The Spanning Laceability of $k$-ary $n$-cubes when $k$ is Even

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Abstract—$Q^k_n$ has been shown as an alternative to the hypercube family. For any even integer $k \geq 4$ and any integer $n \geq 2$, $Q^k_n$ is a bipartite graph. In this paper, we will prove that given any pair of vertices, $u$ and $v$, from different partite sets of $Q^k_n$, there exist $2n$ internally disjoint paths between $u$ and $v$, denoted by $P_i$, $0 \leq i \leq 2n - 1$, such that $\bigcup_{i=0}^{2n-1} P_i$ covers all vertices of $Q^k_n$. The result is optimal since each vertex of $Q^k_n$ has exactly 2n neighbors.

Keywords—container, Hamiltonian, $k$-ary $n$-cube, $m^*$-connected.

I. INTRODUCTION

The $k$-ary $n$-cube, denoted by $Q^k_n$, has been proposed as an alternative to the hypercube since it shares many nice properties of $Q_n$ such as regular degrees, vertex symmetry, recursive structure, etc. The underlying topology of many machines is based on $k$-ary $n$-cubes, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1], [7], [11], [17]. Many researchers have been working on $k$-ary $n$-cubes. For example, Stewart and Xiang [20] proved that the $k$-ary $n$-cube is edge-bipancyclic and bipanconnected for $k \geq 3$ and $n \geq 2$ and $k$ being even. Namely, any edge of a $k$-ary $n$-cube $Q^k_n$ lies on a cycle of any even length $r$ for $4 \leq r \leq |Q^k_n|$, where $|Q^k_n|$ is the total number of vertices of $Q^k_n$. Besides, given two vertices $u$ and $v$ of $Q^k_n$, there exists a path of any even length $r$ between $u$ and $v$ for $d(u, v) \leq r \leq |Q^k_n|$, where $d(u, v)$ is the distance between $u$ and $v$. Other studies about fault tolerance on $k$-ary $n$-cubes can be found in [8], [23]. Recently, there are many studies about the spanning connectivity for interconnection networks and graphs [9]. A graph $H = (B \cup W, E)$ is bipartite if $V(H)$ is the union of two disjoint sets $B$ and $W$ such that every edge joins $B$ with $W$. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected except $K_2$. Note that any (nontrivial) bipartite graph except $K_2$ cannot be hamiltonian connected, whereas a bipartite graph is hamiltonian laceable if there exists a hamiltonian path between any two vertices $u$, $v$ with $u \in B$ and $v \in W$ [22]. A graph $H = (B \cup W, E)$ is a balanced bipartite graph if $|V(B)| = |V(W)|$. Throughout this thesis, we only work on $Q^k_n$ with $k \geq 4$ and even integer and $n \geq 2$, which are balanced bipartite graphs. A bipartite graph $H = (B \cup W, E)$ is $m^*$-laceable if given a white vertex $w \in W$ and a black vertex $b \in B$, there exist(s) $m$ internal disjoint paths between $w$ and $b$, denoted by $P_i$, for $0 \leq i \leq m - 1$, such that $\bigcup_{i=0}^{m-1} P_i$ covers $V$. The spanning laceability of a graph $H$, $\kappa^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for every $m$ with $1 \leq m \leq k$. A higher spanning connectivity/laceability of the interconnection network implies a more efficient communication between processors. About the spanning connectivity and the spanning laceability, readers can refer to [6], [7], [12]–[15].

In this paper, we want to show the spanning laceability of $k$-ary $n$-cubes for any even integer $k \geq 4$. More precisely, we show that given a white vertex $w$ and a black vertex $b$ of a $k$-ary $n$-cube $Q^k_n$, there exist(s) $m$ internally disjoint path(s) between $w$ and $b$ whose union covers all vertices of $Q^k_n$ for $1 \leq m \leq 2n$. The result is optimal since any vertex in $Q^k_n$ has exactly $2n$ neighbors. This paper is organized as follows. In Section 2, we introduce the graph terminologies and symbols that will be used in the paper and the definition of $Q^k_n$. In Section 3, we show our main results.

II. PRELIMINARIES

Throughout this paper, we follow [3] for the graph definitions and notations. The sets of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. If $u$, $v$ are vertices of a graph $G$ such that there is an edge $e = (u, v) \in E(G)$ between $u$ and $v$, then we say that the vertices $u$ and $v$ are adjacent in $G$. The degree of any vertex $x$ is the number of distinct vertices adjacent to $x$. A path $P$ between two vertices $v_0$ and $v_k$ is represented by $P = \langle v_0, v_1, \ldots, v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We use $P^{-1}$ to denote the path $\langle v_k, v_{k-1}, \ldots, v_2, v_1, v_0 \rangle$. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $P(v_0, v_1, \ldots, v_k)$. A hamiltonian path between two vertices $u$ and $v$, where $u$ and $v$ are two distinct vertices of $G$, is a path joining $u$ to $v$ that visits every vertex of $G$ exactly once. A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of $G$ is a cycle that traverses every vertex of $G$ exactly once. A hamiltonian graph is a graph with a hamiltonian cycle. A graph $G$ is connected if there is a path between any two distinct vertices in $G$ and is hamiltonian connected if there is a hamiltonian path between any two distinct vertices in $G$ [18]. A graph $H = (W \cup B, E)$ is bipartite if $V(H) = W \cup B$ and $E(H)$ is a subset of $\{ (u, b) | u \in W, b \in B \}$. A bipartite graph $H$ is hamiltonian laceable if there is a hamiltonian path between any two distinct vertices from different partite sets in $H$.

