The Spanning Laceability of $k$-ary $n$-cubes when $k$ is Even
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Abstract—$Q^n_k$ has been shown as an alternative to the hypercube family. For any even integer $k \geq 4$ and any integer $n \geq 2$, $Q^n_k$ is a bipartite graph. In this paper, we will prove that given any pair of vertices, $w$ and $b$, from different partite sets of $Q^n_k$, there exist $2n$ internally disjoint paths between $w$ and $b$, denoted by $(P_i | 0 \leq i \leq 2n - 1)$, such that $\bigcup_{i=0}^{2n-1} P_i$ covers all vertices of $Q^n_k$. The result is optimal since each vertex of $Q^n_k$ has exactly 2n neighbors.

Keywords—container, Hamiltonian, $k$-ary $n$-cube, $m^*$-connected.

I. INTRODUCTION
The $k$-ary $n$-cube, denoted by $Q^n_k$, has been proposed as an alternative to the hypercube since it shares many nice properties of $Q^n$, such as regular degrees, vertex symmetry, edge symmetry, recursive structure, etc. The underlying topology of many machines is based on $k$-ary $n$-cubes, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1], [4], [11], [17]. Many researchers have been working on $k$-ary $n$-cubes. For example, Stewart and Xiang [20] proved that the $k$-ary $n$-cube is edge-bipancyclic and bipanconnected for $k \geq 3$ and $n \geq 2$ and $k$ being even. Namely, any edge of a $k$-ary $n$-cube $Q^n_k$ lies on a cycle of any even length $r$ for $4 \leq r \leq |Q^n_k|$, where $|Q^n_k|$ is the total number of vertices of $Q^n_k$. Besides, given two vertices $u$ and $v$ of $Q^n_k$, there exists a path of any even length $r$ between $u$ and $v$ for $d(u, v) \leq r \leq |Q^n_k|$, where $d(u, v)$ is the distance between $u$ and $v$. Other studies about fault tolerance on $k$-ary $n$-cubes can be found in [8], [23]. Recently, there are many studies about the spanning connectivity for interconnection networks and graphs [9]. A graph $H = (B \cup W, E)$ is bipartite if $V(H)$ is the union of two disjoint sets $B$ and $W$ such that every edge joins $B$ with $W$. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected except $K_2$. Note that any (nontrivial) bipartite graph except $K_2$ cannot be hamiltonian connected, whereas a bipartite graph is hamiltonian laceable if there exists a hamilton path between any two vertices, $u$ and $v$, with $u \in B$ and $v \in W$ [22]. A graph $H = (B \cup W, E)$ is a balanced bipartite graph if $|V(B)| = |V(W)|$. Throughout this thesis, we only work on $Q^n_k$ with $k \geq 4$ an even integer and $n \geq 2$, which are balanced bipartite graphs. A bipartite graph $H = (B \cup W, E)$ is $m^*$-laceable if given a white vertex $w \in W$ and a black vertex $b \in B$, there exist(s) $m$ internal disjoint paths between $w$ and $b$, denoted by $P_i$ for $0 \leq i \leq m - 1$, such that $\bigcup_{i=0}^{m-1} P_i$ covers $V$. The spanning laceability of a graph $H$, $\kappa^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for every $m$ with $1 \leq m \leq k$. A higher spanning connectivity/laceability of the interconnection network implies a more efficient communication between processors. About the spanning connectivity and the spanning laceability, readers can refer to [6], [7], [12]–[15].

In this paper, we want to show the spanning laceability of $k$-ary $n$-cubes for any even integer $k \geq 4$. More precisely, we show that given a white vertex $w$ and a black vertex $b$ of a $k$-ary $n$-cube $Q^n_k$, there exist(s) $m$ internally disjoint path(s) between $w$ and $b$ whose union covers all vertices of $Q^n_k$ for $1 \leq m \leq 2n$. The result is optimal since any vertex in $Q^n_k$ has exactly $2n$ neighbors. This paper is organized as follows. In Section 2, we introduce the graph terminologies and symbols that will be used in the paper and the definition of $Q^n_k$. In Section 3, we show our main results.

II. PRELIMINARIES
Throughout this paper, we follow [3] for the graph definitions and notations. The sets of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. If $u, v$ are vertices of a graph $G$ such that there is an edge $e = (u, v) \in E(G)$ between $u$ and $v$, then we say that the vertices $u$ and $v$ are adjacent in $G$. The degree of any vertex $x$ is the number of distinct vertices adjacent to $x$. A path $P$ between two vertices $v_0$ and $v_k$ is represented by $P = \langle v_0, v_1, \ldots, v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We use $P^{-1}$ to denote the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0 \rangle$. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We use $P^{-1}$ to denote the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0 \rangle$. A hamiltonian path between $u$ and $v$, where $u$ and $v$ are two distinct vertices of $G$, is a path joining $u$ to $v$ that visits every vertex of $G$ exactly once. A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of $G$ is a cycle that traverses every vertex of $G$ exactly once. A hamiltonian graph is a graph with a hamiltonian cycle. A graph $G$ is connected if there is a path between any two distinct vertices in $G$ and is hamiltonian connected if there is a hamiltonian path between any two distinct vertices in $G$ [18]. A graph $H = (W \cup B, E)$ is bipartite if $V(H) = W \cup B$ and $E(H)$ is a subset of $\{ \langle u, b \rangle | u \in W, b \in B \}$. A bipartite graph $H$ is hamiltonian laceable if there is a hamiltonian path between any two distinct vertices from different partite sets in $H$.

