The Spanning Laceability of $k$-ary $n$-cubes when $k$ is Even

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Abstract—$Q^k_n$ has been shown as an alternative to the hypercube family. For any even integer $k \geq 4$ and any integer $n \geq 2$, $Q^k_n$ is a bipartite graph. In this paper, we will prove that any pair of vertices, $w$ and $b$, from different partite sets of $Q^k_n$, there exist $2n$ internally disjoint paths between $w$ and $b$, denoted by $\{P_i \mid 0 \leq i \leq 2n-1\}$, such that $\bigcup_{i=0}^{2n-1} P_i$ covers all vertices of $Q^k_n$. The result is optimal since each vertex of $Q^k_n$ has exactly 2n neighbors.

Keywords—container, Hamiltonian, $k$-ary $n$-cube, $m^*$-connected.

I. INTRODUCTION

The $k$-ary $n$-cube, denoted by $Q^k_n$, has been proposed as an alternative to the hypercube since it shares many nice properties of $Q_n$ such as regular degrees, vertex symmetry, edge symmetry, recursive structure, etc. The underlying topology of many machines is based on $k$-ary $n$-cubes, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1], [4], [11], [17]. Many researchers have been working on $k$-ary $n$-cubes. For example, Stewart and Xiang [20] proved that the $k$-ary $n$-cube is edge-bipancyclic and bipancyclic for $k \geq 3$ and $n \geq 2$ and $k$ being even. Namely, any edge of a $k$-ary $n$-cube $Q^k_n$ lies on a cycle of any even length $r$ for $4 \leq r \leq |Q^k_n|$, where $|Q^k_n|$ is the total number of vertices of $Q^k_n$. Besides, given two vertices $u$ and $v$ of $Q^k_n$, there exists a path of any even length $r$ between $u$ and $v$ for $d(u, v) \leq r \leq |Q^k_n|$, where $d(u, v)$ is the distance between $u$ and $v$. Other studies about fault tolerance on $k$-ary $n$-cubes can be found in [8], [23]. Recently, there are many studies about the spanning connectivity for interconnection networks and graphs [9]. A graph $H = (B \cup W, E)$ is bipartite if $V(H)$ is the union of two disjoint sets $B$ and $W$ such that every edge joins $B$ with $W$. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected except $K_2$. Note that any (nontrivial) bipartite graph except $K_2$ cannot be hamiltonian connected, whereas a bipartite graph is hamiltonian laceable if there exists a hamiltonian path between any two vertices $u$, $v$ with $u \in B$ and $v \in W$ [22]. A graph $H = (B \cup W, E)$ is a balanced bipartite graph if $|V(B)| = |V(W)|$. Throughout this thesis, we only work on $Q^k_n$ with $k \geq 4$ an even integer and $n \geq 2$, which are balanced bipartite graphs. A bipartite graph $G = (B \cup W, E)$ is $m^*$-laceable if given a white vertex $w \in W$ and a black vertex $b \in B$, there exist(s) $m$ internal disjoint paths between $w$ and $b$, denoted by $P_i$ for $0 \leq i \leq m-1$, such that $\bigcup_{i=0}^{m-1} P_i$ covers $V$. The spanning laceability of a graph $H$, $L^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for every $m$ with $1 \leq m \leq k$. A higher spanning connectivity/laceability of the interconnection network implies a more efficient communication between processors. About the spanning connectivity and the spanning laceability, readers can refer to [6], [7], [12]–[15].

In this paper, we want to show the spanning laceability of $k$-ary $n$-cubes for any even integer $k \geq 4$. More precisely, we show that given a white vertex $w$ and a black vertex $b$ of a $k$-ary $n$-cube $Q^k_n$, there exists(s) $m$ internally disjoint path(s) between $w$ and $b$ whose union covers all vertices of $Q^k_n$ for $1 \leq m \leq 2n$. The result is optimal since any vertex in $Q^k_n$ has exactly $2n$ neighbors. This paper is organized as follows. In Section 2, we introduce the graph terminologies and symbols that will be used in the paper and the definition of $Q^k_n$. In Section 3, we show our main results.

II. PRELIMINARIES

Throughout this paper, we follow [3] for the graph definitions and notations. The sets of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. If $u, v$ are vertices of a graph $G$ such that there is an edge $e = (u, v) \in E(G)$ between $u$ and $v$, then we say that the vertices $u$ and $v$ are adjacent in $G$. The degree of any vertex $x$ is the number of distinct vertices adjacent to $x$. A path $P$ between two vertices $v_0$ and $v_k$ is represented by $P = \langle v_0, v_1, ..., v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We use $P^{-1}$ to denote the path $\langle v_k, v_{k-1}, ..., v_1, v_0 \rangle$. We also write the path $P = \langle v_0, v_1, ..., v_k \rangle$ as $\langle v_0, v_1, ..., v_i, v_j, v_{j+1}, ..., v_k \rangle$, where $Q$ denotes the path $\langle v_i, v_{i+1}, ..., v_j \rangle$. A hamiltonian path between $u$ and $v$, where $u$ and $v$ are two distinct vertices of $G$, is a path joining $u$ to $v$ that visits every vertex of $G$ exactly once. A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of $G$ is a cycle that traverses every vertex of $G$ exactly once. A hamiltonian graph is a graph with a hamiltonian cycle. A graph $G$ is connected if there is a path between any two distinct vertices in $G$ and is hamiltonian connected if there is a hamiltonian path between any two distinct vertices in $G$ [18]. A graph $H = (W \cup B, E)$ is bipartite if $V(H) = W \cup B$ and $E(H)$ is a subset of $\{\{w, b\} | w \in W, b \in B\}$. A bipartite graph $H$ is hamiltonian laceable if there is a hamiltonian path between any two distinct vertices from different partite sets in $H$.