A graph $G$ is $k$-connected if there exists $V' \subseteq V(G)$ with $|V'| = k$ such that $G - V'$ is disconnected and $G - V''$ is
connected for any $V' \subseteq V(G)$ with $|V'| < k$. It follows from Menger’s Theorem [16] that for every $k$-connected graph $G$, there exist $k$ internally vertex-disjoint paths between any pair of distinct vertices of $G$. A $k$-container $C(u, v)$ in a graph $G$ is a set of $k$ internally vertex-disjoint paths between two distinct vertices $u$ and $v$. We say that a graph $G$ has a spanning $k$-container between $u$ and $v$, denoted by $C(u, v)$, if $C(u, v)$ is a $k$-container that covers all vertices of $G$. A spanning $k$-container is also abbreviated as a $k^*$-container for simplicity. A graph $G$ is $k^*$-connected if there is a $k^*$-container between any pair of vertices of $G$. Obviously, a graph $G$ is Hamiltonian connected if and only if $G$ is $1^*$-connected, and $G$ is Hamiltonian if and only if $G$ is $2^*$-connected. Lin et al. [13] defined the concept of spanning connectivity. The spanning connectivity of a graph $G$, $\kappa^*(G)$, is the largest integer $k$ such that $G$ is $w^*$-connected for all $1 \leq w \leq k$. Similarly, a bipartite graph $H$ is $k^*$-laceable if there is a $k^*$-container between any pair of two vertices from different partite sets of $H$. Also, a bipartite graph $H$ is Hamiltonian laceable if and only if $H$ is $1^*$-laceable, and $H$ is Hamiltonian if and only if $H$ is $2^*$-laceable. So, the spanning laceability of a bipartite graph $H$, $\kappa^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for all $1 \leq m \leq k$.

The $k$-ary $n$-cube, $Q_n^k$, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass $Q_n^0$ is the well-studied hypercube family. The subclass $Q_n^k$ with $k \geq 3$ is defined as the cycle of length $k$. The $k$-ary $n$-cube, $Q_n^k$, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u(0), u(1), \ldots, u(n-1))$, where $0 \leq u(i) \leq k-1$. Two vertices $u$ and $v$ are adjacent if and only if $|u(i) - v(i)| = 1$ or $k - 1$ for some $i$ and $u(j) = v(j)$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is shown that $Q_n^k$ is bipartite if $k$ is even [10]. Here we mention some properties of $Q_n^k$ that will be used in this paper.

$Q_n^k$ is vertex symmetric (and edge symmetric) [10]. It means that given any two distinct vertices $v$ and $v'$ of $Q_n^k$, there is an automorphism of $Q_n^k$ mapping $v$ to $v'$. Note that each vertex of $Q_n^k$ is represented by a n-bit tuple. We will call the $d$th bit the $d$th dimension. We can partition $Q_n^k$ over dimension $d$ by fixing the $d$th element of any vertex tuple at some value $a$ for every $a \in \{0, 1, \ldots, k-1\}$. This results in $k$ copies of $Q_{n-1}^k$, denoted by $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \ldots, Q_{n-1}^{k,k-1}$, with corresponding vertices in $Q_{n-1}^{k,0} \cup Q_{n-1}^{k,1} \cup \ldots \cup Q_{n-1}^{k,k-1}$ joined in a cycle of length $k$ (in dimension $d$) [19].

In this article, we always partition $Q_n^k$ over the 0-th dimension by letting $V(Q_n^{k,0}) = \{(i, v(1), v(2), \ldots, v(n-1)) | 0 \leq v(j) \leq k-1, 1 \leq j \leq n-1\}$ for $0 \leq i \leq k-1$. Given a vertex $x = (x(0), x(1), \ldots, x(n-1)) \in V(Q_n^k)$, the symbol $x^j = ((j), x(1), x(2), \ldots, x(n-1))$, where $0 \leq j \leq k-1$, is defined to be the vertex corresponding to $x$ in $Q_{n-1}^{k,j}$ for simplicity. So, if $P = (x_0, x_1, \ldots, x_{n-1})$, $P^j$ is represented by $(x_0^j, x_1^j, \ldots, x_{n-1}^j)$. Throughout this paper, let $n \geq 2$ be an integer and $k \geq 4$ an even integer.

Theorem 1. [10] For any even integer $k \geq 4$, $Q_n^k$ is Hamiltonian laceable for $n \geq 2$. In other words, $Q_n^k$ is $1^*$-laceable.

Theorem 2. [5] The graph $Q_n^k$ is Hamiltonian. In other words, $Q_n^k$ is $2^*$-laceable.

III. MAIN RESULTS

Lemma 1. Given $Q_n^k$ and its $k$ subcubes, $Q_{n-1}^{k,i}$, where $0 \leq i \leq k-1$. Let $j$ and $j'$ be two integers satisfying $0 \leq j \leq j' \leq k-1$, $w \in V(Q_{n-1}^{k,i})$ an arbitrary white vertex, and $b \in V(Q_{n-1}^{k,j'})$ an arbitrary black vertex. Then there exists a path between $w$ and $b$ that visits each vertex in $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'}$ exactly once.

Proof: There are three cases.

Case 1. $j = j'$. W.L.O.G., let $j = j' = 0$. By Theorem 1, $Q_n^k$ is Hamiltonian laceable. Thus, there is a Hamiltonian path between $w$ and $b$ that visits each vertex of $Q_{n-1}^{k,0}$ exactly once.

