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Extremal Properties of Generalized Class of Close-to-convex Functions

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***Abstract*—**Let $G_{\alpha,\beta}(\gamma,\delta)$ denote the class of function $f(z)$, $f(0)=f'(0)-1=0$ which satisfied $\operatorname{Re} e^{i\delta}\{\alpha f'(z)+\beta z f''(z)\}>\gamma$ in the open unit disk $D=\{z\in\square : |z|<1\}$ for some $\alpha\in\square\ (\alpha\neq 0)$, $\beta\in\square$ and $\gamma\in\square\ (0\le\gamma<\alpha)$ where $|\delta|\le\pi$ and $\alpha\cos\delta-\gamma>0$. In this paper, we determine some extremal properties including distortion theorem and argument of $f'(z)$.

***Keywords*—**Argument of $f'(z)$, Carathéodory Function, Close-to-convex Function, Distortion Theorem, Extremal Properties

I. Introduction

We denote $G_{\alpha,\beta}(\gamma,\delta)$ the class of normalized analytic function f in the open unit disk, $D=\{z\in\square : |z|<1\}$ where

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

satisfying $\operatorname{Re} e^{i\delta}\{\alpha f'(z)+\beta z f''(z)\}>\gamma,\ z\in D$ for some $\alpha\in\square\ (\alpha\neq 0)$, $\beta\in\square$ and $\gamma\in\square\ (0\le\gamma<\alpha)$.

Many of the subclasses of $G_{\alpha,\beta}(\gamma,\delta)$ have been studied by some other researchers as [1] for $G_{\alpha,\beta}(\gamma,0)$ of some $\alpha\in\square$, $\beta\in\square\ (\beta\neq 0)$ and $\gamma\in\square\ (0\le\gamma<\alpha)$, [2] for $G_{1,\beta}(\gamma,0)$ where $\alpha>0,\beta<1$, [3] for $G_{1,1}(\gamma,0)$, [4] for $G_{1,1}(0,0)$, [5] for $G_{1,0}(\gamma,\delta)$ where $|\delta|\le\pi$ and $\cos\delta-\gamma>0$, [6] for $G_{1,0}(0,\delta)$ where $|\delta|<\frac{\pi}{2}$ and [7] for $G_{1,0}(0,0)$.

There is a relationship of the class P in the form of $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ with the extremal information of each selected classes. Writing

$$\frac{e^{i\delta}(\alpha f'(z)+\beta z f''(z))-\gamma-i\alpha\sin\delta}{(a\cos\delta-\gamma)}=p(z)\quad(z\in D)$$

clearly $f\in G_{\alpha,\beta}(\gamma,\delta)$ if $p\in P$, the class of functions with positive real parts.

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We make use the result of representation theorem

$$f(z)=\int_{|x|=1}\left[-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)z-\frac{2e^{-i\delta}A}{(n\beta+\alpha)}\bar{x}\log(1-xz)\right]d\mu(x)$$

where $A=(\alpha\cos\delta-\gamma)$ given by [8] in order to determine the distortion theorem and argument of $f'(z)$ for this class of function.

II. Extremal Properties

We begin by finding the radius and centre of $G_{\alpha,\beta}(\gamma,\delta)$ that will be used for later results.

Theorem 3.1 *Let $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$. Then $f'(z)$ maps $|z|\le r$ into disc D_r with centre and radius*

$$-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)+\frac{2e^{-i\delta}AM}{1-r^2}\ \ and\ \ \frac{2AMr}{1-r^2}$$

where $A=\alpha\cos\delta-\gamma$, $M=\dfrac{1}{n\beta+\alpha}$ respectively.

Proof. If a and b are complex numbers with $|b|<1$ and if $0<r<1$, the range of the function $(1+arw)/(1+brw)$ where $|w|\le 1$ is a disc with center and radius respectively.

$$\frac{1-a\bar{b}r^2}{1-|b|^2r^2}\ ,\ \ \frac{|a-b|r}{1-|b|^2r^2}$$

By taking $a=\bar{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)xr$ and $b=xr$ where $|x|=1$, we see that maps $|z|\le r$ onto D_r. By convexity, any linear combination of functions of this form also maps D onto D_r. Since for some probability measure μ, we have

$$B\left\{\frac{1+\bar{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)xz}{1-xz}\right\}$$

$$f'(z)=\int_{|x|=1} B\left\{\frac{1+\overline{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)xz}{(1-xz)}\right\}d\mu(x)$$

where $B=\dfrac{\frac{2\gamma}{\alpha}n\beta e^{-i\delta}-n\beta e^{-2i\delta}+\alpha}{n\beta+\alpha}$, so, the result follows.