A graph $G$ is $k$-connected if there exists $V' \subseteq V(G)$ with $|V'| = k$ such that $G - V'$ is disconnected and $G - V''$ is
connected for any $V'' \subseteq V(G)$ with $|V''| < k$. It follows from Menger’s Theorem [16] that for every $k$-connected graph $G$, there exist $k$ internally vertex-disjoint paths between any pair of distinct vertices of $G$. A $k$-container $C(u,v)$ in a graph $G$ is a set of $k$ internally vertex-disjoint paths between two distinct vertices $u$ and $v$. We say that a graph $G$ has a spanning $k$-container between $u$ and $v$, denoted by $C(u,v)$, if $C(u,v)$ is a $k$-container that covers all vertices of $G$. A spanning $k$-container is also abbreviated as a $k^*$-container for simplicity. A graph $G$ is $k^*$-connected if there is a $k^*$-container between any pair of vertices of $G$. Obviously, a graph $G$ is hamiltonian connected if and only if $G$ is $1^*$-connected, and $G$ is hamiltonian if and only if $G$ is $2^*$-connected. Lin et al. [13] defined the concept of spanning connectivity. The spanning connectivity of a graph $G$, $\kappa'(G)$, is the largest integer $k$ such that $G$ is $w^*$-connected for all $1 \leq w \leq k$. Similarly, a bipartite graph $H$ is $k^*$-laceable if there is a $k^*$-container between any pair of two vertices from different partite sets of $H$. Also, a bipartite graph $H$ is hamiltonian laceable if and only if $H$ is $1^*$-laceable, and $H$ is hamiltonian if and only if $H$ is $2^*$-laceable. So, the spanning laceability of a bipartite graph $H$, $\kappa'(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for all $1 \leq m \leq k$.

The $k$-ary $n$-cube, $Q_n^k$, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass $Q_n^k$ is the well-studied hypercube family. The subclass $Q_n^k$ with $k \geq 3$ is defined as the cycle of length $k$. The $k$-ary $n$-cube, $Q_n^k$, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u(0), u(1), \ldots, u(n-1))$, where $0 \leq u(i) \leq k-1$. Two vertices $u$ and $v$ are adjacent if and only if $|u(i) - v(i)| = 1$ or $k-1$ for some $i$ and $u(j) = v(j)$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is shown that $Q_n^k$ is bipartite if $k$ is even [10]. Here we mention some properties of $Q_n^k$ that will be used in this paper.

$Q_n^k$ is vertex symmetric (and edge symmetric) [10]. It means that given any two distinct vertices $u$ and $v$ of $Q_n^k$, there is an automorphism of $Q_n^k$ mapping $u$ to $v$. Note that each vertex of $Q_n^k$ is represented by a $n$-bit tuple. We will call the $d$th bit the $d$th dimension. We can partition $Q_n^k$ over dimension $d$ by fixing the $d$th element of any vertex tuple at some value $a$ for every $a \in \{0,1,\ldots,k-1\}$. This results in $k$ copies of $Q_{n-1}^k$, denoted by $Q_{n-1}^{k,0}$, $Q_{n-1}^{k,1}$, ..., $Q_{n-1}^{k,k-1}$, with corresponding vertices in $Q_{n-1}^{k-1,0}, Q_{n-1}^{k-1,1}, \ldots, Q_{n-1}^{k-1,k-1}$ joined in a cycle of length $k$ (in dimension $d$) [19].

In this article, we always partition $Q_n^k$ over the $0$-th dimension by letting $V(Q_n^{k,0}) = \{(0,1,1,1, \ldots, 1), (0,0,1,1, \ldots, 1), \ldots, (0,0, \ldots, 0,1,1, \ldots, 1)\}$ for all $1 \leq j \leq k-1$. Given a vertex $x = (x(0), x(1), \ldots, x(n-1)) \in V(Q_n^k)$, the symbol $x^j = ((j), x(1), x(2), \ldots, x(n-1))$, where $0 \leq j \leq k-1$, is defined to be the vertex corresponding to $x$ in $Q_n^{k,j}$ for simplicity. So, if $P = (x_0, x_1, \ldots, x_{n-1})$, $P^j$ is represented by $(x_0^j, x_1^j, \ldots, x_{n-1}^j)$. Throughout this paper, let $n \geq 2$ be an integer and $k \geq 4$ an even integer.

Theorem 1. [10] For any even integer $k \geq 4$, $Q_n^k$ is hamiltonian laceable for $n \geq 2$. In other words, $Q_n^k$ is $1^*$-laceable.

Theorem 2. [5] The graph $Q_n^k$ is hamiltonian. In other words, $Q_n^k$ is $2^*$-laceable.

III. MAIN RESULTS

Lemma 1. Given $Q_n^k$ and its $k$ subcubes, $Q_{n-1}^{k,j}$, where $0 \leq i \leq \ell \leq k-1$. Let $j$ and $j'$ be two integers satisfying $0 \leq j \leq j' \leq k-1$, $w \in V(Q_{n-1}^{k,j})$ an arbitrary white vertex, and $b \in V(Q_{n-1}^{k,j'})$ an arbitrary black vertex. Then there exists a path between $w$ and $b$ that visits each vertex in $Q_{n-1}^{k,j} \cup Q_{n-1}^{k,j'} \cup Q_{n-1}^{k,j'+2} \cup \cdots$, $Q_{n-1}^{k,j'}$ exactly once.

Proof: There are three cases.

Case 1. $j = j'$. W.L.O.G., let $j = j' = 0$. By Theorem 1, $Q_{n-1}^{k}$ is hamiltonian laceable. Thus, there is a hamiltonian path between $w$ and $b$ that visits each vertex of $Q_{n-1}^{2,0}$ exactly once.