Case 2. $j = j' = 1$. W.L.O.G., we can let $j = 0$ and $j' = 1$. Let $w$ be a white vertex in $Q_{n-1}^{k,0}$ and $b$ a black vertex in $Q_{n-1}^{k,1}$. We can find a pair of adjacent vertices $x^0$ and $x^1$ where $x^0$ is a black vertex of $Q_{n-1}^{k,0}$ and $x^1$ is a white vertex of $Q_{n-1}^{k,1}$. By Theorem 1, there exists a Hamiltonian path $P_0$ of $Q_{n-1}^{k,0}$ between $w$ and $x^0$, and a Hamiltonian path $P_1$ of $Q_{n-1}^{k,1}$ between $x^1$ and $b$. Let $P = (w, P_0, x^0, P_1, b)$. Hence $P$ is the path between $w$ and $b$ that visits every vertex of $Q_{n-1}^{k,0}$ and $Q_{n-1}^{k,1}$ exactly once.

Case 3. $j - j' \geq 2$. Let $w$ be a white vertex in $Q_{n-1}^{k,j}$ and $b$ be a black vertex in $Q_{n-1}^{k,j'}$. There are $j - j' + 1$ k-ary $n - 1$-cubes, $Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'}$ and $Q_{n-1}^{k,j'}$. There are $j'$ pairs of adjacent vertices $x^r$ in $Q_{n-1}^{k,j'}$ and $y^{r+1}$ in $Q_{n-1}^{k,j}$ where $x^r$ is a black vertex and $y^{r+1}$ is a white vertex for $j \leq r \leq j'$ - 1. By Theorem 1, there is a Hamiltonian path $R$, of $Q_{n-1}^{k,r}$ joining $y^{r}$ to $x^{r}$, where $j + 1 \leq r \leq j'$. Again, with Theorem 1, there exists a Hamiltonian path $T$ of $Q_{n-1}^{k,j}$ joining $w$ to $x^0$, and a Hamiltonian path $U$ of $Q_{n-1}^{k,j}$ joining $y^{j}$ to $b$. Let $P = (w, T, x^j, y^{j+1}, R_{j+1}, x^{j+1}, y^{j+2}, R_{j+2}, x^{j+2}, \ldots, y^{j-1}, R_{j-1}, x^{j-1}, y^j, U, b)$. Therefore, $P$ is a path covering all the vertices of $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, Q_{n-1}^{k,2}, \ldots, Q_{n-1}^{k,j'}$ for $0 \leq j \leq j' \leq k-1$ between $w$ and $b$. Please see Figure 1 for an illustration.

![Fig. 1. The illustration for Case 3 of Lemma 1.](image-url)

Lemma 2. Given $Q_n^k$ and its $k$ subcubes $Q_{n-1}^{k,i}$ for $0 \leq i \leq k-1$. Let $w$ be a white vertex, $b$ a black vertex in $Q_{n-1}^{k,1}$, and $j$ an integer with $0 \leq j \leq k-1$. There exists a path between $w$ and $b$ that covers all the vertices of $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, \ldots, Q_{n-1}^{k,k-1}$.

Proof: We consider the following two cases.
Case 1. \( j = i \). There is only one \( k \)-ary \( (n - 1) \)-cube \( Q_{n-1}^{k,i} \). By Theorem 1, the lemma holds in this case.

Case 2. \( j \neq i \). There are \( j - i + 1 \) \( k \)-ary \( (n - 1) \)-cubes. According to Theorem 1, there is a hamiltonian path \( P_{i,j} \) that covers all the vertices of \( Q_{n-1}^{k,i} \) between \( w \) and \( b \) of the form \( \langle w, x, y, z, T, w \rangle \), where \( \{x, y\} \) is an edge of \( Q_{n-1}^{k,i} \) with \( \{x, y\} \cap \{w, b\} = \emptyset \). Notice that by Theorem 1, \( Q_{n-1}^{k,i} \) is hamiltonian laceable and hence there exists a hamiltonian path \( P_{i,j} \) between \( x \) and \( y \) of the form \( \langle x, y, z, w, T, w \rangle \) for \( i + 1 \leq r \leq j \). Let the required path between \( w \) and \( b \) be \( R \), we have the following two subcases.

**Case 2.1.** \( j - i + 1 \) is even, then

\[
R = \langle w, S_1, \ldots, z_{i+1}, (S_{i+2})^{-1}, w \rangle_1, x, y, \ldots, z_j, w, T, y, T, \ldots, y, w, b, T \rangle_1, w, T, \ldots, y, T, b \rangle_1.
\]

Please see Figure 2 for an illustration.

**Case 2.2.** \( j - i + 1 \) is odd, then

\[
R = \langle w, S_1, \ldots, z_{i+1}, (S_{i+2})^{-1}, \ldots, z_{i+4}, (S_{i+5})^{-1}, \ldots, z_j, w, T, y, T, \ldots, y, w, b, T \rangle_1, w, T, \ldots, y, T, b \rangle_1.
\]

Please see Figure 2 for an illustration.

