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Fuzzy Ideals in Near-subtraction Semigroups

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***Abstract*—** In this paper,we introduce a notion of fuzzy ideals in near-subtraction semigroups and study their related properties.

***Keywords*—** subtraction algebra, subtraction semigroup, an ideal, near—subtraction semigroup, fuzzy level set, fuzzy ideal, fuzzy homomorphism.

I. Introduction

THE systems of the form Φ, where $(\Phi; \circ, \backslash)$, considered by B. M. Schein [7], is a set of functions closed under the composition "$\circ$" of functions (and hence (Φ; $\circ$) is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B.Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results.Near-ring theory has been developed by Pilz[6].Based on near-ring theory, Dheena at el. [2],introduced the near-subtraction semigroups and strongly regular near-subtraction semigroups.

The concept of fuzzy subset was introduced by L.A.Zadeh [8]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set.K.J. Lee and C.H. Park[5] introduced the notion of a fuzzy ideal in subtraction algebras, and give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras.In this paper,we introduce the notion of fuzzy ideal in near-subtraction semigroup and have studied their related properties.

II. Preliminaries

Definition 2.1: A non-empty set X together with a binary operation "–"is said to be a subtraction algebra if it satisfies the following:
$(1)\, x-(y-x)=x$.
$(2)\, x-(x-y)=y-(y-x)$.
$(3)\, (x-y)-z=(x-z)-y$,for all $x,y,z \in X$.

Example 2.2: Let $X=\{0,a,b,1\}$ in which "–" is defined by

–	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Then $(X,-)$ is a subtraction algebra.

In a subtraction algebra the following holds:
$(P1)\, x-0=x$ and $0-x=0$.
$(P2)\, (x-y)-x=0$.
$(P3)\, (x-y)-y=x-y$.
$(P4)\, (x-y)-(y-x)=x-y$,where $0=x-x$ is an element that does not depend on the choice of $x \in X$.

Following [9],we have the following definition of subtraction semigroup.

Definition 2.3: A non-empty set X together with the binary operations "–" and "." is said to be a subtraction semigroup if it satisfies the following:
$(SS1)\, (X;-)$ is a subtraction algebra.
$(SS2)\, (X;.)$ is a semigroup.
$(SS3)\, x(y-z)=xy-xz$ and $(x-y)z=xz-yz$,for all $x,y,z \in X$.

Example 2.4: [2] Let $X=\{0,a,b,1\}$ in which "–" and "." are defined by

–	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

.	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then $(X,-,.)$ is a subtraction semigroup.
Now we have the following definition of near-subtraction semigroup.

Definition 2.5: A non-empty set X together with the binary operations "–" and "." is said to be a near-subtraction semigroup if it satisfies the following:
$(NS1)\, (X;-)$ is a subtraction algebra.
$(NS2)\, (X;.)$ is a semigroup.
$(NS3)\, (x-y)z=xz-yz$,for all $x,y,z \in X$.

It is clear that $0x=0$,for all $x \in X$.Similarly we can define a near-subtraction semigroup (left).Hereafter a near-subtraction semigroup means it is a near-subtraction semigroup(right) only.

Example 2.6: [2] Let $X=\{0,a,b,1\}$ in which "–" and "." are defined by

–	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

.	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

Then $(X,-,.)$ is a near-subtraction semigroup.

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Definition 2.7: A near-subtraction semigroup X is said to be zero-symmetric if $x0 = 0$ for every $x \in X$.

Definition 2.8: A near-subtraction semigroup X is said have an identity if there exists an element $1 \in X$ such that $1.x = x.1 = x$,for every $x \in X$.

Definition 2.9: A non-empty subset S of a subtraction algebra X is said to be a subalgebra of X,if $x-y \in S$,whenever $x, y \in S$.

Definition 2.10: Let $(X, -, .)$ be a near-subtraction semigroup . A non-empty subset I of X is called
$(I1)$ ***a left ideal*** if I is a subalgebra of $(X, -)$ and $xi - x(y - i) \in I$ for all $x, y \in X$ and $i \in I$.
$(I2)$ ***a right ideal*** if I is a subalgebra of $(X, -)$ and $IX \subseteq I$.
$(I3)$ ***an ideal*** if I is both a left and right ideal. $IX \subseteq I$.

