Extremal Properties of Generalized Class of Close-to-convex Functions

Norlyda Mohamed, Daud Mohamad, and Shaharuddin Cik Soh

Abstract—Let \( G_{a,b}(\gamma, \delta) \) denote the class of function \( f(z) = e^{\gamma z} + \sum_{n=1}^{\infty} a_n z^n \) satisfying \( \Re e^{i\delta} \{ af'(z) + bzf'(z) \} > 0, z \in D \) for some \( \alpha \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \). In this paper, we determine some extremal properties including distortion theorem and argument of \( f'(z) \).

Keywords—Argument of \( f'(z) \), Carathéodory Function, Close-to-convex Function, Distortion Theorem, Extremal Properties

I. INTRODUCTION

We denote \( G_{a,b}(\gamma, \delta) \) the class of normalized analytic function \( f \) in the open unit disk, \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) where

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

satisfying \( \Re e^{i\delta} \{ af'(z) + bzf'(z) \} > \gamma, z \in D \) for some \( \alpha \in \mathbb{R} \), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \) \( (0 < \gamma < \alpha) \).

Many of the subclasses of \( G_{a,b}(\gamma, \delta) \) have been studied by some other researchers as [1] for \( G_{a,b}(\gamma, \delta) \) of some \( \alpha \in \mathbb{R}, \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \) \( (0 \leq \gamma < \alpha) \), [2] for \( G_{1,0}(\gamma, \delta) \) where \( \alpha > 0, \beta < 1 \), [3] for \( G_{1,1}(\gamma, \delta) \), [4] for \( G_{1,0}(0, \delta) \) where \( |\delta| < \pi \) and \( \cos \delta - \gamma > 0 \), [6] for \( G_{0,0}(0, \delta) \) where \( |\delta| < \pi/2 \) and [7] for \( G_{1,0}(0, 0) \).

There is a relationship of the class \( P \) in the form of

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \]

with the extremal information of each selected classes. Writing

\[
\frac{e^{i\delta} (af'(z) + bzf'(z)) - \gamma - i\alpha \sin \delta}{\alpha \cos \delta - \gamma} = p(z) \quad (z \in D)
\]

clearly \( f \in G_{a,b}(\gamma, \delta) \) if \( p \in P \), the class of functions with positive real parts.

II. EXTREMAL PROPERTIES

We begin by finding the radius and centre of \( G_{a,b}(\gamma, \delta) \) that will be used for later results.

Theorem 3.1 Let \( f(z) \in G_{a,b}(\gamma, \delta) \). Then \( f'(z) \) maps \( |z| \leq r \) into disc \( D_r \), with centre and radius

\[
-e^{-i\delta} \left( e^{-i\delta} - \frac{2\gamma}{\alpha} \right) + \frac{2e^{-i\delta} A}{1-r^2} \quad \text{and} \quad \frac{2AMr}{1-r^2}
\]

where \( A = \alpha \cos \delta - \gamma, M = \frac{1}{n\beta + \alpha} \) respectively.

Proof. If \( a \) and \( b \) are complex numbers with \( |a| < 1 \) and if \( 0 < r < 1 \), the range of the function \( (1+avr)/(1+bwr) \) where \( |w| \leq 1 \) is a disc with center and radius respectively.

\[
\frac{1-ar^2}{1-|a|^2}, \quad \frac{|a|-|b|}{1-|a|^2}
\]

By taking \( a = Be^{-i\delta} \left( e^{-i\delta} - \frac{2\gamma}{\alpha} \right) x \) and \( b = xr \) where \( |x|=1 \), we see that maps \( |z| \leq r \) onto \( D_r \). By convexity, any linear combination of functions of this form also maps \( D \) onto \( D_r \). Since for some probability measure \( \mu \), we have

\[
B \left\{ \frac{1+Be^{-i\delta} \left( e^{-i\delta} - \frac{2\gamma}{\alpha} \right)xz}{1-xz} \right\}
\]
Proof. By Theorem 3.1, we can write

$$f'(z) - \left(-e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right) \leq \frac{2AMr}{1-r^2}$$

(1)