**Lemma 3.** The graph \( Q_2^4 \) is \( 3^* \)-laceable and \( 4^* \)-laceable.

**Proof:** The proof is by brute force. Reader can refer to Appendix A.

**Lemma 4.** The graph \( Q_2^k \) is \( 3^* \)-laceable and \( 4^* \)-laceable.

**Proof:** By brute force, we constructed all spanning containers. Please see Appendix B.

**Lemma 5.** The graph \( Q_2^k \) is \( 3^* \)-laceable and \( 4^* \)-laceable for any even integer \( k \geq 6 \).

**Proof:** With Lemma 4, we have shown that \( Q_2^k \) is \( 3^* \)-laceable and \( 4^* \)-laceable. Now we will present a recursive algorithm that uses a \( 3^* \)-container (resp. \( 4^* \)-container) of \( Q_2^k \) to construct a \( 3^* \)-container (resp. \( 4^* \)-container) of \( Q_2^{k+2} \). Let \( R \) be a subset of \( V(Q_2^k) \cup E(Q_2^k) \). We define a function, \( f \), which maps \( R \) from \( Q_2^k \) into \( Q_2^{k+2} \) in the following way:

1. If \((i, j) \in R \cap V(Q_2^k)\), where \( 0 \leq i, j \leq k - 1 \), then

   \[
f((i, j)) = \begin{cases} 
   (i, j) & \text{if } 0 \leq i, j \leq k - 2; \\
   (i + 2, j) & \text{if } i = k - 1, 0 \leq j \leq k - 2; \\
   (i, j + 2) & \text{if } j = k - 1, 0 \leq i \leq k - 2; \\
   (i + 2, j + 2) & \text{if } i = k - 1 = j.
   \end{cases}
\]

2. If \((i, j), (i', j') \in R \cap E(Q_2^k)\), where \( i \leq i', j \leq j'\), then

   \[
f([(i, j), (i', j')]) = \begin{cases} 
   ((i, j), (i', j')) & \text{if } 0 \leq i, j, i', j' \leq k - 3, \\
   ((i + 2, j), (i', j' + 2)) & \text{if } j = k - 1, 0 \leq j \leq k - 3, \\
   ((i, j + 2), (i', j' + 2)) & \text{if } j = k - 1, 0 \leq i \leq k - 3, \\
   ((i + 2, j), (i', j' + 2)) & \text{if } j = k - 1, 0 \leq i \leq k - 3, \\
   (i, j) & \text{if } i = k - 1 = j.
   \end{cases}
\]

Let \( w \) be a white vertex and \( b \) be a black vertex of \( Q_2^k \). We say that a \( 3^* \)-container (resp. \( 4^* \)-container) \( C(u, v) \) of \( Q_2^k \) is regular if \( C(u, v) \) contains some edges in \( \{(u, (k - 2), (a, k - 1)) | 0 \leq a \leq k - 1 \} \cup \{(a, (k - 2), (k - 1, \beta)) | 0 \leq a \leq k - 1 \} \) for all \( a \). For example, all \( 3^* \)-containers and \( 4^* \)-containers of \( Q_2^k \) constructed in Lemma 4 are regular. Let \( C(u, v) \) be a regular \( 3^* \)-container (resp. \( 4^* \)-container) of \( Q_2^k \) with the endvertex set \( P = \{w = (0, 0), b = (x, y)\} \). We construct a regular \( 3^* \)-container (resp. \( 4^* \)-container) of \( Q_2^{k+2} \) with the endvertex set \( f(P) \) using the following algorithm. Please see Figure 4 for an illustration.

\[
\begin{align*}
\text{(a) } & w = (0, 0), b = (2, 1) \\
\text{(b) } & w = (0, 0), b = (2, 1) \\
\end{align*}
\]
Case 1.2. Let $C(w, b)$ be the image in $Q^{k+2}$ of $C(w, b) - \{(v_i, k-2), (v_i, k-1)\}$ is an edge of $C(w, b)$ for $0 \leq i \leq k-1$, \((v_i, k-2), (v_i, k-1)\) is an edge of $C(w, b)$. 

Step 2. Let $C(w, b)$ be the image in $Q^{k+2}$ of $C(w, b) - \{(v_i, k-2), (v_i, k-1)\} [0 \leq i \leq k-1] \cup \{(k-2, h_j), (k-1, h_j)\}$ under the function $f$. Please see Figure 5 for an illustration.

![Figure 5](https://example.com/figure5.png)

**Fig. 5.** Using function $f$ to map a subset of edges and vertices of $Q^2$ into $Q^2$.

Step 3. For any two positive integers $r$ and $d$, we use $[r]_d$ to denote $r \mod d$. In $Q^{k+2}$, define the following path patterns, where $r_1, r_2$ are integers:

$I_0(r_1, r_2) = (\{r_1, \alpha\}, \{r_1 + 1, k + 2, \alpha\}, \ldots, (r_2, \alpha))$;

$I_0^+(r_1, r_2) = (\{r_2, \alpha\}, \{r_2 - 1, k + 2, \alpha\}, \ldots, (r_1, \alpha))$;

$H_0(r_1, r_2) = (\{\beta, r_1\}, \{\beta + 1, k + 2, \alpha\}, \ldots, (\beta, r_2))$;

$H_0^+(r_1, r_2) = (\{\beta, r_2\}, \{\beta + 1, k + 2, \alpha\}, \ldots, (\beta, r_1))$.

Let $t_i = v_i + 2$ if $v_i = k - 1$ and $t_i = v_i$ if $0 \leq v_i \leq k - 2$, and $t_j = h_j + 2$ if $h_j = k - 1$ and $t_j = h_j$ if $0 \leq h_j \leq k - 2$.

Case 1. $t_0 = k - 1$.

Let $P_0 = \{(k+1, k-2), (k+1, k-1), (0, k-1), (k-1, 0), (k-1, k-2), (k-2, k-1), (k-2, k), (k-1, k), (k+1, k), (k+1, k+1)\}$.

Case 1.1. $s = 1$.

Let $P_0 = \{(k+2, t_0), (k-1, t_0), H_{k-1}^{-1}(t_0, t_0 + k+2), (k-1, t_0 + k+2), (t_0 + k+2), (t_0 + k+2), \}$.