Remark 2.11: (i) Suppose if X is a subtraction semigroup and I is a left ideal of X,then for $i \in X$ and $x, y \in X$, we have $xi - x(y - i) = xi - (xy - xi) = xi \in I$ by Property 1 of subtraction algebra.Thus we have $XI \subseteq I$.
(ii) If X is a zero symmetric near-subtraction semigroup,then for $i \in I$ and $x \in X$,we have $xi-x(0-i) = xi-0 = xi \in X$.

For the sake of completeness, now we study some concepts of fuzzy theory.

A mapping $\mu : X \to [0, 1]$ is called *fuzzy set* of X and the *complement* of a fuzzy set μ, denoted by μ' is the fuzzy set in X given by $\mu'(x) = 1 - \mu(x)$ for all $x \in X$.The *level set* of a fuzzy set μ of X is defined as $U(\mu; t) = \{x \in X|\mu(x) \geq t\}$,for all $0 \leq t \leq 1$.

III. Fuzzy ideals

In what follows, let X denote a near-subtraction semigroups,unless otherwise specified.

Definition 3.1: A fuzzy set μ in X is called a *fuzzy ideal* of X if it satisfies the following conditions:
(FI1) $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$,
(FI2) $\mu(ax - a(b - x)) \geq \mu(x)$ for all $a, b, x \in X$ and
(FI3) $\mu(xy) \geq \mu(x)$,for all $x, y \in X$.

Note that μ is a *fuzzy left ideal* of X if it satisfies(FI1)and(FI2), and μ is a *fuzzy right ideal* of X if it satisfies (FI1) and (FI3).

Example 3.2: Let $X = \{0, a, b, 1\}$ in which "–" and "." are defined by

–	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

.	0	a	b
0	0	0	0
a	0	a	0
b	a	0	b

Then $(X, -, .)$ is a near-subtraction semigroup.Let μ be a fuzzy set on X defined by $\mu(0) = 0.8, \mu(a) = 0.5$ and $\mu(b) = 0.3$.Then by routine calculation,it is easy to prove that μ is a fuzzy ideal of X.

Theorem 3.3: Let μ be a fuzzy left (resp. right) of X.Then the set

$$X_\mu = \{x \in X|\mu(x) = \mu(0)\}$$

is a left(resp.right) ideal of X.

Proof: Suppose μ is a fuzzy left ideal of X and let $x, y \in X_\mu$.Then

$$\mu(x - y) \geq min\{\mu(x), \mu(y)\} = \mu(0).$$

Thus $x - y \in X_\mu$.
For every $a, b \in X$ and $x \in X_\mu$,we have

$$\mu(ax - a(b - x)) \geq \mu(x) = \mu(0).$$

Thus $ax - a(b - x) \in X_\mu$.Hence,X_μ is a left ideal of X.Similarly,we have the desired result for the right case. ■

Theorem 3.4: Let A be a non-empty subset of X and μ_A be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} s \ , \ if \ x \in A, \\ t \ , \ otherwise. \end{cases}$$

for all $x \in X$ and $s, t \in [0, 1]$ with $s > t$.Then μ_A is a fuzzy ideal of X if and only if A is an ideal of X.Moreover $X_{\mu_A} = A$.

Proof: Suppose μ_A is a fuzzy ideal of X.Let $x, y \in A$.Then

$$\mu(x - y) \geq min\{\mu(x), \mu(y)\} = s.$$

Thus,$x - y \in A$.
For every $a, b \in X$ and $x \in A$,we have

$$\mu(ax - a(b - x)) \geq \mu(x) = s.$$

Thus $ax - a(b - x) \in A$.
For all $x, y \in A$.Then

$$\mu(xy) \geq \mu(x) = s.$$

Thus, $xy \in A$.Hence , μ_A is an ideal of X.
Conversely, assume that A is an ideal of X.Let $x, y \in X$.If at least one of X and y does not belong to A,then

$$\mu_A(x - y) \geq t = min\{\mu_A(x), \mu_A(y)\}.$$

If $x, y \in A$ then $x - y \in A$,we have

$$\mu_A(x - y) \geq s = min\{\mu_A(x), \mu_A(y)\}.$$

Let $a, b, x \in X$ and if $x \in A$ such that $ax - a(b - x) \in A$,we have

$$\mu_A(ax - a(b - x)) \geq s = \mu_A(x).$$

If $x \notin A$ such that $ax - a(b - x) \notin A$,we have

$$\mu_A(ax - a(b - x)) \geq t = \mu_A(x).$$

For all $x, y \in A$ then $xy \in A$,we have

$$\mu_A(xy) \geq s = \mu(x).$$

Suppose $x \notin A$ we have

$$\mu_A(xy) \geq t = \mu(x).$$

Hence μ_A is a fuzzy ideal of X.Moreover

$$\begin{aligned} X_{\mu_A} &= \{x \in X|\mu_A(x) = \mu_A(0)\} \\ &= \{x \in X|\mu_A(x) = s\} \\ &= \{x \in X|x \in A\} \\ &= A. \end{aligned}$$