A graph $G$ is $k$-connected if there exists $V' \subseteq V(G)$ with $|V'| = k$ such that $G - V'$ is disconnected and $G - V''$ is
connected for any $V'' \subseteq V(G)$ with $|V''| < k$. It follows from Menger’s Theorem [16] that for every $k$-connected graph $G$, there exist $k$ internally vertex-disjoint paths between any pair of distinct vertices of $G$. A $k$-container $C(u, v)$ in a graph $G$ is a set of $k$ internally vertex-disjoint paths between two distinct vertices $u$ and $v$. We say that a graph $G$ has a spanning $k$-container between $u$ and $v$, denoted by $C(u, v)$, if $C(u, v)$ is a $k$-container that covers all vertices of $G$. A spanning $k$-container is also abbreviated as a $k^*$-container for simplicity. A graph $G$ is $k^*$-connected if there is a $k^*$-container between any pair of vertices of $G$. Obviously, a graph $G$ is hamiltonian connected if and only if $G$ is $1^*$-connected, and $G$ is hamiltonian if and only if $G$ is $2^*$-connected. Lin et al. [13] defined the concept of spanning connectivity. The spanning connectivity of a graph $G$, $\kappa^*(G)$, is the largest integer $k$ such that $G$ is $w^*$-connected for all $1 \leq w \leq k$. Similarly, a bipartite graph $H$ is $k^*$-laceable if there is a $k^*$-container between any pair of two vertices from different partite sets of $H$. Also, a bipartite graph $H$ is hamiltonian laceable if and only if $H$ is $1^*$-laceable, and $H$ is hamiltonian if and only if $H$ is $2^*$-laceable. So, the spanning laceability of a bipartite graph $H$, $\kappa^*(H)$, is the largest integer $k$ such that $H$ is $m^*$-laceable for all $1 \leq m \leq k$.

The $k$-ary $n$-cube, $Q_n^k$, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass $Q^2_n$ is the well-studied hypercube family. The subclass $Q^k_n$ with $k \geq 3$ is defined as the cycle of length $k$. The $k$-ary $n$-cube, $Q_n^k$, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q^k_n)$ be represented by $(u(0), u(1), \ldots, u(n-1))$, where $0 \leq u(i) \leq k-1$. Two vertices $u$ and $v$ are adjacent if and only if $|u(i) - v(i)| = 1$ or $k - 1$ for some $i$ and $u(j) = v(j)$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is shown that $Q^k_n$ is bipartite if $k$ is even [10]. Here we just note some properties of $Q_n^k$ that will be used in this paper.

$Q_n^k$ is vertex symmetric (and edge symmetric) [10]. It means that given any two distinct vertices $v$ and $v'$ of $Q_n^k$, there is an automorphism of $Q_n^k$ mapping $v$ to $v'$. Note that each vertex of $Q_n^k$ is represented by a n-bit tuple. We will call the $d$thbit the $d$th dimension. We can partition $Q_n^k$ over dimension $d$ by fixing the $d$th element of any vertex tuple at some value $a$ for every $a \in \{0, 1, \ldots, k-1\}$. This results in $k$ copies of $Q_{n-1}^{k_0}$, denoted by $Q_{n-1}^{k_0,1}, Q_{n-1}^{k_0,2}, \ldots, Q_{n-1}^{k_0,k-1}$, with corresponding vertices in $Q_n^k, Q_n^k, \ldots, Q_n^k$, joined in a cycle of length $k$ (in dimension $d$) [19].

In this article, we always partition $Q_n^k$ over the 0-th dimension by letting $V(Q_n^{k,0}) = \{(i, v(1), v(2), \ldots, v(n-1)) | 0 \leq v(j) \leq k-1, 1 \leq j \leq n-1\}$ for $0 \leq i \leq k-1$. Given a vertex $x = (x(0), x(1), x(2), \ldots, x(n-1)) \in V(Q_n^{k,0})$, the symbol $x^j = (x(0), x(1), x(2), \ldots, x(n-1))$, where $0 \leq j \leq k-1$, is defined to be the vertex corresponding to $x$ in $Q_n^{k,0}$ for simplicity. So, if $P = (x_0, x_1, \ldots, x_{n-1})$, $P_j$ is represented by $(x_0^j, x_1^j, \ldots, x_{n-1}^j)$. Throughout this paper, let $n \geq 2$ be an integer and $k \geq 4$ an even integer.

**Theorem 1.** [10] For any even integer $k \geq 4$, $Q_n^k$ is hamiltonian laceable for $n \geq 2$. In other words, $Q_n^k$ is $1^*$-laceable.

**Theorem 2.** [5] The graph $Q_n^k$ is hamiltonian. In other words, $Q_n^k$ is $2^*$-laceable.

**III. MAIN RESULTS**

**Lemma 1.** Given $Q_n^k$ and its $k$ subcubes, $Q_n^{k,i}$, where $0 \leq i \leq k-1$. Let $j$ and $j'$ be two integers satisfying $0 \leq j \leq j' \leq k-1$, $w \in V(Q_n^{k,j})$ an arbitrary white vertex, and $b \in V(Q_n^{k,j'})$ an arbitrary black vertex. Then there exists a path between $w$ and $b$ that visits each vertex in $Q_n^{k,j}, Q_n^{k,j'+1}, Q_n^{k,j'+2}, \ldots, Q_n^{k,j'}$, exactly once.

**Proof:** There are three cases.

**Case 1.** $j = j'$. W.L.O.G., let $j = j' = 0$. By Theorem 1, $Q_n^{k,0}$ is hamiltonian laceable. Thus, there is a hamiltonian path between $w$ and $b$ that visits each vertex of $Q_n^{k,0}$ exactly once.