So that

$$\frac{2AMr}{1-r^2} \leq \text{Re} \left[f'(z) + e^{-i\delta} \left(e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right) \right] \leq \frac{2AMr}{1-r^2}$$

and

$$-\frac{2AMr}{1-r^2} \leq \text{Re} \left[f'(z) - \left(-e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right) \right] \leq \frac{2AMr}{1-r^2}$$

that gives;

$$-\frac{2AMr}{1-r^2} \leq \text{Re} \left[f'(z) - \left(-e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right) \right] \leq \frac{2AMr}{1-r^2}$$

and

$$-\frac{2AMr}{1-r^2} \leq \text{Im} \left[f'(z) + e^{-i\delta} \left(e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right) \right] \leq \frac{2AMr}{1-r^2}$$

that gives

$$-\frac{2AMr}{1-r^2} \leq \text{Im} \left[f'(z) + e^{-i\delta} \left(e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right) \right] \leq \frac{2AMr}{1-r^2}$$

Since \( \cos \delta = \frac{A+\gamma}{\alpha} \) and \( \sin \delta = \sqrt{1 - \left(\frac{A+\gamma}{\alpha}\right)^2} \), we can write the inequalities in this form

$$-\frac{2AMr}{1-r^2} \leq \text{Re} \left(f'(z) - \left(1 + 2A(A+\gamma) \left(\frac{Ma-1}{\alpha^2} \right) + r \left( \frac{A+\gamma}{\alpha} \right) \right) \right) \leq \frac{2AMr}{1-r^2}$$

and

$$-\frac{2AMr}{1-r^2} \leq \text{Im} \left(f'(z) - \left(1 + 2A(A+\gamma) \left(\frac{Ma-1}{\alpha^2} \right) + r \left( \frac{A+\gamma}{\alpha} \right) \right) \right) \leq \frac{2AMr}{1-r^2}$$

Proof. By Theorem 3.1, we can write

$$f'(z) = \int_{|z|=1} B \left(1 + \frac{Be^{-i\delta}}{1-z} \right) \frac{e^{-i\delta} \left(e^{-\alpha z} - \frac{2\gamma}{\alpha} + \frac{2e^{-i\delta} AM}{1-r^2} \right)}{1-z} \, dz$$

where \( B = \frac{2\gamma}{\alpha} - n\beta e^{-i\delta} + n\beta e^{-2i\delta} + \alpha \), so the result follows.

Corollary 3.1 If \( f(z) \in G_{\alpha,\beta}(\gamma, \delta) \) then

$$f'(z) < B \left(1 + \frac{Be^{-i\delta}}{1-z} \right) \frac{e^{-i\delta} \left(e^{-\alpha z} - \frac{2\gamma}{\alpha} \right)}{1-z}, \ z \in D$$

The simple geometry of a circle enables us to deduce from Theorem 3.2, upper and lower bounds for \( f'(z), \text{Im} \left[f'(z) \right] \) and \( \text{Re} \left[f'(z) \right] \) when \( f(z) = G_{\alpha,\beta}(\gamma, \delta) \).

Theorem 3.2 If \( f(z) \in G_{\alpha,\beta}(\gamma, \delta) \), then

$$\frac{1+B+r^2(2AR-1)-2AMr}{1-r^2} \leq \text{Re} \left[f'(z) \right] \leq \frac{1+B+r^2(2AR-1)+2AMr}{1-r^2}$$

where \( B = \frac{2A(A+\gamma)(Ma-1)}{\alpha^2} \) and \( R = \frac{(A+\gamma)}{\alpha^2} \), and

$$\frac{-2A \left( M-\frac{1}{\alpha} + r(M+rT/\alpha) \right)}{1-r^2} \leq \text{Im} \left[f'(z) \right] \leq \frac{-2A \left( M-\frac{1}{\alpha} + r(M+rT/\alpha) \right)}{1-r^2}$$

where \( T = \sqrt{1 - \left(\frac{A+\gamma}{\alpha}\right)^2} \) and all bounds are sharp for any extreme point \( f(z) \) of \( G_{\alpha,\beta}(\gamma, \delta) \).
$|f| = r$ when $f$ is an extreme point.