■

Corollary 3.5: Let χ_A be the characteristic function of a subset $A \subseteq X$.Then χ_A is a fuzzy left*(resp. right)* ideal if and only if A is a left(*resp. right*) ideal.

Theorem 3.6: Let μ be a fuzzy subset of X.Then μ is a fuzzy ideal of X if and only if each non-empty level subset $U(\mu;t)$ of μ is an ideal of X.

Proof: Assume that μ is a fuzzy ideal of X and $U(\mu;t)$ is a non-empty level subset of X.
(i) Since $U(\mu;t)$ is a non-empty level subset of μ, there exists $x, y \in U(\mu;t)$, $\mu(x-y) \geq min\{\mu(x), \mu(y)\} = t$.Thus $x-y \in U(\mu;t)$.
(ii) Let $a, b, x \in U(\mu;t)$,we have $\mu(ax-a(b-x)) \geq \mu(x) \geq t$. Thus $ax - a(b-x) \in U(\mu;t)$.
(iii) Let $x, y \in U(\mu;t)$, such that $\mu(xy) \geq \mu(x) \geq t$.Thus $xy \in U(\mu;t)$.Hence,$L(\mu;t)$ is an ideal of R.
Conversely,suppose that $U(\mu;t)$ is an ideal of X.
(i)Let if possible, $\mu(x_0 - y_0) < min\{\mu(x_0), \mu(y_0)\}$,for some $x_0, y_0 \in U(\mu;t)$,then by taking

$$t_0 = \frac{1}{2}\{\mu(x_0 - y_0) + min\{\mu(x_0), \mu(y_0)\}\},$$

we have $\mu(x_0 - y_0) > t_0$,for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$.Thus $x_0 - y_0 \notin U(\mu;t)$,for some $x_0, y_0 \in U(\mu;t)$.This is a contradiction,and so $\mu(x-y) \geq min\{\mu(x), \mu(y)$,for all$x, y \in U(\mu;t)$.
(ii)Let if possible, for some $x_0 \in U(\mu;t)$ $\mu(ax-(a(b-x)) < \mu(x_0)$, for all $a, b \in X$ and ,then by taking

$$t_0 = \frac{1}{2}\{\mu(ax_0 - a(b-x_0)) + \mu(x_0)\},$$

we have $\mu(ax_0 - a(b-x_0)) > t_0$,for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$.Thus $ax_0 - a(b-x_0) \notin U(\mu;t)$,for some $x_0 \in U(\mu;t)$ and for all $a, b \in X$.This is a contradiction,and so $\mu(ax - a(b-x)) \geq \mu(x)$,for all $x \in U(\mu;t)$ and $a, b \in X$.
(iii)Let if possible, $\mu(x_0 y_0) < \mu(x_0)$,for some $x_0, y_0 \in U(\mu;t)$,then by taking

$$t_0 = \frac{1}{2}\{\mu(x_0 y_0) + \mu(x_0)\},$$

we have $\mu(x_0 y_0) > t_0$,for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$.Thus $x_0 y_0 \notin U(\mu;t)$,for some $x_0, y_0 \in U(\mu;t)$.This is a contradiction,and so $\mu(xy) \geq \mu(x)$,for all$x, y \in U(\mu;t)$.Hence $U(\mu;t)$ is a fuzzy ideal of X. ■

Definition 3.7: Let X be a near-subtraction semigroup and a family of fuzzy sets $\{\mu_i | i \in I\}$ in X.Then the intersection $\left(\bigwedge_{i\in I} \mu_i\right)$ of $\{\mu_i | i \in I\}$ is defined by

$$\left(\bigwedge_{i\in I} \mu_i\right)(x) = \inf\{\mu_i(x) | i \in I\}$$

Theorem 3.8: If $\{\mu_i | i \in I\}$ is a family of fuzzy ideal of X,then $\left(\bigwedge_{i\in I} \mu_i\right)(x)$ is a fuzzy ideal of X.