Then $C(w, b) \cup P_0 \cup P_0$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}$.

Case 1.2. $s = 2$.

Let $P_0 = \{(k+2, t_0), (k-1, t_0), H_{k-1}^{-1}(t_0, t_0 + k+2), (k-1, t_0 + k+2), (t_0 + k+2), (t_0 + k+2), \}$

for $0 \leq i \leq s - 2$ and $P_{s-1} = \{(k-2, t_{s-1}), (k-1, t_{s-1}), H_{k-1}^{-1}(t_{s-1}, t_0 + k+2), (k-1, t_0 + k+2), (t_0 + k+2), (t_0 + k+2), \}$.

Then $C(w, b) \cup P_0 \cup \{P_i\}$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}$.

Case 2. $v_1 \leq k - 2$ and $(k-2, k-1), (k-1, k-1) \in E(C(w, b))$ in $Q^2$.

Case 2.1. $t = 1$.

Let $P_0 = \{(t_0, k-2), (t_0, k-1), I_{k-1}^{-1}(t_0, k-2), (k-2, k-1), (k-2, k), I_{k-1}^{-1}(k-2, k), (t_0, k), (t_0, k+1)\}$.

Case 2.1.1 $s = 1$.

Let $P_0 = \{(k+2, t_0), (k-1, t_0), H_{k-1}^{-1}(t_0, t_0 + k+2), (k-1, t_0 + k+2), (t_0 + k+2), (t_0 + k+2), \}$.

Then $C(w, b) \cup P_0 \cup \{P_i\}$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}$.
for $0 \leq i \leq s - 2$, and $\overline{P}_{i+1} = ((k-2,\overline{I}_{i+1}),(k-1,\overline{I}_{i+1}),H_{i+1}^{-1}(\overline{I}_{i+1},0), (k-1,0), (k,0), H_{i}(0,\overline{I}_{i+1},(k-1,\overline{I}_{i+1})$. Then $\overline{C}(w,b) \cup P_0 \cup \overline{P} \cup \{P_i \mid 0 \leq i \leq s - 1\}$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.

**Case 3.2.** $t \geq 2$.

Let $P_1 = (1,1,k), I_{k+1} = (1,0,k), H_{i,k,0}^{-1}(1,0,k), (1,0,k), (1,0,k)$ for $0 \leq i \leq t - 2$, and $P_{t-1} = ((k-2,\overline{I}_{t-1},k-1), I_{k-1}^{-1}(\overline{I}_{t-1},k-1), (k-2,k-1), H_{i,k,0}^{-1}(\overline{I}_{t-1},k-1), (k-2,k-1), H_{k-1}^{-1}(0,0,k), (0,0,k), (0,0,k), (0,0,k))$.

**Case 3.2.1.** $s = 1$.

Using the same $\overline{P}_0$ as in Case 3.1.1., then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t - 1\} \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.

**Case 3.2.2.** $s \geq 2$.

Using the same $\{P_i \mid 0 \leq i \leq s - 1\}$ as in Case 3.2.1., then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t - 1\} \cup \overline{P}_0 \cup \{P_i \mid 0 \leq i \leq s - 1\}$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.\]  

**Case 4.** $v_{t-1} = k - 1$ for some $t \geq 2$ and $v_0 = \theta$.

**Case 4.1.** $t = 2$.

Let $P_0 = ((1,0,k), I_{k-1}^{-1}(1,0,k), (1,0,k), (1,0,k))$ and $P_1 = ((k-1,k-2), (k-2,k-1), (k-2,k-1), H_{I_{k-1}^{-1}(1,0,k), (1,0,k), (1,0,k)}$.

**Case 4.1.1.** $s = 1$.

Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(w,b) \cup P_0 \cup P_1 \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.

**Case 4.1.2.** $s \geq 2$.

Using the same $\{P_i \mid 0 \leq i \leq s - 1\}$ as in Case 1.2., then $\overline{C}(w,b) \cup P_0 \cup P_1 \cup \overline{P}_0 \cup \{P_i \mid 0 \leq i \leq s - 1\}$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.\]

**Case 4.2.** $t \geq 3$.

Let $P_i = ((1,0,k), I_{k-1}^{-1}(1,0,k), (1,0,k), (1,0,k))$ for $0 \leq i \leq t - 3$; $P_{t-2} = ((k-2,k-2), (k-2,k-1), (k-2,k-1), I_{k-1}^{-1}(k-2,k-2), (k-2,k-1), (k-2,k-1), I_{k-1}^{-1}(k-2,k-2), (k-2,k-1), (k-2,k-1))$, and $P_{t-1} = ((k+1,k+1), (k+1,k+1), (k+1,k+1), (k+1,k+1))$.

**Case 4.2.1.** $s = 1$.

Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t - 1\} \cup \overline{P}_0$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.

**Case 4.2.2.** $s \geq 2$.

Using the same $\{P_i \mid 0 \leq i \leq s - 1\}$ as in Case 1.2., then $\overline{C}(w,b) \cup \{P_i \mid 0 \leq i \leq t - 1\} \cup \overline{P}_0 \cup \{P_i \mid 0 \leq i \leq s - 1\}$ is the $3^*$-container (or $4^*$-container) of $Q^{k+2}_{2}$.

**Theorem 3.** For any integer $n \geq 2$ and any even integer $k \geq 4$, the graph $Q^k_n$ is $m^*$-laceable where $1 \leq m \leq 2n$.