Our next result is to obtain a distortion theorem for $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$.

**Theorem 3.3** If $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$, then

$$|f'(z)| \leq |f(r)| + \frac{2AMr}{1-r^2}$$

where

$$C(r) = \left(1 - \frac{4\gamma A}{\alpha^2} + \frac{4AM}{\alpha(1-r^2)}(\frac{AM}{1-r^2}) + \gamma - A\right)^{\frac{1}{2}}$$

(2)

**Proof.** Let

$$\Gamma(r) = -e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right) + \frac{2e^{-i\delta} AM}{1-r^2}$$

and from (1), we have

$$|f'(z)| \leq |\Gamma(r)| \leq \frac{2AMr}{1-r^2}$$

so that

$$|f'(z)| \leq |f(r)| + \frac{2AMr}{1-r^2}$$

$\Gamma(r)$ as required.

If $\gamma \geq 0$, then $f'$ is non-zero throughout $D$ and has continuous argument whereas if $\gamma < 0$ and $f_0$ is any extreme function of $G_{\alpha,\beta}(\gamma, \delta)$, then at some points in $D$, $f_0$ has a zero, thus, there is no argument. We next obtain bounds for $f(z), \phi_\delta$ when $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$ with restricted value of $|z|$ for the case of $\gamma < 0$. Furthermore, we will use the following property for argument: for given $\delta$ in $[-\pi, \pi]$ and as $x$ varies in some interval $[0, c]$, so that $e^{i\delta} + x \neq 0, \phi_\delta(x)$ is continuous argument of $e^{i\delta} + x \neq 0$ for which $\phi_\delta(0) = \delta$. We have

$$\phi_\delta(x) = \begin{cases} \tan^{-1}\left(\frac{\sin \delta}{\cos \delta + x}\right), & x + \cos \delta > 0 \\ \pi + \tan^{-1}\left(\frac{\sin \delta}{\cos \delta + x}\right), & x + \cos \delta < 0 \\ \frac{\pi}{2}, & x + \cos \delta = 0 \end{cases}$$

when $0 < \delta < \pi$ and $-\pi < \delta < 0$ for $\delta = 0, \pm \pi$.

**Theorem 3.4** Let $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$ and put

$$x(r) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)(0 \leq r \leq 1).$$

Let

$$r_0 = \begin{cases} 1, & \gamma \geq 0 \\ 1 - \frac{4\alpha AM(\gamma AM - A + \gamma)}{(4\gamma \alpha - A^2)}), & \gamma < 0 \end{cases}$$

Then, for $0 < |z| = r < r_0$, and for a suitable determination of argument

$$\arg f'(z) + \delta - \phi_\delta(x(r)) \leq \sin^{-1}\left(\frac{2AM}{1-r^2} C(r)\right)$$

where $\phi_\delta(x)$ is defined on $[0, x(r_0)]$ as above and $C(r)$ is given by (2). The result is sharp.

**Proof.** To make sure that $f'(z) \neq 0$, we restrict the values of $|z|$ by the condition

$$\left|\frac{2AM}{1-r^2} - \left(-\frac{2\gamma}{\alpha}\right)\right| > \frac{2AM}{1-r^2}$$

Squaring both sides, we have

$$\left|\frac{4A^2 M^2}{1-r^2} + \frac{4AM}{1-r^2}\right| \frac{2\gamma}{\alpha} - \cos \delta \leq \frac{4\gamma (\cos \delta - \frac{\gamma}{\alpha})}{\alpha} > 0$$

and since $A = \alpha \cos \delta - \gamma$, hence

$$1 + \frac{4AM}{1-r^2}\left(\frac{AM - (A - \gamma)}{\alpha}\right) - \frac{4A\gamma}{\alpha^2} > 0$$

The inequality holds for all $r$ in $[0, 1]$ if $\gamma \geq 0$ and for $0 \leq r > \sqrt{1 - \frac{4\alpha AM(\gamma AM - A + \gamma)}{(4\gamma \alpha - A^2)}}$ if $\gamma < 0$. This establishes the restricted on $|z|$ in the statement of the theorem.