Case 2. $j = j' = 1$. W.L.O.G., we can let $j = 0$ and $j' = 1$. Let $w$ be a white vertex in $Q_{n-1}^{0,0}$ and $b$ a black vertex in $Q_{n-1}^{0,1}$. We can find a pair of adjacent vertices $x^0$ and $x^1$ where $x^0$ is a black vertex of $Q_{n-1}^{0,0}$ and $x^1$ is a white vertex of $Q_{n-1}^{0,1}$. By Theorem 1, there exists a hamiltonian path $P_0$ of $Q_{n-1}^{0,1}$ between $w$ and $x^0$, and a hamiltonian path $P_1$ of $Q_{n-1}^{1,1}$ between $x^1$ and $b$. Let $P = (w, P_0, x^0, x^1, P_1, b)$. Hence $P$ is the path between $w$ and $b$ that visits every vertex of $Q_{n-1}^{0,0}$ and $Q_{n-1}^{0,1}$ exactly once.

Case 3. $j = j' \geq 2$. Let $w$ be a white vertex in $Q_{n-1}^{k,j}$ and $b$ be a black vertex in $Q_{n-1}^{k,j'}$. There are $j = j' + 1$ k-ary $n-1$-cubes, $Q_{n-1}^{k,j'+1}, Q_{n-1}^{k,j'+2}, \ldots, Q_{n-1}^{k,j'-1}$ and $Q_{n-1}^{k,j'}$. There are $j' - j$ pairs of adjacent vertices $x^r \in Q_{n-1}^{r,j}$ and $y^{r+1} \in Q_{n-1}^{r+1,j'}$ where $x^r$ is a black vertex and $y^{r+1}$ is a white vertex for $j \leq r < j' - 1$. By Theorem 1, there is a hamiltonian path $R_r$, of $Q_{n-1}^{r,k}$ joining $y^r$ to $x^{r+1}$, where $j + 1 \leq r < j' - 1$. Again, with Theorem 1, there exists a hamiltonian path $T$ of $Q_{n-1}^{k-1}$ joining $w$ to $x^1$, and a hamiltonian path $U$ of $Q_{n-1}^{k-1}$ joining $y^0$ to $b$. Let $P = (w, T, x^1, y^1, R_{j'+1}, x^{j'+2}, y^{j'+2}, R_{j'+2}, x^{j'+3}, \ldots, y^{j'+1}, R_{j'+1}, x^{j'}, y^j, U, b)$. Therefore, $P$ is a path covering all the vertices of $Q_{n-1}^{k,j}, Q_{n-1}^{k,j'+1}, Q_{n-1}^{k,j'+2}, \ldots, Q_{n-1}^{k,j'}$ for $0 \leq j \leq j' \leq k-1$ between $w$ and $b$. Please see Figure 1 for an illustration.

Fig. 1. The illustration for Case 3 of Lemma 1.

Lemma 2. Given $Q_n^k$ and its $k$ subcubes $Q_{n-1}^k$ for $0 \leq i \leq k-1$. Let $w$ be a white vertex, $b$ a black vertex in $Q_{n-1}^{k,j}$, and $j$ an integer with $0 \leq i \leq j \leq k-1$. There exists a path between $w$ and $b$ that covers all the vertices of $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \ldots, Q_{n-1}^{k,k-1}$.

Proof: We consider the following two cases.
Lemma 4. j = i. There is only one k-ary \((n - 1)\)-cube \(Q^{k,i}_{n-1}\). By Theorem 1, the lemma holds in this case.

Case 2. \(j \neq i\). There are \(j = i + 1 k\)-ary \((n - 1)\)-cubes. According to Theorem 1, there is a hamiltonian path \(P_j\) that covers all the vertices of \(Q^{k,i}_{n-1}\) between \(w\) and \(b\) of the form \((w, z^r, y^s, T, b)\), where \(\{x^i, y^j\}\) is an edge of \(Q^{k,i}_{n-1}\) with \(\{x^i, y^j\} \in \{w, b\}\). Notice that by Theorem 1, \(Q^{k,i}_{n-1}\) is hamiltonian laceable and hence there exists a hamiltonian path \(P_j\) between \(x^i\) and \(y^j\) of the form \((x^r, S, z^t, w^r, T, y^s)\) for \(i + 1 \leq r \leq j\). Let the required path between \(w\) and \(b\) be \(R\), we have the following two subcases.

Case 2.1. If \(j = i + 1\) is even, then

\[ R = (w, x_i^1, x_i^2, x_i^3, z_i^4, z_i^5, z_i^6, z_i^7, z_i^8, z_i^9, z_i^{10}, z_i^{11}, z_i^{12}, z_i^{13}, z_i^{14}, S, z_i^{15}, z_i^{16}, z_i^{17}, z_i^{18}, z_i^{19}, z_i^{20}, z_i^{21}, z_i^{22}, z_i^{23}, z_i^{24}, z_i^{25}, z_i^{26}, z_i^{27}, z_i^{28}, z_i^{29}, z_i^{30}, T, b) \]

Please see Figure 2 for an illustration.

Case 2.2. If \(j = i + 1\) is odd, then

\[ R = (w, x_i^1, x_i^2, x_i^3, z_i^4, z_i^5, z_i^6, z_i^7, z_i^8, z_i^9, z_i^{10}, z_i^{11}, z_i^{12}, z_i^{13}, z_i^{14}, S, z_i^{15}, z_i^{16}, z_i^{17}, z_i^{18}, z_i^{19}, z_i^{20}, z_i^{21}, z_i^{22}, z_i^{23}, z_i^{24}, z_i^{25}, z_i^{26}, z_i^{27}, z_i^{28}, z_i^{29}, z_i^{30}, T, b) \]

Please see Figure 2 for an illustration.