**Proof:** According to Theorem 2-3 and Lemma 3-5, the theorem holds for any even integer $k \geq 4$ when $n = 2$. We will give the proof of the theorem by mathematical induction on $n$. By induction hypothesis, assume that $Q^{k+1}_{n-1}$ is $m^*$-laceable for $1 \leq m \leq 2n - 2$, where $0 \leq k \leq k - 1$. Given a white vertex $w \in V(Q^{k}_{n-1})$ and a black vertex $b \in V(Q^{k}_{n-1})$. We will show that we can use the $m^*$-containers of $Q^{k}_{n-1}$ to construct a $(m + 2)^*$-container of $Q^{k}_{n}$ between $w$ and $b$.

**Case 1.** For $j = j'$. Without loss of generality, we let $j = j' = 0$.

In this case, we have $\{w,b\} \in Q^{k}_{n-1}$. By induction hypothesis, there are $m$ internal disjoint paths $\{P_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q^{k}_{n-1}$ between $w$ and $b$ for $1 \leq m \leq 2n - 2$. By Lemma 2, the exists a path $S$ covering all vertices of $Q^{k}_{n-1}$ for $1 \leq i \leq k - 2$ between $w$ and $b$. We can let $P_m = (w,w^1,S,b^1,b)$. In $Q^{k-1}_{n-1}$, there exist a hamiltonian path $R$ joining from $w^{k-1}$ to $b^{k-1}$ by Theorem 1. Also, we can let $P_{m+1} = (w,w^1,R,b^{k-1},b)$. Therefore, there are $m + 2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q^{k}_{n}$ between $w$ and $b$. Please see Figure 6 for an illustration.

**Case 2.** For $j' - j \neq 1$. With no loss of generality, we let $j = 0$ and $j' = 1$.

We have the following two cases.

**Case 2.1.** Suppose that $d(w,b) = 1$. It is easy to see that we can let $P_m = \langle w,b \rangle$.

**Case 2.1.1.** If $m = 1$.

Let $z$ be any black vertex of $Q^{k}_{n-1}$. By Theorem 1, there exist a hamiltonian path $S$ of $Q^{k}_{n-1}$ from $w$ to $z$, and a hamiltonian path $T$ of $Q^{k}_{n-1}$ from $z$ to $b$. So we set $P_0 = \langle w,S,z,T,b \rangle$. According to Lemma 1, a hamiltonian path $R$ between $w^k \in Q^{k}_{n-1}$ and $b^k \in Q^{k}_{n-1}$ covers all vertices of $Q^{k}_{n-1}$ for $2 \leq i \leq k - 1$. We can write $P_1$ as $\langle w^k,R,b^k,b \rangle$. Hence, there are $3$ internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q^{k}_{n}$ between $w$ and $b$. Please see Figure 7 for an illustration.

**Case 2.1.2.** If $m \geq 2$.

According to the induction hypothesis, given any black vertex $z \in V(Q^{k}_{n-1})$, there exist $m$ internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q^{k}_{n-1}$ between $w$ and $z$ for $2 \leq m \leq 2n - 2$. Let $R_i = \langle w,S_i,y_i,z \rangle$ for $0 \leq i \leq m - 1$. We set $P_0 = \langle w,S_0,y_0,z \rangle$ and $P_i = \langle w,S_i,y_i,z \rangle$ for $1 \leq i \leq m - 1$. By
Lemma 1, there is a hamiltonian path $T$ between $u_{k-1}^i \in Q_{k,i-1}^k$ and $b_{i} \in Q_{k,i+1}^k$ covering all vertices of $Q_{k,i}^k$ for $2 \leq i \leq k - 1$. Set $P_n = \{w, w_{k-1}^i, T, b_{i}^k\}$. Consequently, there are $m + 2$ internal disjoint paths $\{P_{k,i}^{m+1}\}$ whose union covers all vertices of $Q_k^k$ between $w$ and $b$. Please see Figure 8 for an illustration.

According to Lemma 1, there is a hamiltonian path $U$ between $g_{k-2}$ and $b_{i}$ covering all vertices of $Q_{k,i}^k$ for $2 \leq i \leq k - 2$. We can set $P_0 = \{w, x_0^i, y_0^i, z_0^i, y_{m-1}, T_{m-1}, b_{i}\}$, $P_1 = \{w, w_{k-1}^i, T_{m-1}^i, b_{i}\}$, $P_2 = \{w, w_{k-1}^i, R_{i}^0, e_{i}^k, e, S_0^i, y_{m-1}, z_0, y_0^i, (S_{m-1}^i)^{-1}, f, f_{k-1}, R_{i}^{k-1}, g, g_{k-2}, U, b_{i}^k, b_{i}\}$, $P_3 = \{w, S_m^i, b_{i}^k, y_{m-1}, z_0, y_0^i, (S_{m}^i)^{-1}, f, f_{k-1}, R_{i}^{k-1}, g, g_{k-2}, U, b_{i}^k, b_{i}\}$, and $P_4 = \{w, S_{m-1}^i, b_{i}^k, y_{m-1}, z_0, y_0^i, (S_{m-1}^i)^{-1}, f, f_{k-1}, R_{i}^{k-1}, g, g_{k-2}, U, b_{i}^k, b_{i}\}$ for $4 \leq i \leq m + 1$. So, there are $m + 2$ internal disjoint paths $\{P_{k,i}^{m+1}\}$ whose union covers all vertices of $Q_k^k$ between $w$ and $b$. Please see Figure 10 for an illustration.

Case 2.2. Suppose that $d(w, b) \geq 3$.

Case 2.2.1. If $m = 1$.