**Case 2.** $j - j' = 1$. W.L.O.G., we can let $j = 0$ and $j' = 1$. Let $w$ be a white vertex in $Q_n^{k,0}$, and $b$ be a black vertex in $Q_n^{k,1}$. We can find a pair of adjacent vertices $x^0, y^0$ and $x^1, y^1$ where $x^0$ is a black vertex of $Q_n^{k,0}$ and $x^1$ is a white vertex of $Q_n^{k,1}$. By Theorem 1, there exists a hamiltonian path $P_0$ of $Q_n^{k,1}$ between $w$ and $x^1$, and a hamiltonian path $P_1$ of $Q_n^{k,1}$ between $x^1$ and $b$. Let $P = (w, P_0, x^1, P_1, b)$. Hence $P$ is the path between $w$ and $b$ that visits every vertex of $Q_n^{k,1}$ and $Q_n^{k,0}$ exactly once.

**Case 3.** $j - j' \geq 2$. Let $w$ be a white vertex in $Q_n^{k,j}$, and $b$ be a black vertex in $Q_n^{k,j'}$. There are $j - j' + 1$ any $n - 1$-cubes, $Q_n^{k,j+1}, Q_n^{k,j+2}, \ldots, Q_n^{k,j'}$, and $Q_n^{k,j'}$. There are $j - j'$ pairs of adjacent vertices $x^r = Q_n^{k,r+1}$ and $y^r+1 = Q_n^{k,r+1}$ where $x^r$ is a black vertex and $y^r+1$ is a white vertex for $j' \leq r < j' - 1$. By Theorem 1, there is a hamiltonian path $R_{j'}$ of $Q_n^{k,j'}$ joining $y^r$ to $x^r$, where $j + 1 \leq r \leq j' - 1$. Again, with Theorem 1, there exists a hamiltonian path $T$ of $Q_n^{k,j'}$ joining $w$ to $x^1$, and a hamiltonian path $U$ of $Q_n^{k,j}$ joining $y^0$ to $b$. Let $P = (w, T, x^1, y^1, R_{j+1}, x^{j'1}, y^{j'+1}, R_{j'+2}, x^{j'2}, y^{j'+2}, \ldots, y^{j'+1}, R_{j'+1}, x^1, y^0, U, b)$. Therefore, $P$ is a path covering all the vertices of $Q_n^{k,j}, Q_n^{k,j+1}, Q_n^{k,j+2}, \ldots, Q_n^{k,j'}$, for $0 \leq j \leq j' \leq k-1$ between $w$ and $b$. Please see Figure 1 for an illustration.

**Fig. 1.** The illustration for Case 3 of Lemma 1.

**Lemma 2.** Given $Q_n^k$ and its $k$ subcubes $Q_n^{k,i}$ for $0 \leq i \leq k-1$. Let $w$ be a white vertex, $b$ a black vertex in $Q_n^{k,i}$, and $j$ an integer with $0 \leq j \leq k-1$. There exists a path between $w$ and $b$ that covers all the vertices of $Q_n^{k,j}, Q_n^{k,j+1}, \ldots, Q_n^{k,j}$.

**Proof:** We consider the following two cases.
Lemma 4. If $i \langle -i, y_i w_i, z_j w_j, x_j \rangle_i T_i = (14)$, then $i, j \leq k - 1$, then

$$ f((i, j)) = \begin{cases} 
(i, j) & \text{if } 0 \leq i, j \leq k - 2; \\
(i + 2, j) & \text{if } i = k - 1, 0 \leq j \leq k - 2; \\
(i, j + 2) & \text{if } j = k - 1, 0 \leq i \leq k - 2; \\
(i + 2, j + 2) & \text{if } i = k - 1 = j.
\end{cases} $$

Lemma 3. The graph $Q^4_2$ is 3'-laceable and 4'-laceable.

Proof: The proof is by brute force. Reader can refer to Appendix A.

Lemma 4. The graph $Q^4_2$ is 3'-laceable and 4'-laceable.

Proof: By brute force, we constructed all spanning containers. Please see Appendix B.

Lemma 5. The graph $Q^3_k$ is 3'-laceable and 4'-laceable for any even integer $k \geq 6$.

Proof: With Lemma 4, we have shown that $Q^3_k$ is 3'-laceable and 4'-laceable. Now we will present a recursive algorithm that uses a 3'-container (resp. 4'-container) of $Q^3_k$ to construct a 3'-container (resp. 4'-container) of $Q^3_{k+2}$. Let $R$ be a subset of $V(Q^3_k) \cup E(Q^3_k)$. Define a function, $f$, which maps $R$ from $Q^3_k$ into $Q^3_{k+2}$ in the following way:

1. If $(i, j) \in R \cap V(Q^3_k)$, where $0 \leq i, j \leq k - 1$, then

$$ f((i, j)) = \begin{cases} 
(i, j) & \text{if } 0 \leq i, j \leq k - 3; \\
(i + 2, j) & \text{if } j = k - 1, 0 \leq i \leq k - 3; \\
(i, j + 2) & \text{if } i = k - 1, 0 \leq j \leq k - 3; \\
(i + 2, j + 2) & \text{if } i = k - 1 = j.
\end{cases} $$

2. If $(i, j), (i', j') \in R \cap E(Q^3_k)$, where $i \leq i', j \leq j'$, then $f(((i, j), (i', j'))) = ((i, j), (i', j'))$.