Proof: Let $\{\mu_i | i \in I\}$ be a family of fuzzy ideal of X.

(i)For all $x, y \in X$,we have

$$\left(\bigwedge_{i\in I} \mu_i\right)(x-y) = \inf\{\mu_i(x-y) | i \in I\}$$

$$\geq \inf\{min\left(\mu_i(x), \mu_i(y)\right) | i \in I\}$$

$$= min\{\inf\left(\mu_i(x) | i \in I\right), \inf\left(\mu_i(y) | i \in I\right)\}$$

$$= min\left\{\left(\bigwedge_{i\in I} \mu_i\right)(x), \left(\bigwedge_{i\in I} \mu_i\right)(y)\right\}$$

(i)For all $a, b, x \in X$,we have

$$\left(\bigwedge_{i\in I} \mu_i\right)(ax - a(b-x)) = \inf\{\mu_i(ax - a(b-x)) | i \in I\}$$

$$\geq \inf\{\mu_i(x) | i \in I\}$$

$$= \{\inf\left(\mu_i(x) | i \in I\right)\}$$

$$= \left(\bigwedge_{i\in I} \mu_i\right)(x).$$

(iii) For all $x, y \in X$,we have

$$\left(\bigwedge_{i\in I} \mu_i\right)(xy) = \inf\{\mu_i(xy) | i \in I\}$$

$$\geq \inf\{min\left(\mu_i(x)\right) | i \in I\}$$

$$= \left(\bigwedge_{i\in I} \mu_i\right)(x)$$

Hence $\left(\bigwedge_{i\in I} \mu_i\right)$ is a fuzzy ideal of X.

■

Definition 3.9: Let $f : X \longrightarrow X'$ be a mapping ,where X and X' are non-empty sets and μ is a fuzzy subset of X.The *preimage* of μ under f written μ^f,is a fuzzy subset of X defined by $\mu^f = \mu(f(x))$,for all $x \in X$.

Theorem 3.10: Let $f : X \longrightarrow X'$ be a homomorphism of near-subtraction semigroups. If μ is a fuzzy ideal of X',then μ^f is a fuzzy ideal of X.

Proof: Suppose μ is a fuzzy ideal of X',then
(i) For all $x, y \in X$,we have

$$\begin{aligned}
\mu^f(x-y) &= \mu\left(f(x-y)\right) = \mu\left(f(x) - f(y)\right) \\
&\geq min\{\mu\left(f(x)\right), \mu\left(f(y)\right)\} \\
&= min\left\{\mu^f(x), \mu^f(y)\right\}.
\end{aligned}$$

(ii) For all $a, b, x \in X$,we have

$$\begin{aligned}
\mu^f(ax - a(b-x)) &= \mu\left(f\left(ax - a(b-x)\right)\right) \\
&= \mu\left(f(ax) - f(a(b-x))\right) \\
&= \mu\left(f(a)f(x) - f(a)(f(b) - f(x))\right) \\
&\geq \mu\left(f(x)\right) \\
&= \mu^f(x).
\end{aligned}$$

(iii)For all $x, y \in X$,we have

$$\begin{aligned} \mu^f(xy) &= \mu(f(xy)) \\ &= \mu(f(x)f(y)) \\ &\geq \mu(f(y)) \\ &= \mu^f(y). \end{aligned}$$

■

Hence μ^f is a fuzzy ideal of X.

Theorem 3.11: Let $f : X \longrightarrow X'$ be a homomorphism of near-subtraction semigroup . If μ^f is a fuzzy ideal of X ,then μ is fuzzy ideal of X'.

Proof: Suppose μ is a fuzzy ideal of X',then

(i)Let $x', y' \in X'$,there exists $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$,we have

$$\begin{aligned} \mu(x' - y') &= \mu(f(x) - f(y)) \\ &= \mu(f(x - y)) \\ &= \mu^f(x - y) \\ &\geq min\{\mu^f(x), \mu^f(y)\} \\ &= min\{\mu(f(x)), \mu(f(y))\} \\ &= min\{\mu(x'), \mu(y')\}. \end{aligned}$$