Lemma 3. The graph \(Q_2^3\) is 3-\(\ell\)-laceable and 4-\(\ell\)-laceable.

Proof: The proof is by brute force. Reader can refer to Appendix A.

Lemma 4. The graph \(Q_2^4\) is 3-\(\ell\)-laceable and 4-\(\ell\)-laceable.

Proof: By brute force, we constructed all spanning containers. Please see Appendix B.

Lemma 5. The graph \(Q_2^k\) is 3-\(\ell\)-laceable and 4-\(\ell\)-laceable for any even integer \(k \geq 6\).

Proof: With Lemma 4, we have shown that \(Q_2^k\) is 3-\(\ell\)-laceable and 4-\(\ell\)-laceable. Now we will present a recursive algorithm that uses a 3-\(\ell\)-container (resp. 4-\(\ell\)-container) of \(Q_2^3\) to construct a 3-\(\ell\)-container (resp. 4-\(\ell\)-container) of \(Q_2^k\). Let \(R\) be a subset of \(V(Q_2^3) \cup E(Q_2^3)\). We define a function, \(f\), which maps \(R\) from \(Q_2^3\) into \(Q_2^k\) in the following way:

1. If \((i, j) \in R \cap V(Q_2^3)\), where \(0 \leq i, j \leq k - 1\), then
   \[ f((i, j)) = \begin{cases} 
   (i, j) & \text{if } 0 \leq i, j \leq k - 3, \\
   (i + 2, j) & \text{if } i = k - 1, 0 \leq j \leq k - 2, \\
   (i, j + 2) & \text{if } j = k - 1, 0 \leq i \leq k - 2, \\
   (i + 2, j + 2) & \text{if } i = k - 1, j = k - 1. 
   \end{cases} \]

2. If \((i, j, (i', j')) \in R \cap E(Q_2^3)\), where \(i \leq i', j \leq j'\), then
   \[ f(((i, j), (i', j'))) = \begin{cases} 
   ((i, j), (i', j')) & \text{if } 0 \leq i, j \leq k - 3, \\
   ((i + 2, j), (i', j' + 2)) & \text{if } i = k - 1, 0 \leq j \leq k - 3, \\
   ((i, j + 2), (i', j' + 2)) & \text{if } j = k - 1, 0 \leq i \leq k - 3, \\
   ((i + 2, j), (i' + 2, j' + 2)) & \text{if } i = k - 1, j = k - 1. 
   \end{cases} \]

Let \(w\) be a white vertex and \(b\) be a black vertex of \(Q_2^4\). We say that a 3-\(\ell\)-container (resp. 4-\(\ell\)-container) \(C(w, v)\) of \(Q_2^4\) is regular if \(C(w, v)\) contains some edges in \(\{(\alpha, k - 2), (\alpha, k - 1)\} \mid 0 \leq \alpha \leq k - 1\) and \(\{(k - 2, \beta), (k - 1, \beta)\} \mid 0 \leq \beta \leq k - 1\). For example, all 3-\(\ell\)-containers and 4-\(\ell\)-containers of \(Q_2^4\) constructed in Lemma 4 are regular. Let \(C(w, b)\) be a regular 3-\(\ell\)-container (resp. 4-\(\ell\)-container) of \(Q_2^4\) with the endvertex set \(P = \{w = (0, 0), b = (x, y)\}\). We construct a regular 3-\(\ell\)-container (resp. 4-\(\ell\)-container) of \(Q_2^k\) with the endvertex set \(f(P)\) using the following algorithm. Please see Figure 4 for an illustration.

Step 1. In \(Q_2^k\), let \(\{v_0, v_1, \ldots, v_{k-1}\}\) and \(\{h_0, h_1, \ldots, h_{k-1}\}\) be finite sequences of indices satisfying the following requirements:

1. \(0 \leq v_0 < v_1 < \ldots < v_{k-1} \leq k - 1\) and \(k - 1 \geq h_0 > h_1 > \ldots > h_{k-1} \geq 0\).
Case 1.2. Let $C(w, b)$ be the image in $Q^{k+2}$ of $C(w, b) - \{(v_i, k-2), (v_i, k-1)\}$ (0 ≤ i ≤ k-1) under the function $f$. Please see Figure 5 for an illustration.

Step 3. For any two positive integers $r$ and $d$, we use $\lfloor r \mod d \rfloor$ to denote $r$ (mod $d$). In $Q^{k+2}$, define the following path patterns, where $r_1, r_2$ are integers:

$I_0(r_1, r_2) = (\langle r_1, 0 \rangle, \langle r_1 + 1 \rangle, \ldots, \langle r_2, 0 \rangle)$

$I_0^*(r_1, r_2) = (\langle r_2, 0 \rangle, \langle r_2 + 1 \rangle, \ldots, \langle r_1, 0 \rangle)$

$H_\beta(r_1, r_2) = (\langle \beta, r_1 \rangle, \langle \beta, r_1 + 1 \rangle, \ldots, \langle \beta, r_2 \rangle)$

$H_\beta^*(r_1, r_2) = (\langle \beta, r_2 \rangle, \langle \beta, r_2 + 1 \rangle, \ldots, \langle \beta, r_1 \rangle)$.

Let $v_i = v_i + 2$ if $v_i = k - 1$ and $w_j = v_i$ if 0 ≤ $v_i$ ≤ k - 2, and $w_j = h_j + 2$ if $h_j = k - 1$ and $w_j = h_j$ if $h_j < k - 2$.

Case 1. $v_0 = k - 1$.