Corollary 3.1 *If* $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$ *then*

$$f'(z)\prec B\left\{\frac{1+\overline{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)z}{1-z}\right\},\ z\in D$$

The simple geometry of a circle enables us to deduce from Theorem 3.2, upper and lower bounds for $\operatorname{Re} f'(z)$, $\operatorname{Im} f'(z)$, $|f'(z)|$ and $\arg f'(z)$ when $f(z)=G_{\alpha,\beta}(\gamma,\delta)$.

Theorem 3.2 *If* $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$, *then*

$$\frac{1+B+r^2(2AR-1)-2AMr}{1-r^2}\le \operatorname{Re} f'(z)\le\frac{1+B+r^2(2AR-1)+2AMr}{1-r^2}$$

where $B=\dfrac{2A(A+\gamma)(M\alpha-1)}{\alpha^2}$ *and* $R=\dfrac{(A+\gamma)}{\alpha^2}$ *, and*

$$\frac{-2A\left(T\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{rT}{\alpha}\right)\right)}{1-r^2}\le \operatorname{Im} f'(z)\le\frac{2A\left(T\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{rT}{\alpha}\right)\right)r}{1-r^2}$$

where $T=\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}$ *and all bounds are sharp for any extreme point* $f(z)$ *of* $G_{\alpha,\beta}(\gamma,\delta)$.

Proof. By Theorem 3.1, we can write

$$\left|f'(z)-\left\{-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)+\frac{2e^{-i\delta}AM}{1-r^2}\right\}\right|\le\frac{2AMr}{1-r^2} \qquad (1)$$

So that

$$\frac{2AMr}{1-r^2}\le \operatorname{Re}\left\{f'(z)+e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)-\frac{2e^{-i\delta}AM}{1-r^2}\right\}\le\frac{2AMr}{1-r^2}$$

that gives ;

$$-\frac{2AMr}{1-r^2}\le \operatorname{Re} f'(z)-\left\{\frac{2AM\cos\delta-(1-r^2)\left(2\cos^2\delta-1-\frac{2\gamma}{\alpha}\cos\delta\right)}{1-r^2}\right\}$$

$$\le\frac{2AMr}{1-r^2}$$

and

$$-\frac{2AMr}{1-r^2}\le \operatorname{Im}\left\{f'(z)+e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)-\frac{2e^{-i\delta}AM}{1-r^2}\right\}\le\frac{2AMr}{1-r^2}$$

that gives

$$\frac{-2A\left(\sin\delta\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{r\sin\delta}{\alpha}\right)\right)}{1-r^2}\le \operatorname{Im} f'(z)\le$$

$$\frac{2A\left(\sin\delta\left(\frac{1}{\alpha}-M\right)+r\left(M-\frac{r\sin\delta}{\alpha}\right)\right)}{1-r^2}$$

Since $\cos\delta=\dfrac{A+\gamma}{\alpha}$ and $\sin\delta=\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}$, we can write the inequalities in this form

$$-\frac{2AMr}{1-r^2}\le \operatorname{Re} f'(z)-\left\{\frac{1+2A(A+\gamma)\left(\frac{M\alpha-1}{\alpha^2}\right)+r^2\left(\frac{2A(A+\gamma)}{\alpha^2}-1\right)}{1-r^2}\right\}\le\frac{2AMr}{1-r^2}$$

and

$$\frac{-2A\left(\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{r\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}}{\alpha}\right)\right)}{1-r^2}\le \operatorname{Im} f'(z)\le$$

$$\frac{2A\left(\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}\left(\frac{1}{\alpha}-M\right)+r\left(M-\frac{r\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}}{\alpha}\right)\right)}{1-r^2}$$

Letting $B=\dfrac{2A(A+\gamma)(M\alpha-1)}{\alpha^2}$, $R=\dfrac{(A+\gamma)}{\alpha^2}$ and $T=\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}$, we obtain the above inequalities as required. It is clear that each inequality is sharp for some z on

$|z| = r$ when f is an extreme point.

Our next result is to obtain a distortion theorem for $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$.