Given any black vertex $z$ in $Q_{k,0}^k$, by Theorem 1, there is a hamiltonian path $R$ of $Q_{k,0}^k$ joining from $w$ to $z$. So there is also a hamiltonian path $S$ of $Q_{k,1}^k$ between $w$ to $z$. We can set $S = (w, S_1^i, b, S_2^i, z)$. By Lemma 1, there exists a hamiltonian path $T$ between $w_{k-1}^i \in Q_{k-1,i}^k$ and $b_{i} \in Q_{k,i}^k$ covering all vertices of $Q_{k,i}^k$ for $2 \leq i \leq k - 1$. We let $P_0 = \{w, R, z, S_1^i, (S_2^i)^{-1}, b, P_1 = \{w, w_{k-1}^i, S_1^i, b\}$, and $P_2 = \{w, w_{k-1}^i, T, b_{i}^k, b\}$. Therefore, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_k^k$ between $w$ and $b$. Please see Figure 9 for an illustration.

Case 2.2.2. If $m \geq 2$.

Let $z$ be a black vertex of $V(Q_{k,0}^k - N(w))$. In $Q_{k,1}^k$, according to the induction hypothesis, there exist $m$ internal disjoint paths $\{S_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{k,1}^k$ between $w$ and $z$ for $2 \leq m \leq 2n - 2$. So as in $Q_{k,1}^k$, there exist $m$ internal disjoint paths $\{T_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{k,1}^k$ between $z$ and $b$ for $2 \leq m \leq 2n - 2$. Let $T_0 = (z, y_0^i, T_0^i, w, T_0^i, b)$ and $T_1 = (z, y_0^i, T, b_{i}^k)$ for $1 \leq i \leq m - 1$ in $Q_{k,1}^k$. Case 2.2.2.1. If $b_{i}^k \notin V(S_{m-1})$. Without loss of generality, let $b_{i}^k \notin V(S_{m-1})$. In $Q_{k,1}^k$, we also let $S_0 = (w, x_0^i, e, S_0^i, y_0^i, z), S_1 = (w, S_1^i, y_0^i, z)$ for $1 \leq i \leq m - 2$, and $S_{m-1} = (w, S_{m-1}, b_{i}^k, f, S_{m-1}^i, y_{m-1}, z). A$ hamiltonian path $R$ is embedded in $Q_{k,1}^k$ between $w_{k-1}^i$ and $f_{k-1}$ by Theorem 1. Write $R$ as $(w_{k-1}^i, R_{i}^0, e_{i}^k, g, R_{i}^{k-1}, f_{k-1}). Notice that g_{k-2}$ is a black vertex and $b_{i}^k$ is a white vertex.
Let $R^i_p = \langle w^i, x^i_p, U^i_p, y^i_p, b^i \rangle$ for $0 \leq p \leq m - 1$ and $0 \leq i \leq j'$. According to Lemma 2, a hamiltonian path $S$ covers all vertices of $Q^k_{n-1}$ for $j' + 1 \leq k \leq 2$, joining from $w^j+1$ to $b^j+1$. There is a hamiltonian path $T$ of $Q^k_{n-1}$ from $w^k-1$ to $b^k-1$ by Theorem 1. Hence, we can write $P_\mu = \langle w = w^1, x^0_p, y^0_p, (U^1_p)^{-1}, x^1_p, y^1_p, b^1 \rangle$ for $0 \leq \mu \leq m - 1$, $P_m = \langle w = w^0, w^2, w^3, \ldots, w^j+1, S, b^j+1, b^j = b \rangle$, and $P_{m+1} = \langle w = w^{j+1}, T, b^{j+1}, b^j, \ldots, b^j+1, b^j = b \rangle$. Therefore, there are $m + 2$ internal disjoint paths $\{P_\mu\}_{\mu=0}^{m+1}$ whose union covers all vertices of $Q^k_{n}$ between $w$ and $b$. Please see Figure 12 for an illustration.

Fig. 12. The illustration for Case 3 of Theorem 3.

**Case 4.** For $|j' - j| \geq 2$. Without loss of generality, we let $j = 0$ and $1 \leq j' \leq \frac{j}{2} + 1$ be odd.

**Case 4.1.** If $m = 1$.

Choosing a black vertex $z$ of $Q^k_{n-1}$, Theorem 1, there is a hamiltonian path $R$ of $Q^k_{n-1}$ joining from $w$ to $z$. In $Q^{k+1}_{n-1}$, there exists a hamiltonian path $S$ of $Q^{k+1}_{n-1}$ between $w^{k+1}$ and $z$. We can let $S = \langle w^{k+1}, S', e, b^{k+1}, S', z \rangle$, where $b^{k+1}$ is a black vertex of $Q^{k+1}_{n-1}$, so $e$ is a white vertex of $Q^{k+1}_{n-1}$. By Theorem 1, there is a hamiltonian path $T$ of $Q^{k+2}_{n-1}$ joining from $w^{k+2}$ to $b^{k+2}$. Let $T = \langle w^{k+2}, T, W, f^{k+2}, b^{k+2} \rangle$. In $Q^{k+1}_{n-1}$, we also have a hamiltonian path $T'$ joining from $e$ to $b'$ for $1 \leq i \leq 3$, so we let $T' = \langle e, T, W, f', b' \rangle$. According to Lemma 1, there is a hamiltonian path $U$ between a black vertex $w^1 \in Q^k_{n-1}$ and a white vertex $b^j+1 \in Q^k_{n-1}$ covering all vertices of $Q^k_{n-1}$ for $2 \leq i \leq j' - 1$. We set $P_0 = \langle w, w^1, U, b^j+1, b \rangle$, $P_1 = \langle w, R, z, z^{x+1}, (S')^{-1}, b^k-1, b^k, b^k-1, b^k+1, b^k = b \rangle$, and $P_2 = \langle w, w^{k+1}, S', e, e^{k+2}, W, f^{k+2}, f^{k+3}, (W^{k+3})^{-1}, e^{k+3}, e^{k+4}, W^{k+4}, f^{k+4}, \ldots, f^{k+1}, f', W', f', b' = b \rangle$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ which union covers all vertices of $Q^k_{n}$ between $w$ and $b$. Please see Figure 13 for an illustration.