Let $P_0 = \{(k + 1, k - 2), (k + 1, k - 1), (0, k - 1), (k - 1, 0), (0, k - 2), (k - 2, k), (k - 2, k - 1), (k + 1, k), (k + 1, k + 1)\}$.

Case 1.1. s = 1.

Let $P_0 = \{(k - 2, h_0), (k - 1, h_0), (k - 1, h_0), (h_0, k - 1), (h_0, k - 2), (h_0, k - k), (h_0, k), (h_0, k + 1)\}$. Then $C(w, b) \cup P_0 \cup P_0$ is the 3rd-container (or 4th-container) of $Q^{k+2}$.

Case 2. $v_0 + 1 ≤ k - 2$ and $((k - 2, k - 1), (k - 1, k - 1)) \notin E(C(w, b))$ in $Q^{k+2}$.

Case 2.1. $t = 1$.

Let $P_0 = \{(v_0, k - 2), (v_0, k - 1), (k - 1, v_0), (k - 2, k - 1), (k - 2, k), (k - 2, k - 1), (k - 1, v_0), (k - 1, v_0 + 1)\}$. Then $C(w, b) \cup P_0 \cup P_0$ is the 3rd-container (or 4th-container) of $Q^{k+2}$.

Case 2.1.1. s = 1.

Let $P_0 = \{(k - 2, h_0), (k - 1, h_0), (h_0, k - 1), (h_0, k - 2), (h_0, k - k), (h_0, k), (h_0, k + 1)\}$. Then $C(w, b) \cup P_0 \cup P_0$ is the 3rd-container (or 4th-container) of $Q^{k+2}$.

Case 2.1.2. s = 2.

Let $P_0 = \{(k - 2, h_0), (k - 1, h_0), (h_0, k - 1), (h_0, k - 2), (h_0, k - k), (h_0, k), (h_0, k + 1)\}$. Then $C(w, b) \cup P_0 \cup P_0$ is the 3rd-container (or 4th-container) of $Q^{k+2}$.

Case 2.2. $v_0 + 1 ≤ k - 2$ and $((k - 2, k), (k - 1, k - 1)) \notin E(C(w, b))$ in $Q^{k+2}$.

Case 2.2.1. $t = 1$.

Let $P_0 = \{(v_0, k - 2), (v_0, k - 1), (k - 1, v_0), (k - 2, k - 1), (k - 2, k), (k - 2, k - 1), (k - 1, v_0), (k - 1, v_0 + 1)\}$. Then $C(w, b) \cup P_0 \cup P_0$ is the 3rd-container (or 4th-container) of $Q^{k+2}$.

Case 2.2.1.1. s = 1.

Let $P_0 = \{(k - 2, h_0), (k - 1, h_0), (h_0, k - 1), (h_0, k - 2), (h_0, k - k), (h_0, k), (h_0, k + 1)\}$. Then $C(w, b) \cup P_0 \cup P_0$ is the 3rd-container (or 4th-container) of $Q^{k+2}$.
for $0 \leq i \leq s - 2$, and $P_{i-1} = ((k - 2, \tau_{i-1}), (k - 1, \tau_{i-1}), H_{i-1}^{-1}(\tau_{i-1}, 0), (k - 1, 0), (k, 0), H_k(0, \tau_{i-1}, 0), (k, 1), (k + 1, \tau_{i-1})).$ Then $\mathcal{C}(w, b) \cup P_0 \cup P_i$ for $0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Case 3.2. $t \geq 2$.

Let $P_0 = ((\tau_0, k - 2), (\tau_0, k - 1), I_{k-1}(\tau_0, k - 2), (\tau_0, k - 1), (k - 2, k - 1), (k - 2, 0), I_{k-1}^{-1}((k - 1, \tau_0), (k, \tau_0)), (k, 0), (k, k), (k + 1, k), (k + 1, k + 1))$, and $P_i = ((k + 1, k - 2), (k + 1, k - 1), (k + 1, k), (k + 1, k + 1))$.

Case 3.2.1. $s = 1$.

Using the same $\mathcal{P}_0$ as in Case 3.1.1, then $\mathcal{C}(w, b) \cup \bigcup_{i=1}^{s} P_i$ for $0 \leq i \leq t - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Case 3.2.2. $s \geq 2$.

Using the same $\mathcal{P}_0$ as in Case 3.1.1, then $\mathcal{C}(w, b) \cup \bigcup_{i=1}^{s-1} P_i \cup \bigcup_{i=1}^{t} P_i$ for $0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Case 4. $v_{t-1} = k - 1$ for some $t \geq 2$ and $v_0 = 0$.

Case 4.1. $t = 2$.

Let $P_0 = ((\tau_0, k - 2), (\tau_0, k - 1), I_{k-1}(\tau_0, k - 2), (\tau_0, k - 1), (k - 2, k - 1), (k - 2, 0), I_{k-1}^{-1}((k - 1, \tau_0), (k, \tau_0)), (k, 0), (k, k), (k + 1, k), (k + 1, k + 1))$, and $P_1 = ((k + 1, k - 2), (k + 1, k - 1), (k + 1, k), (k + 1, k + 1))$.

Case 4.1.1. $s = 1$.

Using the same $\mathcal{P}_0$ as in Case 1.1, then $\mathcal{C}(w, b) \cup \bigcup_{i=1}^{s} P_i$ for $0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Case 4.1.2. $s \geq 2$.

Using the same $\mathcal{P}_0$ as in Case 1.1, then $\mathcal{C}(w, b) \cup \bigcup_{i=1}^{s-1} P_i \cup \bigcup_{i=1}^{t} P_i$ for $0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Case 4.2. $t \geq 3$.