Let $w$ be a white vertex and $b$ be a black vertex of $Q^3_k$. We say that a 3'-container (resp. 4'-container) $C(u, v)$ of $Q^3_k$ is regular if $C(w, b)$ contains some edges in $\{(\alpha, k - 2, \beta, k - 1) \mid 0 \leq \alpha, \beta \leq k - 1\}$ and $\{(\alpha - 2, \beta, k - 1, \beta) \mid 0 \leq \beta \leq k - 1\}$. For example, all 3'-containers and 4'-containers of $Q^3_2$ constructed in Lemma 4 are regular. Let $C(w, b)$ be a regular 3'-container (resp. 4'-container) of $Q^3_2$ with the endvertex set $P = \{w = (0, 0), b = (x, y)\}$. We construct a regular 3'-container (resp. 4'-container) of $Q^3_{k+2}$ with the endvertex set $f(P)$ using the following algorithm. Please see Figure 4 for an illustration.

Fig. 4. Using the 4'-container of $Q^3_k$ to construct the 4'-container of $Q^3_{k+2}$.

Step 1. In $Q^3_k$, let $\{v_0, v_1, \ldots, v_{k-1}\}$ and $\{h_0, h_1, \ldots, h_{k-1}\}$ be finite sequences of indices satisfying the following requirements:

1. $0 \leq v_0 < v_1 < \ldots < v_{k-1} \leq k - 1$ and $k - 1 \geq h_0 > h_1 > \ldots > h_{k-1} \geq 0$;
(2) for $0 \leq i \leq k - 1$, $(v_i, k-2), (v_i, k-1)$) is an edge of $C(w, b)$; for $0 \leq j \leq k - 1$, $(k-2, h_j), (k-1, h_j))$ is an edge of $C(w, b)$.

**Step 2.** Let $\overline{C}(w, b)$ be the image in $Q^{k+2}$ of $C(w, b) - \{(v_i, k-2), (v_i, k-1)) | 0 \leq i \leq k-1 \} \cup \{(k-2, h_j), (k-1, h_j)) | 0 \leq j \leq k-1 \}$ under the function $f$. Please see Figure 5 for an illustration.

**Step 3.** For any two positive integers $r$ and $d$, we use $[r]_d$ to denote $r \pmod{d}$. In $Q^{k+2}$, define the following path patterns, where $r_1, r_2$ are integers:

$$I_\alpha(r_1, r_2) = \langle (r_1, \alpha), ([r_1] + 1)_{k+2}, \alpha, \ldots, (r_2, \alpha) \rangle$$

$$I_\alpha^*(r_1, r_2) = \langle (r_2, \alpha), ([r_2] - 1)_{k+2}, \alpha, \ldots, (r_1, \alpha) \rangle$$

$$H_{\beta}(r_1, r_2) = \langle (\beta, r_1), (\beta, [r_1] + 1)_{k+2}, \ldots, (\beta, r_2) \rangle$$

$$H_{\beta}^*(r_1, r_2) = \langle (\beta, r_2), (\beta, [r_2] - 1)_{k+2}, \ldots, (\beta, r_1) \rangle$$

Let $v_i = v_i + 2$ if $v_i = k - 1$ and $\pi_i = v_i$ if $0 \leq v_i \leq k - 2$, and $h_j = h_j + 2$ if $h_j = k - 1$ and $h_j = h_j + 1$ if $h_j < k - 2$.

**Case 1.** $v_0 = k - 1$.

Let $P_0 = \{(k+1, k-2), (k+1, k-1), (0, k-1), h_0 = (0, k-2), (k-2, k), (k-2, k), I_{\alpha}^*(k-2, 0), (0, k), (k, k+1, k), (k+1, k+1)\}$.

**Case 1.1.** $s = 1$.

Let $P_0 = \{(k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, \bar{h}_0 + 1)_{k+2}, (k-1, \bar{h}_0 + 1)_{k+2}, (\bar{h}_0, \bar{h}_0 + 1)_{k+2}, H_{k-1}^{-1}(\bar{h}_0, \bar{h}_0 + 1)_{k+2}, (k, \bar{h}_0 + 1)_{k+2}, H_{k-1}^{-1}(\bar{h}_0, \bar{h}_0 + 1)_{k+2}, (k, \bar{h}_0)\}$. Then $\overline{C}(w, b) \cup P_0 \cup P_0$ is the $3^\text{rd}$-container (or $4^\text{th}$-container) of $Q^{k+2}$.

**Case 1.2.** $s \geq 2$.

Let $P_0 = \{(k-2, \bar{h}_0), (k-1, \bar{h}_0), H_{k-1}^{-1}(\bar{h}_0, \bar{h}_0 + 1)_{k+2}, (k-1, \bar{h}_0 + 1), H_{k-1}^{-1}(\bar{h}_0, \bar{h}_0 + 1)_{k+2}, (k, \bar{h}_0 + 1)_{k+2}, H_{k-1}^{-1}(\bar{h}_0, \bar{h}_0 + 1)_{k+2}, (k, \bar{h}_0)\}$. Then $\overline{C}(w, b) \cup P_0 \cup P_0$ is the $3^\text{rd}$-container (or $4^\text{th}$-container) of $Q^{k+2}$.

**Case 2.** $v_1 = k - 2$ and $(k-2, k-1), (k-1, k-1) \in E(C(w, b))$ in $Q^{k+2}$.

**Case 2.1.** $t = 1$.

Let $P_0 = \{(\bar{v}_0, k-2), (\bar{v}_0, k-1), I_{k-1}^{-1}(\bar{v}_0, k-2), (k-2, k-1), (k-2, k), I_{k-1}^{-1}(k-2, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1)\}$.

**Case 2.1.1.** $s = 1$.

Let $P_0 = \{(k-2, \bar{v}_0), (k-1, \bar{v}_0), H_{k-2}^{-1}(\bar{v}_0, \bar{v}_0), (0, k), (k, k-1), (k, k-1), I_{k-1}^{-1}(k-1, \bar{v}_0), (\bar{v}_0, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1)\}$.