**Case 4.2.** If $m \geq 2$.

Given a white vertex $z$ in $Q^k_{n-1}$ such that $z$ is adjacent to $b$. So $z^0$ is a black (resp. white) vertex and $w^1$ is a white (resp. black) vertex of $Q^k_{n-1}$ if $0 \leq i \leq j' - 1$ when $i$ is even (resp. odd). By the induction hypothesis, there exist $m$ internal disjoint paths $\{R^i_{m-1}\}_{i=0}^{m-1}$ of $Q^k_{n-1}$ between $w$ and $z^0$. We write $R_0 = \langle w, x_0(1), x_0(2), \ldots, x_0(\alpha), z^0 \rangle$ and $R_m = \langle w, x_0, S, y_0, z^0 \rangle$ for $1 \leq p \leq m - 1$. Again, by the induction hypothesis, there exist $m$ internal disjoint paths $\{T^i_{p=m-1}\}_{i=0}^{m-1}$ of $Q^k_{n-1}$ between $w^i$ and $z^i$ for $2 \leq i \leq j' - 1$. We let $T_p = \langle w^i, x_p, U^i_p, t^i_p, z^i \rangle$ for $0 \leq p \leq m - 1$ and $2 \leq i \leq j' - 1$. Notice that $b^j+1$ is adjacent to $z^j+1$, without loss of generality, we let $t^{j+1}_m = 1$. In $Q^{j+1}_{n-1}$, there are $m$ internal disjoint paths $\{W_i\}_{i=0}^{m-1}$ from $b$ to $z$ by the induction hypothesis. We can write $W_p = \langle z, t^j_p, y_p, b \rangle$ for $0 \leq p \leq m - 2$ and $W_{m-1} = \langle z, b \rangle$. According to Lemma 1, there is a hamiltonian path $V$ between $w^{j+1} \in Q^{j+1}_{n-1}$ and $b^{j+1} \in Q^{j+1}_{n-1}$ covering all vertices of $Q^{j+1}_{n-1}$ for $j' + 1 \leq i \leq k - 1$. Set $P_0 = \langle w, w^{j+1}, V, b^{j+1}, b \rangle$, $P_1 = \langle w, w^1, U, b^{j+1}, b \rangle$, $P_2 = \langle w, w^{j+1}, V, b^{j+1}, b \rangle$, $P_3 = \langle w, w^1, U, b^{j+1}, b \rangle$, $P_4 = \langle w, w^{j+1}, V, b^{j+1}, b \rangle$, and $P_5 = \langle w, w^1, U, b^{j+1}, b \rangle$. Hence, there are 3 internal disjoint paths $\{P_{1}, P_2, P_3\}$ whose union covers all vertices of $Q^k_{n}$ between $w$ and $b$. Please see Figure 14 for an illustration.

**APPENDIX A**

**PROOF OF LEMMA 3**

Notice that $Q^2_1$ is vertex symmetric. W.L.O.G. let $w = (0, 0)$. There are only two cases for $b$. That is, $b \in \{(1, 0), (2, 1)\}$.

**Case 1.** To prove that $Q^2_1$ is 3*-laccceable.

Case 1.1. Let $b = (1, 0)$.

The three disjoint paths $\{P_1, P_2, P_3\}$ between $w$ and $b$ whose
union covers all vertices of $Q^2_3$ are $P_1 = \{(0,0),(1,0)\}$, $P_2 = \{(0,0),(0,1),(1,1),(1,0)\}$, and $P_3 = \{(0,0),(0,0),(0,0)\}$.

Case 1.2. Let $b = (2,1)$. The three disjoint paths $\{R_1, R_2, R_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_3$ are $R_1 = \{(0,0),(1,1),(2,0),(2,1)\}$, $R_2 = \{(0,0),(0,1),(1,1),(2,1)\}$, and $R_3 = \{(0,0),(0,0),(3,0),(2,0),(1,2),(2,2),(2,1)\}$.

Case 2. To prove that $Q^2_3$ is 4*-laceable.

Proof of Lemma 4

Notice that $Q^2_3$ is vertex symmetric. W.L.O.G, let $w = (0,0)$. There are four cases for $b$. That is, $b \in \{(1,0),(2,1),(3,0),(3,2)\}$.

Case 1. To prove that $Q^2_3$ is 3*-laceable.

Case 1.1. Let $b = (1,0)$. The three disjoint paths $\{P_1, P_2, P_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_3$ are $P_1 = \{(0,0),(1,0)\}$, $P_2 = \{(0,0),(0,1),(1,1),(1,0)\}$, and $P_3 = \{(0,0),(0,0),(0,0)\}$.

Case 1.2. Let $b = (2,1)$. The three disjoint paths $\{R_1, R_2, R_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_3$ are $R_1 = \{(0,0),(3,0),(3,1),(2,1)\}$, $R_2 = \{(0,0),(0,1),(2,0),(2,1)\}$, and $R_3 = \{(0,0),(0,0),(3,0),(2,0),(1,2),(1,3),(2,3),(3,3),(3,2),(2,2),(2,1)\}$.

APPENDIX B

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