Let $P_0 = ((\tau_0, k - 2), (\tau_0, k - 1), I_{k-1}(\tau_0, k - 2), (\tau_0, k - 1), (k - 2, k - 1), (k - 2, 0), I_{k-1}^{-1}((k - 1, \tau_0), (k, \tau_0)), (k, 0), (k, k), (k + 1, k), (k + 1, k + 1))$, and $P_i = ((k + 1, k - 2), (k + 1, k - 1), (k + 1, k), (k + 1, k + 1))$.

Case 4.2.1. $s = 1$.

Using the same $\mathcal{P}_0$ as in Case 1.1, then $\mathcal{C}(w, b) \cup \bigcup_{i=1}^{s} P_i$ for $0 \leq i \leq t - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Case 4.2.2. $s \geq 2$.

Using the same $\mathcal{P}_0$ as in Case 1.1, then $\mathcal{C}(w, b) \cup \bigcup_{i=1}^{s-1} P_i \cup \bigcup_{i=1}^{t} P_i$ for $0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_n^{k+2}$.

Theorem 3. For any integer $n \geq 2$ and any integer $k \geq 4$, the graph $Q_n^k$ is $m^*$-laceable where $1 \leq m \leq 2n$.

Proof: According to Theorem 2-3 and Lemma 3-5, the theorem holds for any even integer $k \geq 4$ when $n = 2$. We will give the proof of the theorem by mathematical induction on $n$. By induction hypothesis, assume that $Q_n^{k-1}$ is $m^*$-laceable for $1 \leq m \leq 2n - 2$, where $0 \leq i \leq k - 1$. Given a white vertex $w \in V(Q_n^{k-1})$ and a black vertex $b \in V(Q_n^{k-1})$.

Case 1. For $j = j'$. Without loss of generality, we let $j = j' = 0$.

In this case, we have $\{w, b\} \in Q_n^{k-1}$. By induction hypothesis, there are $m$ internal disjoint paths $\{P_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_n^{k-1}$ between $w$ and $b$.

Let $P_m = (w, w^1, b, b^1, \ldots, b^{m-1}, b)$. In $Q_n^{k-1}$, there exist a Hamiltonian path $P$ joining from $w^1$ to $b^{m-1}$ by Theorem 1. Also, we can let $P_m+1 = (w, w^{k-1}, R, b^1, b^2, \ldots, b^{m-1}, b)$. Therefore, there are $m + 2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 6 for an illustration.

Fig. 6. The illustration for Case 1 of Theorem 3.

Case 2. For $|j' - j| = 1$. Without loss of generality, we let $j = 0$ and $j' = 1$.

We have the following two cases.

Case 2.1. Suppose that $d(w, b) = 1$. It is easy to see that we can let $P_m = (w, b)$.

Case 2.1.1. If $m = 1$.

Let $z$ be any black vertex of $Q_n^{k-1}$. By Theorem 1, there exist a Hamiltonian path $P$ of $Q_n^{k-1}$ from $w$ to $z$, and a Hamiltonian path $P$ of $Q_n^{k-1}$ from $z$ to $b$. So we set $P_0 = (w, S, z, T, b)$. According to Lemma 1, a Hamiltonian path $R$ between $w^{k-1}$ and $b^2$ covers all vertices of $Q_n^{k-1}$ for $2 \leq i \leq k - 1$. We can write $P_1$ as $(w, w^{k-1}, R, b^2, b)$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 7 for an illustration.

Fig. 7. The illustration for Case 2.1.1 of Theorem 3.

Case 2.1.2. If $m \geq 2$.

According to the induction hypothesis, given any black vertex $z \in V(Q_n^{k-1})$, there exist $m$ internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_n^{k-1}$ between $w$ and $z$ for $2 \leq m \leq 2n - 2$. Let $R_i = (w, S_i, y_i, z, z', S_i^{-1}, b)$ and $P_i = (w, S_i, y_i, y_i', (S_i^{-1})^{-1}, b)$ for $1 \leq i \leq m - 1$. By
Lemma 1, there is a hamiltonian path $T$ between $w^k - 1 \in Q_{n-1}^{k-1}$ and $b^2 \in Q_{n-2}^{k-2}$ covering all vertices of $Q_n^{k-1}$ for $2 \leq i \leq k - 1$. Set $P_m = \langle w, w^k - 1, T, b^2, b \rangle$. Consequently, there are $m + 2$ internal disjoint paths $\{P_m\}_{m=0}^{+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 8 for an illustration.

According to Lemma 1, there is a hamiltonian path $U$ between $g^k - 2$ and $b^2$ covering all vertices of $Q_n^{k-1}$ for $2 \leq i \leq k - 2$. We can set $P_0 = \langle w, x_0, 0, (T_0)_i - 1, y_0, z_1, y_m, T_{m-1}, b \rangle$, $P_1 = \langle w, w^1, T_0^1, b \rangle$, $P_2 = \langle w, w^{k-1}, R, e^{k-1} - 1, e, S_{m-1}^{0}, y_0, z, y_0^{m}, (S_{m-1}^{0})^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$, $P_3 = \langle w, S_{m-1}, y_0^{m} - 1, z, y_0^{m}, (S_{m-1}^{0})^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$, and $P_4 = \langle w, S_{m-1}^{0}, y_0^{m} - 1, y_m, T_{m-3}, b \rangle$ for $4 \leq i \leq m + 1$. So, there are $m + 2$ internal disjoint paths $\{P_m\}_{m=0}^{+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 10 for an illustration.

Case 2.2. Suppose that $d(w, b) \geq 3$.

Case 2.2.1. If $m = 1$.