**Case 2.1.2.** $s \geq 2$.

Let $P_0 = \{(k-2, \bar{v}_0), (k-1, \bar{v}_0), H_{k-2}^{-1}(\bar{v}_0, \bar{v}_0), (0, k), (k, k-1), (k, k-1), I_{k-1}^{-1}(k-1, \bar{v}_0), (\bar{v}_0, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1)\}$.

**Case 3.** $v_1 \leq k - 2$ and $(k-2, k-1), (k-1, k-1) \not\in E(C(w, b))$ in $Q^{k+2}$.

**Case 3.1.** $t = 1$.

Let $P_0 = \{(\bar{v}_0, k-2), (\bar{v}_0, k-1), I_{k-1}^{-1}(\bar{v}_0, k-2), (k-2, k-1), (k-2, k), I_{k-1}^{-1}(k-2, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1)\}$.

**Case 3.1.1.** $s = 1$.

Let $P_0 = \{(k-2, \bar{v}_0), (k-1, \bar{v}_0), H_{k-2}^{-1}(\bar{v}_0, \bar{v}_0), (0, k), (k, k-1), (k, k-1), I_{k-1}^{-1}(k-1, \bar{v}_0), (\bar{v}_0, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1)\}$.

**Case 3.1.2.** $s \geq 2$.

Let $P_0 = \{(k-2, \bar{v}_0), (k-1, \bar{v}_0), H_{k-2}^{-1}(\bar{v}_0, \bar{v}_0), (0, k), (k, k-1), (k, k-1), I_{k-1}^{-1}(k-1, \bar{v}_0), (\bar{v}_0, \bar{v}_0), (\bar{v}_0, k), (\bar{v}_0, k+1)\}$.
for $0 \leq i \leq s - 2$, and $\mathcal{P}^m_{i+1} = \{(k-2, \tau_{i+1}), (k-1, \tau_{i+1}), H_{k-1}^- (\tau_{i+1} - 0), (k-1, 0), (k, 0), H_{k} (0, \tau_{i+1} - 1), (k, \tau_{i+1} - 1), (k+1, \tau_{i+1})\}$. Then $\mathcal{C}(w, b) \cup \mathcal{P}_0 \cup \mathcal{P}^m_{i+1} \{0 \leq i \leq s - 1\}$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 3.2. $t \geq 2$.

Let $P_t = \{(\tau_i, k - 2), (\tau_i, k - 1), I_{k-1} (\tau_i, k - 2), (\tau_i, k - 1), I_{k-1} (\tau_i, k - 1), (k-1, k - 1), H_{k}^- (\tau_i - 1), (k, \tau_i + 1), (k, \tau_i), (k, 1), (k-1, \tau_i)\}$ for $0 \leq i \leq t - 2$, and $P_{t+1} = \{(\tau_i, k - 2), (\tau_i, k - 1), I_{k-1} (\tau_i, k - 1), (k-1, k - 1), H_{k}^- (\tau_i - 1), (k, \tau_i + 1), (k, \tau_i), (k, 1), (k-1, \tau_i)\}$ (for $0 \leq i \leq s - 1$) is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 3.2.1. $s = 1$.

Using the same $\mathcal{P}_0$ as in Case 3.1.1, then $\mathcal{C}(w, b) \cup \{P_t \mid 0 \leq t \leq 1\} \cup \mathcal{P}_0$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 3.2.2. $s \geq 2$.

Using the same $\mathcal{P}_1 \mid 0 \leq i \leq s - 1$ as in Case 3.2.2., then $\mathcal{C}(w, b) \cup \{P_t \mid 0 \leq t \leq 1\} \cup \mathcal{P}_0 \mid 0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 4. $\nu_t = k - 1$ for some $t \geq 2$ and $\nu_0 = 0$.

Case 4.1. $t = 2$.

Let $P_0 = \{(\tau_0, k - 2), (\tau_0, k - 1), I_{k-1} (\tau_0, k - 2), (\tau_0, k - 1), I_{k-1} (\tau_0, k - 1), (k-2, k - 1), (k-1, k), (k, \nu_0 + 1), (k, \nu_0), (k, \nu_0 - 1), (k-1, k), (k, 1), (k, 0), (k, 1), (k, 1), (k, k)\}$ and $P_1 = \{(k-1, k - 2), (k, k - 1), (k, 1), (k, 0), (k, 1), (k, 1), (k, k)\}$.

Case 4.1.1. $s = 1$.

Using the same $\mathcal{P}_0$ as in Case 1.1, then $\mathcal{C}(w, b) \cup \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_0$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 4.1.2. $s \geq 2$.

Using the same $\mathcal{P}_1 \mid 0 \leq i \leq s - 1$ as in Case 1.2., then $\mathcal{C}(w, b) \cup \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_0 \mid 0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 4.2. $t \geq 3$.

Let $P_t = \{(\tau_i, k - 2), (\tau_i, k - 1), I_{k-1} (\tau_i, k - 2), (\tau_i, k - 1), I_{k-1} (\tau_i, k - 1), (k-1, k - 1), (k-1, k), (k, \nu_0 + 1), (k, \nu_0), (k, \nu_0 - 1), (k-1, k), (k, 1), (k, 0), (k, 1), (k, 1), (k, k)\}$ and $P_{t-1} = \{(k-1, k - 2), (k-2, k - 1), (k-2, k), (k-2, k-1), (k-2, k-2), (k-2, k-3), (k-2, k-4)\}$.

Case 4.2.1. $s = 1$.

Using the same $\mathcal{P}_0$ as in Case 1.1, then $\mathcal{C}(w, b) \cup \{P_t \mid 0 \leq t \leq 1\} \cup \mathcal{P}_0$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Case 4.2.2. $s \geq 2$.