Theorem 3.3 *If* $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$, *then* $|f'(z)| \le |\Gamma(r)| + \dfrac{2AMr}{1-r^2}$ *where*

$$C(r) = \left(1 - \frac{4\gamma A}{\alpha^2} + \frac{4AM}{\alpha(1-r^2)}\left\{\frac{AM\alpha}{(1-r^2)} + \gamma - A\right\}\right)^{\frac{1}{2}} \tag{2}$$

Proof. Let

$$\Gamma(r) = -e^{-i\delta}\left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right) + \frac{2e^{-i\delta}AM}{1-r^2} \tag{3}$$

and from (1), we have

$$|f'(z) - \Gamma(r)| \le \frac{2AMr}{1-r^2}$$

so that

$$|f'(z)| \le |\Gamma(r)| + \frac{2AMr}{1-r^2} = C(r) + \frac{2AMr}{1-r^2}$$

as required.

If $\gamma \ge 0$, then f' is non-zero throughout D and has continuous argument whereas if $\gamma < 0$ and f_0 is any extreme function of $G_{\alpha,\beta}(\gamma,\delta)$, then at some points in D, f_0' has a zero, thus, there is no argument. We next obtain bounds for $\arg f'(z)$ when $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$ with restricted value of $|z|$ for the case of $\gamma < 0$. Furthermore, we will use the following property for argument: for given δ in $[-\pi,\pi]$ and as x varies in some interval $[0,c]$, so that $e^{i\delta} + x \ne 0$, $\phi_\delta(x)$ is continuous argument of $e^{i\delta} + x \ne 0$ for which $\phi_\delta(0) = \delta$. We have

$$\phi_\delta(x) = \begin{cases} \tan^{-1}\left(\dfrac{\sin\delta}{\cos\delta + x}\right) & x + \cos\delta > 0 \\ \pi + \tan^{-1}\left(\dfrac{\sin\delta}{\cos\delta + x}\right) & x + \cos\delta < 0 \\ \dfrac{\pi}{2} & x + \cos\delta = 0 \end{cases}$$

when $0 < \delta < \pi$ and $-\pi < \delta < 0$ for $\delta = 0, \pm\pi$.

Theorem 3.4 *Let* $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$ *and put* $x(r) = 2\left(\dfrac{AM}{1-r^2} - \dfrac{A}{\alpha}\right)$ $(0 \le r \le 1)$. *Let*

$$r_0 = \begin{cases} 1 & \gamma \ge 0 \\ \sqrt{1 - \dfrac{4\alpha AM(\alpha AM - A + \gamma)}{(4A\gamma - \alpha^2)}} & \gamma < 0 \end{cases}$$

Then, for $0 < |z| = r < r_0$, *and for a suitable determination of argument*

$$|\arg f'(z) + \delta - \phi_\delta(x(r))| \le \sin^{-1}\frac{2AMr}{(1-r^2)C(r)}$$

where $\phi_\delta(x)$ *is defined on* $[0, x(r_0))$ *as above and* $C(r)$ *is given by (2). The result is sharp.*

Proof. To make sure that $f'(z) \ne 0$, we restrict the values of $|z| = r$ by the condition

$$\left|\frac{2AM}{1-r^2} - \left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right)\right| > \frac{2AMr}{1-r^2}$$

Squaring both sides, we have

$$\frac{4A^2M^2}{(1-r^2)} + 1 + \frac{4AM}{(1-r^2)}\left(\frac{2\gamma}{\alpha} - \cos\delta\right) - \frac{4\gamma\left(\cos\delta - \dfrac{\gamma}{\alpha}\right)}{\alpha} > 0$$

and since $A = \alpha\cos\delta - \gamma$, hence

$$1 + \frac{4AM}{(1-r^2)}\left(AM - \frac{(A-\gamma)}{\alpha}\right) - \frac{4A\gamma}{\alpha^2} > 0$$

The inequality holds for all r in $[0,1)$ if $\gamma \ge 0$ and for $0 \le r > \sqrt{1 - \dfrac{4\alpha AM(\alpha AM - A + \gamma)}{(4A\gamma - \alpha^2)}}$ if $\gamma < 0$. This establishes the restricted on $|z|$ in the statement of the theorem.

