Extremal Properties of Generalized Class of Close-to-convex Functions
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Abstract—Let $G_{a,b}(γ, δ)$ denote the class of function $f(z)$, $f(0) = f'(0) = 0$ which satisfied $\Re e^{iδ} \{af'(z) + βf''(z)\} > γ$ in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ for some $a \in \mathbb{R} (a \neq 0)$, $β \in \mathbb{R}$ and $γ \in \mathbb{R} (0 \leq γ < α)$. In this paper, we determine some extremal properties including distortion theorem and argument of $f'(z)$.

Keywords—Argument of $f'(z)$, Carathéodory Function, Close-to-convex Function, Distortion Theorem, Extremal Properties

I. INTRODUCTION

We denote $G_{a,b}(γ, δ)$ the class of normalized analytic function $f$ in the open unit disk, $D = \{z \in \mathbb{C} : |z| < 1\}$ where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfying $\Re e^{iδ} \{af'(z) + βf''(z)\} > γ$, $z \in D$ for some $a \in \mathbb{R} (a ≠ 0)$, $β \in \mathbb{R}$ and $γ \in \mathbb{R} (0 \leq γ < α)$.

Many of the subclasses of $G_{a,b}(γ, δ)$ have been studied by some other researchers as [1] for $G_{a,b}(γ, 0)$ of some $a \in \mathbb{R}$, $β \in \mathbb{R}$ and $γ \in \mathbb{R} (0 \leq γ < α)$, [2] for $G_{1,0}(γ, 0)$ where $α > 0, β < 1$, [3] for $G_{1,1}(γ, 0)$, [4] for $G_{1,1}(0, 0)$, [5] for $G_{1,0}(γ, δ)$ where $|δ| ≤ π$ and $cos δ – γ > 0$, [6] for $G_{1,0}(0, δ)$ where $|δ| < π/2$ and [7] for $G_{1,0}(0, 0)$.

There is a relationship of the class $P$ in the form of

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

with the extremal information of each selected classes. Writing

$$e^{iδ} (af'(z) + βf''(z)) - γ - iα sin δ = p(z) \quad (z \in D)$$

clearly $f \in G_{a,b}(γ, δ)$ if $p \in P$, the class of functions with positive real parts.

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We make use the result of representation theorem

$$f(z) = \int_{|z|=1} \left[ -e^{-iδ} \left( e^{-iδ} - \frac{2γ}{α} \right) - e^{-iδ} A \frac{z log(1-xz)}{(nβ + α)} \right] dµ(x)$$

where $A = (α cos δ - γ)$ given by [8] in order to determine the distortion theorem and argument of $f'(z)$ for this class of function.

II. EXTREMAL PROPERTIES

We begin by finding the radius and centre of $G_{a,b}(γ, δ)$ that will be used for later results.

Theorem 3.1 Let $f(z) \in G_{a,b}(γ, δ)$. Then $f'(z)$ maps $|z| ≤ r$ into disc $D_r$ with centre and radius

$$-e^{-iδ} \left( e^{-iδ} - \frac{2γ}{α} \right) + \frac{2e^{-iδ} AM}{1-r^2} and \frac{2AMr}{1-r^2}$$

where $A = α cos δ - γ$, $M = \frac{1}{nβ + α}$ respectively.

Proof. If $a$ and $b$ are complex numbers with $|b| < 1$ and if $0 < r < 1$, the range of the function $(1 + arw)/(1 + brw)$ where $|w| ≤ 1$ is a disc with center and radius respectively.

$$\frac{1 - abr^2}{1 - |b|^2 r^2}, \quad \frac{|a| - |b|}{1 - |b|^2 r^2}$$

By taking $a = B e^{-iδ} \left( e^{-iδ} - \frac{2γ}{α} \right)xr$ and $b = xr$ where $|x| = 1$, we see that maps $|z| ≤ r$ onto $D_r$. By convexity, any linear combination of functions of this form also maps $D$ onto $D_r$. Since for some probability measure $μ$, we have

$$B \left\{ \frac{1 + Be^{-iδ} \left( e^{-iδ} - \frac{2γ}{α} \right)xz}{1 - xz} \right\}$$
Corollary 3.1 If \( f(z) \in G_{\alpha, \beta}(\gamma, \delta) \) then

\[
f'(z) < B \left\{ \frac{1 + \text{Re}^{-i\theta} - 2\gamma}{1 - z} \right\}, \quad z \in D
\]

The simple geometry of a circle enables us to deduce from Theorem 3.2, upper and lower bounds for \( \text{Re} f'(z) \), \( \text{Im} f'(z) \), \( |f'(z)| \) and \( \arg f'(z) \) when \( f(z) = G_{\alpha, \beta}(\gamma, \delta) \).