Using the same $\mathcal{P}_1 \mid 0 \leq i \leq s - 1$ as in Case 1.2., then $\mathcal{C}(w, b) \cup \{P_t \mid 0 \leq i \leq t - 1\} \cup \mathcal{P}_0 \mid 0 \leq i \leq s - 1$ is the 3*-container (or 4*-container) of $Q_k^{*+2}$.

Theorem 3. For any integer $n \geq 2$ and any even integer $k \geq 4$, the graph $Q^n_k$ is $m$*-laceable where $1 \leq m \leq 2n$.

Proof: According to Theorem 2-3 and Lemma 3-5, the theorem holds for any even integer $k \geq 4$ when $n = 2$. We will give the proof of the theorem by mathematical induction on $n$. By induction hypothesis, assume that $Q^{n+1}_k$ is $m$*-laceable for $1 \leq m \leq 2n - 2$, where $0 \leq i \leq k - 1$. Given a white vertex $w \in V(Q^{n+1}_k)$ and a black vertex $b \in V(Q^{n+1}_k)$. We will show that we can use the $m$*-container of $Q^{n+1}_k$, to construct a $(m+2)^*$-container of $Q^n_k$ between $w$ and $b$.

Case 1. For $j = j'$. Without loss of generality, we let $j = j' = 0$.

In this case, we have $(w, b) \in Q^n_k$. By induction hypothesis, there are $m$ internal disjoint paths $\{P_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q^n_k$ between $w$ and $b$. We can let $P_m = (w, w, b, b, b)$. In $Q^n_{k+1}$, there exist a hamiltonian path $R$ joining from $w^{k+1}$ to $b^{k+1}$ by Theorem 1. Also, we can let $P_{m+1} = (w, w^{k+1}, R, b^{k+1}, b)$. Therefore, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q^n_k$ between $w$ and $b$. Please see Figure 6 for an illustration.

![Figure 6](image-url) 

Case 2. For $j' - j = 1$. Without loss of generality, we let $j = 0$ and $j' = 1$.

We have the following two cases.

Case 2.1. Suppose that $d(w, b) = 1$. It is easy to see that we can let $P_{m+1} = (w, b)$.

Case 2.1.1. Suppose that $m = 1$.

Let $z$ be any black vertex of $Q^n_k$. By Theorem 1, there exist a hamiltonian path $S$ of $Q^{k+1}_n$ from $w$ to $z$, and a hamiltonian path $T$ of $Q^{k+1}_n$ from $z$ to $b$. So we set $P_0 = (w, S, z, z, T, b)$. According to Lemma 1, a hamiltonian path $R$ between $w^{k+1}$ and $b^{k+1}$ in $Q^{k+1}_n$. We can write $P_1$ as $(w^{k+1}, R, b^{k+1})$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q^n_k$ between $w$ and $b$. Please see Figure 7 for an illustration.

![Figure 7](image-url) 

**Figure 6.** The illustration for Case 1 of Theorem 3.

**Figure 7.** The illustration for Case 2.1.1 of Theorem 3.
Lemma 1, there is a Hamiltonian path $T$ between $u^{k-1} \in Q_{k-1} \cup b^2 \in Q_{k-1}^2$ covering all vertices of $Q_{k-1}^i$ for $2 \leq i \leq k-1$. Set $P_m = (w, w^{k-1}, T, b^2, b)$. Consequently, there are $m + 2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_{n-1}^i$ between $w$ and $b$. Please see Figure 8 for an illustration.

According to Lemma 1, there is a Hamiltonian path $U$ between $b^3 \in Q_{k-1} \cup b^2$ covering all vertices of $Q_{n-1}^i$ for $2 \leq i \leq k-2$. We can set $P_0 = (w, x^0, T_{m-1}^i, y^i, y^i, y_{m-1}, T_{m-1}, b)$. $P_1 = (w, z^1, T_{n-1}^i, b)$. $P_2 = (w, b^1, T_{n-1}^i, b)$. $P_3 = (w, S_{n-1}, b^3, y^i, y^i, y_{n-1}, T_{n-1}, b)$. $P_4 = (w, S_{n-1}, y^i, y^i, y_{n-1}, T_{n-1}, b)$ for $4 \leq i \leq m + 1$. So there are $m + 2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_{n-1}^i$ between $w$ and $b$. Please see Figure 10 for an illustration.

Case 2.2. Suppose that $d(w, b) \geq 3$.

Case 2.2.1. If $m = 1$.

Given any black vertex $z$ in $Q_{n-1}^i$, by Theorem 1, there is a Hamiltonian path $P$ of $Q_{n-1}^i$ joining from $w$ to $z$. So there is also a Hamiltonian path $Q_{n-1}^i$ between $w$ to $z$. We can set $S = \langle w^1, S^1, b^2, z \rangle$. By Lemma 1, there is a Hamiltonian path $T$ between $w^{k-1} \in Q_{k-1} \cup b^3 \in Q_{k-1}^2$ covering all vertices of $Q_{n-1}^i$ for $2 \leq i \leq k-1$. We let $P_0 = (w, w, z, z^1, S^1_1, b, b_2, z^1)$. By Lemma 1, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of $Q_{n-1}^i$ between $w$ and $b$. Please see Figure 9 for an illustration.

Case 2.2.2. If $b^0 \in V(S_0)$.