From (1) then with $\Gamma(r)$ given by (3) and $C(r) = |\Gamma(r)|$, we have

$$C(r) = \left(1 + \frac{4A^2M}{1-r^2}\left(\frac{M}{1-r^2} - \frac{1}{\alpha}\right) + \frac{4A(\alpha\cos\delta - A)}{\alpha^2}\left(\frac{M\alpha}{1-r^2} - 1\right)\right)^{\frac{1}{2}}$$

and deduced to $|\arg f'(z) - \arg\Gamma(r)| \le \sin^{-1}\dfrac{2AMr}{(1-r^2)C(r)}$

also

$$\arg\Gamma(r) = \arg\left(e^{-i\delta}\left(\frac{2AM}{1-r^2} - e^{-i\delta} + \frac{2\gamma}{\alpha}\right)\right)$$

$$= -\delta + \arg\left(e^{-i\delta} + 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)\right).$$

Put $x(r)=2\left(\frac{AM}{1-r^2}-\frac{A}{\alpha}\right)$, then $\arg\Gamma(r)=-\delta+\phi_\delta(x(r))$.

We obtain another theorem that replaced $\arg f'(z)$ with restricted range of $|z|$ as $\arg(f'(z)+k)$ for some real number k that satisfied $f'(z)+k\neq 0$ for $z\in D$ and $f\in G_{\alpha,\beta}(\gamma,\delta)$. By taking $|\delta|\neq\pi/2$ as any choice of k with $k\cos\delta+\gamma>0$ will ensure that above conditions are fulfilled and this is important for the following result to be valid. In the following theorem, for a given $\delta\in[-\pi,\pi]$ and as x varies in same interval $[0,c)$, so that $(k+1)e^{i\delta}+x\neq 0$, $\psi_\delta(\delta)$ is the continuous argument of $(k+1)e^{i\delta}+x$ for which $\psi_\delta(0)$ is principal.

Theorem 3.5 *For* $|\delta|\neq\pi/2$, $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$ *and put* $x(r)=2\left(\frac{AM}{1-r^2}-\frac{A}{\alpha}\right)$ $(0\le r\le 1)$. *Let* $k\alpha\cos\delta+\gamma>0$ *where k is a real number. Then, for* $\psi_\delta(x)$ *defined on* $[0,\infty)$,

$$\left|\arg(f'(z)+k)+\delta-\psi_\delta(x(r))\right|\le\sin^{-1}\frac{2AMr}{(1-r^2)C_1(r)}$$

where

$$C_1(r)=\left[4AT\left(k\cos\delta+\frac{A}{\alpha}(T+1)+\frac{\gamma}{\alpha}\right)+(k+1)^2\right],\ T=\frac{M}{1-r^2}-\frac{1}{\alpha}$$

Proof. Let $|\delta|\neq\pi/2$ and k satisfied $k\alpha\cos\delta+\gamma>0$. Using (1), we have

$$\left|(f'(z)+k)-(\Gamma(r)+k)\right|\le\frac{2AMr^2}{1-r^2}$$

where

$$\Gamma(r)=-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)+\frac{2e^{-i\delta}AM}{1-r^2}=1+\frac{2Ae^{-i\delta}\left(M-\frac{1}{\alpha}+\frac{r^2}{\alpha}\right)}{1-r^2}$$

Hence

$$\left|\arg(f'(z)+k)-\arg(\Gamma(r)+k)\right|\le\sin^{-1}\frac{2AMr}{(1-r^2)C_1(r)}\qquad(4)$$

where

$$C_1(r)=|\Gamma(r)+k|=\left[\begin{array}{l}(k+1)^2+4A^2\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)^2+k\cos\delta+\cos\delta\\ +\frac{4A}{1-r^2}\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)\left(A\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)\right)\end{array}\right]^{\frac{1}{2}}$$

Let $T=\frac{M}{1-r^2}-\frac{1}{\alpha}$, we have

$$C_1(r)=\left[4AT\left(k\cos\delta+\frac{A}{\alpha}(T+1)+\frac{\gamma}{\alpha}\right)+(k+1)^2\right]^{\frac{1}{2}}$$

Now

$$\begin{aligned}\arg(\Gamma(r)+k)&=\arg\left(e^{-i\delta}\left(\frac{2AM}{1-r^2}-e^{-i\delta}+\frac{2\gamma}{\alpha}+ke^{i\delta}\right)\right)\\&=-\delta+\arg\left((k+1)e^{i\delta}+2A\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)\right)\\&=-\delta+\psi_\delta(x(r))\end{aligned}$$

and with (4) this completes the proof.

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