Theorem 3.2 If \( f(z) \in G_{\alpha, \beta}(\gamma, \delta) \), then

\[
\frac{1 + B + r^2(2AM - 2AM\alpha - 1)}{1 - r^2} \leq \text{Re} f'(z) \leq \frac{1 + B + r^2(2AM - 2AM\alpha - 1) + 2AM\alpha}{1 - r^2}
\]

where \( B = \frac{2A(A + \gamma)(\alpha - 1)}{\alpha^2} \) and \( R = \frac{A + \gamma}{\alpha^2} \), and

\[
-2A \left( M - \frac{1}{\alpha} \right) + r(M + \frac{rT}{\alpha}) - 2A \left( M - \frac{1}{\alpha} \right) + r(M + \frac{rT}{\alpha})
\]

\[
-1 - r^2 \leq \text{Im} f'(z) \leq -\frac{2A}{1 - r^2}
\]

where \( T = \sqrt{1 - \left( \frac{A + \gamma}{\alpha} \right)^2} \) and all bounds are sharp for any extreme point \( f(z) \) of \( G_{\alpha, \beta}(\gamma, \delta) \).

Proof. By Theorem 3.1, we can write

\[
\left| f'(z) - e^{-i\theta}(e^{-i\theta} - 2\gamma) + \frac{2e^{-i\theta}AM}{1 - r^2} \right| \leq \frac{2AM\alpha}{1 - r^2}
\]

That gives:

\[
\frac{2AM\alpha}{1 - r^2} \leq \text{Re} f'(z) + e^{-i\theta}(e^{-i\theta} - 2\gamma) + \frac{2e^{-i\theta}AM}{1 - r^2} \leq \frac{2AM\alpha}{1 - r^2}
\]

and

\[
\frac{2AM\alpha}{1 - r^2} \leq \text{Im} f'(z) + e^{-i\theta}(e^{-i\theta} - 2\gamma) + \frac{2e^{-i\theta}AM}{1 - r^2} \leq \frac{2AM\alpha}{1 - r^2}
\]

That gives:

\[
\frac{-2AM\alpha}{1 - r^2} \leq \text{Re} f'(z) + e^{-i\theta}(e^{-i\theta} - 2\gamma) + \frac{2e^{-i\theta}AM}{1 - r^2} \leq \frac{-2AM\alpha}{1 - r^2}
\]

Since \( \cos \delta = \frac{A + \gamma}{\alpha} \) and \( \sin \delta = \sqrt{1 - \left( \frac{A + \gamma}{\alpha} \right)^2} \), we can write the inequalities in this form

\[
\frac{-2AM\alpha}{1 - r^2} \leq \text{Re} f'(z) - \frac{1 + 2A(A + \gamma)(\alpha - 1) + r^2(2A(A + \gamma)(\alpha - 1) - 1)}{1 - r^2} \leq \frac{2AM\alpha}{1 - r^2}
\]

and

\[
\frac{-2A(1 - (A + \gamma)^2(M - \frac{1}{\alpha}))) + r(M + \frac{rT}{\alpha})}{1 - r^2} \leq \text{Im} f'(z) \leq \frac{2A(1 - (A + \gamma)^2(M - \frac{1}{\alpha}))) + r(M + \frac{rT}{\alpha})}{1 - r^2}
\]

Letting \( B = \frac{2A(A + \gamma)(\alpha - 1)}{\alpha^2} \), \( R = \frac{A + \gamma}{\alpha^2} \) and \( T = \sqrt{1 - \left( \frac{A + \gamma}{\alpha} \right)^2} \), we obtain the above inequalities as required. It is clear that each inequality is sharp for some \( z \) on
Our next result is to obtain a distortion theorem for $f(\zeta) \in G_{a,\beta}(\gamma, \delta)$.