Let $S_0 = \langle w, x^0, e, S^0, b^0, f, S^0, y^0, z \rangle$, and $S_1 = \langle w, S^1, y^1, z \rangle$ for $1 \leq i \leq m$. A Hamiltonian path $R$ is embedded in $Q_{n-1}^i$ between $w^{k-1}$ and $b^1$ by Theorem 1. Let $P_0 = \langle w, x^0, T_{m-1}^i, y^i, y^i, y_{m-1}, T_{m-1}, b \rangle$. $P_1 = \langle w, w^{k-1}, R, e^{k-1}, e, S^1_0, b \rangle$. $P_3 = \langle w, S^1_0, y^i, y^i, y_{m-1}, T_{m-1}, b \rangle$. $P_4 = \langle w, S^1_0, y^i, y^i, y_{m-1}, T_{m-1}, b \rangle$ for $4 \leq i \leq m + 1$. Hence, there are $m + 2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of $Q_{n-1}^i$ between $w$ and $b$. Please see Figure 11 for an illustration.

Case 3. For $|y^i - j| \geq 2$. Without loss of generality, let $j = 0$ and $2 \leq \frac{j}{2}$ be even. Because $b \in Q_{k-1} \cup b^3$ where $j^i$ is even, $b^i$ is a white (resp. black) vertex in $Q_{k-1}^i$ for $0 \leq i \leq k-1$ when $i$ is odd (resp. even). It is easy to see that $w^i$ is a black (resp. white) vertex in $Q_{n-1}^i$ for $0 \leq i \leq k-1$ when $i$ is odd (resp. even). By the induction hypothesis, there exist $m$ internal disjoint paths $\{R^m_{i=0} \}_{i=0}^{m+1}$ of $Q_{n-1}^i$ between $w^i$ and $b^i$ for $0 \leq i \leq j^i$.
Let \( R^i_p = (w^i, x^i_p, U^i_p, y^i_p, b^i_p) \) for \( 0 \leq p \leq m - 1 \) and \( 0 \leq i \leq j' \). According to Lemma 2, a hamiltonian path \( S \) covers all vertices of \( Q_{n-1}^{k,i} \) for \( j' + 1 \leq k \leq k - 2 \) joining from \( w^{j'+1} \) to \( b^{j'+1} \). There is a hamiltonian path \( T \) of \( Q_{n-1}^{k,i} \) from \( w^{j'+1} \) to \( b^{j'+1} \) by Theorem 1. Hence, we can write \( P_{m} = (w = w_0, x_0, y_0, U_0, b_0) \) for \( 0 \leq p \leq m - 1 \), \( P_{m} = (w = w_0, w_1, w_2, \ldots, w_0, w^{j'+1}, b^{j'+1}, b^j = b) \), and \( P_{m+1} = (w = w_0, \ldots, w^{j'+1}, T, b^{j'+1}, b_0, \ldots, b^{j'-1}, b^i = b) \). Therefore, there are \( m + 2 \) internal disjoint paths \( \{P_{m}\}_{i=0}^{m-1} \) whose union covers all vertices of \( Q_{n}^{k,i} \) between \( w \) and \( b \). Please see Figure 12 for an illustration.

**Case 4.** For \( |j' - j| \geq 2 \). Without loss of generality, we let \( j = 0 \) and \( 0 \leq j' \leq \frac{1}{2} + 1 \) be odd.

**Case 4.1.** If \( m = 1 \).

Choosing a black vertex \( z \) of \( Q_{n-1}^{k,0} \), by Theorem 1, there is a hamiltonian path \( R \) of \( Q_{n-1}^{k,0} \) joining from \( w \) to \( z \). In \( Q_{n-1}^{k,0} \), there exists a hamiltonian path \( S \) of \( Q_{n-1}^{k,1} \) between \( w^{k-1} \) and \( z^{k-1} \). We can let \( S = (w^{k-1}, S^i, e, b^{k-1}, S^j, z^{k-1}) \), where \( b^{k-1} \) is a black vertex of \( Q_{n-1}^{k,1} \), so \( e \) is a white vertex of \( Q_{n-1}^{k,1} \). By Theorem 1, there is a hamiltonian path \( T \) of \( Q_{n-1}^{k,1} \) from \( k-2 \) to \( b^{k-2} \). Let \( T = (w^{k-2}, W, f^{k-2}, b^{k-2}) \). In \( Q_{n-1}^{k,1} \), we also have a hamiltonian path \( P \) between \( e^{k-1} \) and \( b' \) for \( j' \leq i \leq k - 3 \), so we let \( T = (e^{k-1}, W, f^{k-1}, b') \). According to Lemma 1, there is a hamiltonian path \( U \) between a black vertex \( w^{k-1} \), and a white vertex \( b^{k-1} \) covering all vertices of \( Q_{n-1}^{k-1} \) for \( 2 \leq i \leq j' - 1 \). We set \( P_0 = (w, w^3, U, b^{j'+1}, b) \), \( P_1 = (w, R, z, w^{k-1}, (S^{j-1}), b^{k-1}, b^{k-1}, e^{k-2}, w^{k-3}, f^{j-1}, f^{j-1}, b') \), \( P_2 = (w, w^{k-1}, S^{j-1}, e^{k-2}, W, f^{j-2}, f^{j-3}, (W^{j-3})^{k-1}, e^{j-3}, W^{k-4}, f^{j-4}, \ldots, e^{j-1}, W^{j-1}, f^{j-1}, f^{j-1}, b') \). Hence, there are 3 internal disjoint paths \( \{P_0, P_1, P_2\} \) whose union covers all vertices of \( Q_{n}^{k,1} \) between \( w \) and \( b \). Please see Figure 13 for an illustration.