From (1) then with $\Gamma(r)$ given by (3) and $C(r) = |\Gamma(r)|$, we have

$$C(r) = \left[1 + 4\frac{A^2 M}{1-r^2}\left(\frac{M}{1-r^2} - \frac{1}{\alpha}\right) + 4\gamma(\alpha \cos \delta - A)(\frac{MA}{1-r^2})\right]^{\frac{1}{2}}$$

and deduced to $|\arg f'(z) - \arg \Gamma(r) | \leq \sin^{-1}\left(\frac{2AM}{1-r^2} C(r)\right)$

also

$$\arg \Gamma(r) = \arg e^{-i\delta} \left(\frac{2AM}{1-r^2} - e^{-i\delta} + \frac{2\gamma}{\alpha}\right)$$

$$= -\delta + \arg e^{-i\delta} + 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right).$$
Put \( x(r) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right) \), then \( \arg\Gamma(r) = -\delta + \phi_\delta(x(r)) \).

We obtain another theorem that replaced \( \arg f'(z) \) with restricted range of \( |z| \) as \( \arg(f'(z)+k) \) for some real number \( k \) that satisfied \( f'(z)+k \neq 0 \) for \( z \in D \) and \( f \in G_{a,\beta}(y,\delta) \). By taking \( |\delta| \neq \pi/2 \) as any choice of \( k \) with \( k\cos\delta + \gamma > 0 \) will ensure that above conditions are fulfilled and this is important for the following result to be valid. In the following theorem, for a given \( \delta \in [-\pi, \pi] \) and as \( x \) varies in same interval \([0, c]\), so that \((k+1)e_\delta^\gamma + x \neq 0\), \( \psi_\delta(\delta) \) is the continuous argument of \((k+1)e_\delta^\gamma + x \) for which \( \psi_\delta(0) \) is principal.

**Theorem 3.5** For \( |\delta| \neq \pi/2 \), \( f(z) \in G_{a,\beta}(y,\delta) \) and put \( x(r) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)(0 \leq r \leq 1) \). Let \( k\alpha\cos\delta + \gamma > 0 \) where \( k \) is a real number. Then, for \( \psi_\delta(x) \) defined on \([0, \alpha]\),

\[
\arg(f'(z)+k) + \psi_\delta(x(r)) \leq \sin^{-1} \frac{2AMr}{1-r^2}C_1(r)
\]

where

\[
C_1(r) = \left[4\alpha T \left(k\cos\delta + \frac{A}{\alpha}(T+1) + \frac{\gamma}{\alpha}\right)(k+1)^2\right]^{1/2}
\]

\[\Gamma(r) = -e^{-i\delta}\left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right) + \frac{2e^{-i\delta}AMr}{1-r^2} = 1 + \frac{2e^{-i\delta}}{1-r^2}\left(\frac{M - 1}{\alpha}\right)\frac{r^2}{1-r^2}
\]

Hence

\[
\arg(f'(z)+k) - \arg\Gamma(r) \leq \sin^{-1} \frac{2AMr}{1-r^2}C_1(r)
\] (4)

where

\[
C_1(r) = \left[\frac{(k+1)^2 + 4\alpha^2\left(\frac{M - 1}{\alpha}\right)}{1-r^2}\right]^{1/2}
\]

\[C_1(r) = \left[4\alpha T \left(k\cos\delta + \frac{A}{\alpha}(T+1) + \frac{\gamma}{\alpha}\right)(k+1)^2\right]^{1/2}
\]

Let \( T = \frac{M - 1}{\alpha} \), we have

\[
C_1(r) = \left[4\alpha T \left(k\cos\delta + \frac{A}{\alpha}(T+1) + \frac{\gamma}{\alpha}\right)(k+1)^2\right]^{1/2}
\]

Now

\[
\arg(\Gamma(r)+k) = \arg\left(e^{-i\delta}\left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right) + \frac{2e^{-i\delta}AMr}{1-r^2} - \frac{1}{\alpha}\right)
\]

\[= -\delta + \psi_\delta(\delta)(x(r))
\]

and with (4) this completes the proof.

**REFERENCES**