**Theorem 3.3** If $f(\zeta) \in G_{a,\beta}(\gamma, \delta)$, then
\[
|f'(\zeta)| \leq |f(\zeta)| + \frac{2AMr}{1-r^2}
\]
where
\[
C(r) = \left(1 - \frac{4\gamma A}{\alpha} + \frac{4AM}{\alpha} \left(\frac{AM}{1-r^2} + \gamma - A\right)\right)^{\frac{1}{2}}
\]

Proof. Let
\[
\Gamma(r) = -e^{-i\delta}\left(e^{-i\delta} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2}\right)
\]
and from (1), we have
\[
|f'(\zeta)| - |\Gamma(r)| \leq \frac{2AMr}{1-r^2}
\]
so that
\[
|f'(\zeta)| \leq |\Gamma(r)| + \frac{2AMr}{1-r^2} = C(r) + \frac{2AMr}{1-r^2}
\]
as required.

If $\gamma \geq 0$, then $f'$ is non-zero throughout $D$ and has continuous argument whereas if $\gamma < 0$ and $f_0$ is any extreme function of $G_{a,\beta}(\gamma, \delta)$, then at some points in $D$, $f_0$ has a zero, thus, there is no argument. We next obtain bounds for $\arg f'(\zeta)$ when $f(\zeta) \in G_{a,\beta}(\gamma, \delta)$ with restricted value of $|\zeta|$ for the case of $\gamma < 0$. Furthermore, we will use the following property for argument: for given $\delta$ in $[-\pi, \pi]$ and as $x$ varies in some interval $[0, c]$, so that $e^{i\delta} + x \neq 0$, $\phi_\delta(x)$ is continuous argument of $e^{i\delta} + x \neq 0$ for which $\phi_\delta(0) = \delta$. We have
\[
\phi_\delta(x) = \begin{cases} \tan^{-1}\left(\frac{\sin \delta}{\cos \delta + x}\right) & x + \cos \delta > 0 \\ \pi + \tan^{-1}\left(\frac{\sin \delta}{\cos \delta + x}\right) & x + \cos \delta < 0 \\ \frac{\pi}{2} & x + \cos \delta = 0 \end{cases}
\]
when $0 < \delta < \pi$ and $-\pi < \delta < 0$ for $\delta = 0, \pm \pi$.

**Theorem 3.4** Let $f(\zeta) \in G_{a,\beta}(\gamma, \delta)$ and put
\[
x(\zeta) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right) (0 \leq r \leq 1). \text{ Let}
\]
Then, for $0 < |\zeta| = r < r_0$, and for a suitable determination of argument
\[
\arg f'(\zeta) + \delta - \phi_\delta(x(\zeta)) \leq \sin^{-1} \frac{2AMr}{1-r^2} C(r)
\]
where $\phi_\delta(x)$ is defined on $[0, x(\tau_0)]$ as above and $C(r)$ is given by (2). The result is sharp.

Proof. To make sure that $f'(\zeta) \neq 0$, we restrict the values of $|\zeta| = r$ by the condition
\[
\frac{2AMr}{1-r^2} \left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right) > \frac{2AMr}{1-r^2}
\]
and obtain
\[
\frac{2AMr}{1-r^2} - \frac{2\gamma}{\alpha} > 0
\]
and since $A = \alpha \cos \delta - \gamma$, hence
\[
1 + \frac{4AM}{1-r^2} \left(\frac{\alpha}{A - \gamma} - \frac{A - \gamma}{\alpha} - 4A\right) > 0
\]
The inequality holds for all $r$ in $[0,1]$ if $\gamma \geq 0$ and for $r > \sqrt{\frac{4AMr \alpha A - \gamma \gamma}{4A A - \gamma^2}}$ if $\gamma < 0$. This establishes the restricted on $|\zeta|$ in the statement of the theorem. From (1) then with $\Gamma(r)$ given by (3) and $C(r) = |\Gamma(r)|$, we have
\[
C(r) = \left|\frac{4AM}{1-r^2} \left(\frac{M}{1-r^2} - \frac{1}{\alpha}\right) + \frac{4A(\alpha \cos \delta - A)}{\alpha} \left(\frac{M}{1-r^2} - 1\right)\right|^{\frac{1}{2}}
\]
and deduced to $|f'(\zeta)| - \arg \Gamma(r) \leq \sin^{-1} \frac{2AMr}{1-r^2} C(r)$ also
\[
\arg \Gamma(r) = \arg \left(e^{-i\delta} - \frac{2AMr}{1-r^2} - e^{-i\delta} + \frac{2\gamma}{\alpha}\right)
\]
and
\[
= -\delta + \arg \left(e^{-i\delta} + 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)\right).
Put \( x(r) = 2 \left( \frac{AM}{1-r^2} - \frac{A}{\alpha} \right) \), then \( \arg \Gamma(r) = -\delta + \phi_\delta(x(r)) \).

