad H_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x^2}\\ \dfrac{\partial^2 \phi_y}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_x}{\partial x\partial y}+\dfrac{\partial^2 \phi_y}{\partial x^2}\end{Bmatrix} \tag{A.12}$$

$$\left\{M^f{}_{xy,y}\right\}=\left[E_{16}\quad E_{26}\quad E_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial y\partial x}\\ \dfrac{\partial^2 v_o}{\partial y^2}\\ \dfrac{\partial^2 u_o}{\partial y^2}+\dfrac{\partial^2 v_o}{\partial y\partial x}\end{Bmatrix}-\left[F_{16}\quad F_{26}\quad F_{66}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^2\partial y}\\ \dfrac{\partial^3 w_o}{\partial y^3}\\ 2.\dfrac{\partial^3 w_o}{\partial y^2\partial x}\end{Bmatrix}+\left[H_{16}\quad H_{26}\quad H_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_y}{\partial y^2}\\ \dfrac{\partial^2 \phi_x}{\partial y^2}+\dfrac{\partial^2 \phi_y}{\partial x\partial y}\end{Bmatrix} \tag{A.13}$$

$$\left\{M_{xx,xx}\right\}=\left[B_{11}\quad B_{12}\quad B_{16}\right]\begin{Bmatrix}\dfrac{\partial^3 u_o}{\partial x^3}\\ \dfrac{\partial^3 v_o}{\partial x^2\partial y}\\ \dfrac{\partial^3 u_o}{\partial x^2\partial y}+\dfrac{\partial^3 v_o}{\partial x^3}\end{Bmatrix}-\left[D_{11}\quad D_{12}\quad D_{16}\right]\begin{Bmatrix}\dfrac{\partial^4 w_o}{\partial x^4}\\ \dfrac{\partial^4 w_o}{\partial x^2\partial y^2}\\ 2.\dfrac{\partial^4 w_o}{\partial x^3\partial y}\end{Bmatrix} \quad \text{(A.14)}$$

$$+\left[F_{11}\quad F_{12}\quad F_{16}\right]\begin{Bmatrix}\dfrac{\partial^3 \phi_x}{\partial x^3}\\ \dfrac{\partial^3 \phi_y}{\partial x^2\partial y}\\ \dfrac{\partial^3 \phi_x}{\partial x^2\partial y}+\dfrac{\partial^3 \phi_y}{\partial x^3}\end{Bmatrix}$$

$$\left\{M_{yy,yy}\right\}=\left[B_{12}\quad B_{22}\quad B_{26}\right]\begin{Bmatrix}\dfrac{\partial^3 u_o}{\partial y^2\partial x}\\ \dfrac{\partial^3 v_o}{\partial y^3}\\ \dfrac{\partial^3 u_o}{\partial y^3}+\dfrac{\partial^3 v_o}{\partial y^2\partial x}\end{Bmatrix}-\left[D_{12}\quad D_{22}\quad D_{26}\right]\begin{Bmatrix}\dfrac{\partial^4 w_o}{\partial x^2\partial y^2}\\ \dfrac{\partial^4 w_o}{\partial y^4}\\ 2.\dfrac{\partial^4 w_o}{\partial y^3\partial x}\end{Bmatrix} \quad \text{(A.15)}$$

$$+\left[F_{12}\quad F_{22}\quad F_{26}\right]\begin{Bmatrix}\dfrac{\partial^3 \phi_x}{\partial x\partial y^2}\\ \dfrac{\partial^3 \phi_y}{\partial y^3}\\ \dfrac{\partial^3 \phi_x}{\partial y^3}+\dfrac{\partial^3 \phi_y}{\partial x\partial y^2}\end{Bmatrix}$$

$$\left\{M_{xy,xy}\right\}=\left[B_{16}\quad B_{26}\quad B_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 u_o}{\partial x^2}\\ \dfrac{\partial^2 v_o}{\partial x\partial y}\\ \dfrac{\partial^2 u_o}{\partial x\partial y}+\dfrac{\partial^2 v_o}{\partial x^2}\end{Bmatrix}-\left[D_{16}\quad D_{26}\quad D_{66}\right]\begin{Bmatrix}\dfrac{\partial^3 w_o}{\partial x^3}\\ \dfrac{\partial^3 w_o}{\partial x\partial y^2}\\ 2.\dfrac{\partial^3 w_o}{\partial x^2\partial y}\end{Bmatrix} \quad \text{(A.16)}$$

$$+\left[F_{16}\quad F_{26}\quad F_{66}\right]\begin{Bmatrix}\dfrac{\partial^2 \phi_x}{\partial x^2}\\ \dfrac{\partial^2 \phi_y}{\partial x\partial y}\\ \dfrac{\partial^2 \phi_x}{\partial x\partial y}+\dfrac{\partial^2 \phi_y}{\partial x^2}\end{Bmatrix}$$