Given any black vertex $z$ in $Q_n^{k-1}$, by Theorem 1, there is a hamiltonian path $R$ of $Q_n^{k-1}$ joining from $w$ to $z$. So there is also a hamiltonian path $S$ of $Q_n^{k-1}$ between $z$ and $w^1$. We can set $S = \langle w, S'_1, z, S'_2, z^1 \rangle$. By Lemma 1, there exists a hamiltonian path $T$ between $w^k - 1 \in Q_{n-1}^{k-1}$ and $b^2 \in Q_{n-2}^{k-2}$ covering all vertices of $Q_{n-1}^{k-1}$ for $2 \leq i \leq k - 1$. We let $P_0 = \langle w, R, z, z^1, (S'_2)^{-1}, b \rangle$, $P_1 = \langle w, w, S'_1, b \rangle$, and $P_2 = \langle w, w^k - 1, T, b^2, b \rangle$. Therefore, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 9 for an illustration.

Fig. 8. The illustration for Case 2.1.2 of Theorem 3.

Case 2.2.2. If $b^0 \in V(S_0)$.

Let $S_0 = \langle w, x_0, 0, e, S_0^0, f, S_0^0, y_0, z, S_{m-1} \rangle$, and $S_t = \langle w, S_t^0, y_0, z \rangle$ for $1 \leq i \leq m - 1$. A hamiltonian path $R$ is embedded in $Q_n^{k-1}$ between $w^k - 1$ and $f^k - 1$ by Theorem 1. $R$ is written as $\langle w^k - 1, R, e^{k-1}, g, R', f^{k-1} \rangle$. Notice that $g^{k-2}$ is a black vertex and $b^2$ is a white vertex. According to Lemma 1, there is a hamiltonian path $U$ between $g^k - 2$ and $b^2$ covering all vertices of $Q_n^{k-1}$ for $2 \leq i \leq k - 2$. We let $P_0 = \langle w, x_0^1, x_0, (T_0^1)_i - 1, y_0, z_1, y_m, T_{m-1}, b \rangle$, $P_1 = \langle w, w^1, T_0^1, b \rangle$, $P_2 = \langle w, w^{k-1}, R, e^{k-1} - 1, e, S_{m-1}^{0}, b, b \rangle$, $P_3 = \langle w, S_{m-1}, y_0^{m} - 1, z, y_0^{m}, (S_{m-1}^{0})^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$, and $P_4 = \langle w, S_{m-1}^{0}, y_0^{m} - 1, y_m, T_{m-3}, b \rangle$ for $4 \leq i \leq m + 1$. Hence, there are $m + 2$ internal disjoint paths $\{P_m\}_{m=0}^{+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 11 for an illustration.

Fig. 9. The illustration for Case 2.2.1 of Theorem 3.

Fig. 10. The illustration for Case 2.2.1.2 of Theorem 3.

Case 3. For $|j' - j| \geq 2$. Without loss of generality, let $j = 0$ and $2 \leq j' \leq \frac{k}{2}$ be even.

Because $b \in Q_n^{k-1}$ where $j'$ is even, $b^i$ is a white (resp. black) vertex in $Q_n^{k-1}$ for $0 \leq i \leq k - 1$ when $i$ is odd (resp. even). It is easy to see that $w^1$ is a black (resp. white) vertex in $Q_n^{k-1}$ for $0 \leq i \leq k - 1$ when $i$ is odd (resp. even).

By the induction hypothesis, there exist $m$ internal disjoint paths $\{R'_p\}_{p=0}^{m-1}$ of $Q_n^{k-1}$ between $w^i$ and $b^i$ for $0 \leq i \leq j'$.
Let $R_p^v = \langle w^i, x^i_p, U^i_p, y^i_p, b^i \rangle$ for $0 \leq p \leq m - 1$ and $0 \leq i \leq j'$. According to Lemma 2, a hamiltonian path $S$ covers all vertices of $Q^{k-1}_{n-1}$ for $j' + 1 \leq i \leq k - 2$ joining from $w^{j'+1}_i$ to $b^{j'+1}$. There is a hamiltonian path $T$ of $Q^{k-1}_{n-1}$ from $w^{k-1}$ to $b^{k-1}$ by Theorem 1. Hence, we can write $R_p^v = \langle w = w^i, x^i_p, U^i_p, y^i_p, b^i = b \rangle$ for $0 \leq p \leq m - 1$. $P_m = \langle w = w^{i_0}, w^{i_1}, w^{i_2}, \ldots, w^{i_{j'+1}}, b' = b \rangle$, and $P_{m+1} = \langle w = w^{i_0}, w^{i_1}, w^{i_2}, \ldots, b' = b \rangle$. Therefore, there are $m + 2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 12 for an illustration.

**Case 4.** For $|j' - j| \geq 2$. Without loss of generality, we let $j = 0$ and $3 \leq j' \leq \frac{1}{2} + 1$ be odd.

**Case 4.1.** If $m = 1$.

Choosing a black vertex $z$ of $Q^{k,0}_{n-1}$, then $z$ is a hamiltonian path $P$ of $Q^{k,0}_{n-1}$ and $z$ is a black vertex of $Q^{k-1}_{n-1}$, so $z$ is a white vertex of $Q^{k-1}_{n-1}$. According to Lemma 1, there is a hamiltonian path $T$ of $Q^{k-2}_{n-1}$ joining from $w^{k-2}$ to $b^{k-2}$. Let $T' = \langle e_i, W, f_{j}' \rangle$. In $Q^{k,0}_{n-1}$, we also have a hamiltonian path $P'$ between $e_i$ and $f_{j}' \leq i \leq k - 3$, so we let $T'' = \langle e_i, W, f_{j}' \rangle$. According to Lemma 1, there is a hamiltonian path $U$ between a black vertex $w^i \in Q^{k-1}_{n-1}$ and a white vertex $e_i \in Q^{k-1}_{n-1}$ covering all vertices of $Q^{k-1}_{n-1}$ for $2 \leq i \leq j' - 1$. Let $P_0 = \langle w, w^{i}, U, b^{j'-1}, b \rangle$, $P_1 = \langle w, w^{i}, U, b^{j'-1}, b \rangle$, and $P_2 = \langle w, w^{i}, U, b^{j'-1}, b \rangle$. Therefore, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 13 for an illustration.