**Case 4.2.** If \( m \geq 2 \).

Given a vertex \( z \) in \( Q_{n-1}^{k,0} \) such that \( z \) is adjacent to \( b \). So \( z^2 \) is a black (resp. white) vertex and \( w^2 \) is a white (resp. black) vertex of \( Q_{n-1}^{k,0} \) if \( 0 \leq i \leq j' - 1 \) when \( i \) is even (resp. odd). By the induction hypothesis, there exist \( m \) internal disjoint paths \( \{R_{m}^{i} \}_{i=0}^{m-1} \) of \( Q_{n-1}^{k,0} \) between \( w \) and \( z^{0} \). We write \( R_0 = (w, x_0(1), x_0(2), \ldots, x_0(a), z^0) \) and \( R_{m} = (w, x_0, S_{m}, y_{m}, z^m) \) for \( 1 \leq p \leq m - 1 \). Again, by the induction hypothesis, there exist \( m \) internal disjoint paths \( \{P_{m}^{i} \}_{i=0}^{m-1} \) of \( Q_{n-1}^{k,1} \) between \( w^{i} \) and \( z^{i} \) for \( 2 \leq i \leq j' - 1 \). We let \( P_{m+1} = (w, x_{m+1}, U_{m+1}, b^{j'+1}, b) \) for \( 0 \leq p \leq m - 1 \) and \( 2 \leq i \leq j' - 1 \). Notice that \( b^{j'+1} \) is adjacent to \( z^{j'+1} \), without loss of generality, we let \( x_{m+1} = b^{j'+1} \). In \( Q_{n}^{k,1} \), there are \( m \) internal disjoint paths \( \{P_{m}^{i} \}_{i=0}^{m-1} \) from \( b \) to \( z \) by the induction hypothesis. We can write \( P_{m} = (z, t_{m}, b) \) for \( 0 \leq p \leq m - 2 \) and \( W_{m+1} = (z, b) \). According to Lemma 1, there is a hamiltonian path \( V \) between \( w^{k-1} \) and \( b^{k-1} \) and \( b^{k-1} \) covering all vertices of \( Q_{n-1}^{k,1} \) for \( j' + 1 \leq i \leq k - 1 \). Set \( P_0 = (w, w^{k-1}, V, b^{j'+1}, b) \), \( P_1 = (w, w^{k-1}, W, f^{k-2}, f^{k-3}, (W^{k-3})^{j-1}, e^{j-3}, W^{k-4}, f^{j-4}, \ldots, e^{j-1}, W^{j-1}, f^{j-1}, f^{j-1}, b') \), and \( P_{m} = (w, w^{k-1}, U_{m}, b^{j'+1}, b) \), \( P_{m+1} = (w, x_{m+1}, U_{m+1}, b^{j'+1}, b) \). Hence, there are \( m + 2 \) internal disjoint paths \( \{P_{m}^{i} \}_{i=0}^{m-1} \) whose union covers all vertices of \( Q_{n}^{k,1} \) between \( w \) and \( b \). Please see Figure 14 for an illustration.
union covers all vertices of $Q^2_d$ are $P_1 = \langle (0,0), (1,0) \rangle$, $P_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, and $P_3 = \langle (0,0), (3,0), (3,1), (3,2), (3,3), (2,3), (1,3), (0,3), (0,2), (1,2), (2,2), (2,1), (2,0), (1,0) \rangle$.

Case 1.2. Let $b = (2,1)$.

The three disjoint paths $\{R_1, R_2, R_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_d$ are $R_1 = \langle (0,0), (1,0) \rangle$, $R_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, and $R_3 = \langle (0,0), (3,0), (3,1), (3,2), (3,3), (2,3), (2,2), (2,1), (2,0), (1,0) \rangle$.

Case 2. To prove that $Q^2_d$ is $4^*$-laceable.

Case 2.1. Let $b = (1,0)$.

The four disjoint paths $\{P_1, P_2, P_3, P_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_d$ are $P_1 = \langle (0,0), (1,0) \rangle$, $P_2 = \langle (0,0), (0,1), (1,1), (1,0) \rangle$, $P_3 = \langle (0,0), (3,0), (3,1), (3,2), (3,3), (2,3), (2,2), (2,1), (2,0), (1,0) \rangle$.

Case 2.2. Let $b = (2,1)$.

The four disjoint paths $\{R_1, R_2, R_3, R_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_d$ are $R_1 = \langle (0,0), (3,0), (3,1), (2,1) \rangle$, $R_2 = \langle (0,0), (1,0) \rangle$, $R_3 = \langle (0,0), (0,1), (1,1) \rangle$, and $R_4 = \langle (0,0), (2,0) \rangle$.

Case 2.3. Let $b = (3,0)$.

The four disjoint paths $\{S_1, S_2, S_3, S_4\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_d$ are $S_1 = \langle (0,0), (1,0) \rangle$, $S_2 = \langle (0,0), (0,1), (1,1) \rangle$, $S_3 = \langle (0,0), (5,0), (5,1), (5,2), (5,3), (5,4), (5,5), (4,5), (4,4), (4,3), (4,2), (4,1), (4,0), (4,3), (3,0), (3,1), (3,2), (3,3), (3,2), (2,2), (2,1), (2,0), (1,0) \rangle$.

Case 1.3. Let $b = (3,0)$.

The three disjoint paths $\{T_1, T_2, T_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_d$ are $T_1 = \langle (0,0), (1,0) \rangle$, $T_2 = \langle (0,0), (1,0), (2,1) \rangle$, $T_3 = \langle (0,0), (2,1) \rangle$.

Case 1.4. Let $b = (3,2)$.

The three disjoint paths $\{T_1, T_2, T_3\}$ between $w$ and $b$ whose union covers all vertices of $Q^2_d$ are $T_1 = \langle (0,0), (1,0) \rangle$, $T_2 = \langle (0,0), (1,0), (2,1) \rangle$, $T_3 = \langle (0,0), (2,1) \rangle$.

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