We obtain another theorem that replaced \( \arg f'(z) \) with restricted range of \( \delta \) as \( \arg(f'(z)+k) \) for some real number \( k \) that satisfied \( f'(z)+k \neq 0 \) for \( z \in D \) and \( f \in G_{a,\beta}(\gamma,\delta) \). By taking \( \delta \neq \pi/2 \) as any choice of \( k \) with \( k \cos \delta + \gamma > 0 \) will ensure that above conditions are fulfilled and this is important for the following result to be valid. In the following theorem, for a given \( \delta \in [-\pi, \pi] \) and as \( x \) varies in same interval \([0, c]\), so that \((k+1)e^{i\delta} + x \neq 0\), \( \psi_\delta(\delta) \) is the continuous argument of \((k+1)e^{i\delta} + x\) for which \( \psi_\delta(0) \) is principal.

**Theorem 3.5** For \( \delta \neq \pi/2 \), \( f(z) \in G_{a,\beta}(\gamma,\delta) \) and put \( x(r) = 2 \left( \frac{AM}{1-r^2} - \frac{A}{\alpha} \right) (0 \leq r \leq 1) \). Let \( k \alpha \cos \delta + \gamma > 0 \) where \( k \) is a real number. Then, for \( \psi_\delta(x) \) defined on \([0, \alpha]\),
\[
\arg(f'(z)+k) - \psi_\delta(x(r)) \leq \sin^{-1} \left( \frac{2AMr}{1-r^2} C_1(r) \right)
\]

where
\[
C_1(r) = \left[ 4AT \left( \cos \delta + \frac{1}{\alpha} \gamma (T+1) + k \right)^2 \right], \quad T = \frac{M}{1-r^2} - \frac{1}{\alpha}
\]

**Proof.** Let \( \delta \neq \pi/2 \) and \( k \) satisfied \( k \alpha \cos \delta + \gamma > 0 \). Using (1), we have
\[
|f'(z)+k - \Gamma(r)+k| \leq \frac{2AMr^2}{1-r^2}
\]

where
\[
\Gamma(r) = -e^{-i\delta} \left( e^{-i\delta} - \frac{2\gamma}{\alpha} \right) + 2e^{-i\delta} \frac{AMr}{1-r^2} = 1 + \frac{2Ae^{-i\delta} \left( M - \frac{1}{\alpha} \right)}{1-r^2}
\]

Hence
\[
\arg(f'(z)+k) - \arg(\Gamma(r)+k) \leq \sin^{-1} \left( \frac{2AMr}{1-r^2} C_1(r) \right) \quad (4)
\]

where
\[
C_1(r) = \left[ (k+1)^2 + 4AT \left( \frac{M}{1-r^2} - \frac{1}{\alpha} \right)^2 + k \cos \delta + \cos \delta \right]^{\frac{1}{2}}
\]

Let \( T = \frac{M}{1-r^2} - \frac{1}{\alpha} \), we have
\[
C_1(r) = \left[ 4AT \left( \cos \delta + \frac{1}{\alpha} \gamma (T+1) + \frac{1}{\alpha} \right)^2 + (k+1)^2 \right]^{\frac{1}{2}}
\]

Now
\[
\arg(\Gamma(r)+k) = \arg \left( -\delta + \gamma \left( k+1 \right) e^{i\delta} + 2A \left( \frac{M}{1-r^2} - \frac{1}{\alpha} \right) \right)
\]

and with (4) this completes the proof.

**REFERENCES**