**Case 4.2.** If $m \geq 2$.

Given a white vertex $z$ in $Q^{k,0}_{n-1}$ such that $z$ is adjacent to $b$. So $z'$ is a black (resp. vertex) and $w$ is a white (resp. black) vertex of $Q^{k,0}_{n-1}$ if $0 \leq i \leq j' - 1$ when $i$ is even (resp. odd). By the induction hypothesis, there exist $m$ internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ of $Q^{k,0}_{n-1}$ between $w$ and $z$. We write $R_0 = \langle w, x^0_1, x^0_2, \ldots, x^0_i, z^0 \rangle$, and $P_{m+1} = \langle w, x^0_1, x^0_2, \ldots, z^0 \rangle$ for $1 \leq p \leq m - 1$. Again, by the induction hypothesis, there exist $m$ internal disjoint paths $\{P_i\}_{i=0}^{m-1}$ of $Q^{k,0}_{n-1}$ between $w^i$ and $z^i$ for $2 \leq i \leq j' - 1$. We let $T_p^i = \langle w^{i}, x^i_p, U^i_p, z^i_p \rangle$ for $0 \leq p \leq m - 1$, and $P_{m+1} = \langle w^{i}, x^i_p, U^i_p, z^i_p, e^{j'+1} \rangle$. According to Lemma 3, there is a hamiltonian path $V$ between $w^{i-1}$ and $b^{j'+1}$ in $Q^{k,0}_{n-1}$, so $T^{j'+1}_1$ is an internal disjoint path $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_n^k$ between $w$ and $b$. Please see Figure 14 for an illustration.

**APPENDIX A**

**PROOF OF LEMMA 3**

Notice that $Q^1_2$ is vertex symmetric. W.L.O.G, let $w = (0, 0)$. There are only two cases for $b$. That is, $b \in \{(1, 0), (2, 1)\}$.

**Case 1.** To prove that $Q^1_2$ is 3°-laceable.

**Case 1.1.** Let $b = (1, 0)$.

The three disjoint paths $\{P_1, P_2, P_3\}$ between $w$ and $b$ whose
union covers all vertices of $Q^3_1$ are $P_1 = \langle (0,0), (1,0) \rangle$, $P_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, and $P_3 = \langle (0,0), (0,3), (3,1), (3,2), (3,3), (3,4), (2,3), (1,3), (0,3), (0,2), (1,2), (2,2), (2,1), (2,0), (1,0) \rangle$.

Case 1.2. Let $b = (2,1)$.

The three disjoint paths $\{R_1, R_2, R_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^3_1$ are $R_1 = \langle (0,0), (1,0), (2,0), (2,1) \rangle$, $R_2 = \langle (0,0), (0,1), (1,1), (2,1) \rangle$, and $R_3 = \langle (0,0), (0,3), (3,1), (3,2), (3,3), (3,4), (2,3), (2,2), (2,1), (2,0) \rangle$, (1,0) \rangle$.

Case 2. To prove that $Q^3_2$ is 4\textsuperscript{-}laceable.

Let $b = (1,0)$.

The four disjoint paths $\{P_1, P_2, P_3, P_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^3_2 = P_1 = \langle (0,0), (1,0), (2,0), (2,1) \rangle$, $P_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, $P_3 = \langle (0,0), (0,3), (3,1), (3,2), (3,3), (3,4), (2,3), (2,2), (2,1), (2,0) \rangle$, (1,0) \rangle$.

Case 2.1. Let $b = (2,1)$.

The four disjoint paths $\{P_1, P_2, P_3, P_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^3_2$ are $P_1 = \langle (0,0), (1,0), (2,0), (2,1) \rangle$, $P_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, $P_3 = \langle (0,0), (0,3), (3,1), (3,2), (3,3), (3,4), (2,3), (2,2), (2,1), (2,0) \rangle$, (1,0) \rangle$.

Case 2.2. Let $b = (2,1)$.

The four disjoint paths $\{R_1, R_2, R_3, R_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^3_2$ are $R_1 = \langle (0,0), (3,0), (3,1), (2,1) \rangle$, $R_2 = \langle (0,0), (0,1), (0,0), (2,0), (2,1) \rangle$, $R_3 = \langle (0,0), (0,1), (1,1), (2,1) \rangle$, and $R_4 = \langle (0,0), (0,0), (0,3), (0,2), (1,2), (1,3), (2,3), (3,3), (3,2), (2,2), (2,1), (2,0) \rangle$.

Case 2.3. Let $b = (3,0)$.

The four disjoint paths $\{S_1, S_2, S_3, S_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^3_2$ are $S_1 = \langle (0,0), (0,1), (0,2), (0,3) \rangle$, $S_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, $S_3 = \langle (0,0), (0,3), (3,1), (3,2), (3,3), (3,4), (2,3), (2,2), (2,1), (2,0) \rangle$, (1,0) \rangle$.

Case 2.4. Let $b = (3,2)$.

The four disjoint paths $\{T_1, T_2, T_3, T_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^3_2$ are $T_1 = \langle (0,0), (1,0), (2,0), (2,1) \rangle$, $T_2 = \langle (0,0), (0,1), (2,0), (2,1) \rangle$, $T_3 = \langle (0,0), (1,1), (2,1), (2,2), (2,3), (1,3), (1,2), (1,1) \rangle$, and $T_4 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$.

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