(ii)Let $a', b', x' \in X'$,there exists $a, b, x \in X$ such that $f(a) = a', f(b) = b'$ and $f(x) = x'$,we have

$$\begin{aligned} \mu(a'x' - b(a' - x')) &= \mu(f(a)f(x) - f(b)(f(a) - f(x))) \\ &= \mu(f(ax) - f(b)f(a - x)) \\ &= \mu(f(ax) - f(b(a - x))) \\ &= \mu(f(ax - b(a - x))) \\ &= \mu^f(ax - b(a - x)) \\ &\geq \mu^f(x) \\ &= \mu(f(x)) \\ &= \mu(x'). \end{aligned}$$

(iii)Let $x', y' \in X'$,there exists $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$,we have

$$\begin{aligned} \mu(x'y') &= \mu(f(x)f(y)) = \mu(f(xy)) \\ &= \mu^f(xy) \\ &\geq \mu^f(x) \\ &= \mu(f(x)) \\ &= \mu(x') \end{aligned}$$

Hence μ is a fuzzy ideal of X'. ■

Definition 3.12: Let f be a mapping defined on X.If ν is a fuzzy subset in $f(X)$,then the fuzzy subset $\mu = \nu \circ f$ in X(i.e., the fuzzy subset defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called the *preimage* of ν under f.

Proposition 3.13: An onto homomorphic preimage of a fuzzy ideal of X is a fuzzy ideal.

Proof: Straight forward. ■

Let μ be a fuzzy subset in X and f be a mapping defined on X.Then the fuzzy subset μ^f in $f(X)$ defined by $\mu^f(y) = \sup\limits_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(X)$ is called the *image* of μ under f.A fuzzy subset μ in X is said to have an *sup property* if for every subset $N \subseteq X$,there exists $n_0 \in N$ such that $\mu(n_0) = \sup\limits_{n \in N} \mu(n)$.

Proposition 3.14: An onto homomorphic image of a fuzzy ideal with sup property is fuzzy ideal.

Proof: Let $f : X \longrightarrow X'$ be an onto homomorphism of near-subtraction semigroup and let μ be a fuzzy ideal of X with the sup property.

(i)Given $x', y' \in X'$,we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n), \;\; \mu(y_0) = \sup_{n \in f^{-1}(y')} \mu(n)$$

respectively.Then , we have

$$\begin{aligned} \mu^f(x' - y') &= \sup_{z \in f^{-1}(x' - y')} \mu(z) \\ &\geq min\{\mu(x_0), \mu(y_0)\} \\ &= min\left\{ \sup_{n \in f^{-1}(x')} \mu(n), \sup_{n \in f^{-1}(y')} \mu(n) \right\} \\ &= min\{\mu^f(x'), \mu^f(y')\} \end{aligned}$$

(ii) Given $a', b', x' \in R'$, we let $a_0 \in f^{-1}(a')$, $b_0 \in f^{-1}(b')$, $x_0 \in f^{-1}(x')$ be such that

$$\begin{aligned} \mu^f(a'x' - a'(b' - x')) &= \sup_{z \in f^{-1}(a'x' - a'(b' - x'))} \mu(z) \\ &\geq \mu(x_0) \\ &= \sup_{n \in f^{-1}(x')} \mu(n) \\ &= \mu^f(x'). \end{aligned}$$

(iii)Given $x', y' \in X'$,we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n), \;\; \mu(y_0) = \sup_{n \in f^{-1}(y')} \mu(n)$$

respectively.Then , we have

$$\begin{aligned} \mu^f(x'y') &= \sup_{z \in f^{-1}(x'y')} \mu(z) \\ &\geq \mu(x_0) \\ &= \sup_{n \in f^{-1}(x')} \mu(n) \\ &= \mu^f(x') \end{aligned}$$

■

Hence,μ^f is a fuzzy ideal of X'.

IV. Chain conditions

Proposition 4.1: Let μ and ν be a fuzzy subset of X.If they are fuzzy ideal of X,then so $\mu \cap \nu$,where $\mu \cap \nu$ is defined by

$(\mu\cap\nu)(x) = min\{\mu(x), \nu(x)\}$ for all $x, \in X$.

Proof: (i) For all $x, y \in X$,we have

$$\begin{aligned}(\mu\cap\nu)(x-y) &= min\{\mu(x-y), \nu(x-y)\}\\ &\geq min\{min\{\mu(x), \mu(y)\},\\ &\quad min\{\nu(x), \nu(y)\}\}\\ &= min\{(\mu\cap\nu)(x), (\mu\cap\nu)(y)\}.\end{aligned}$$

(ii) For all $x, y \in X$,we have

$$\begin{aligned}&(\mu\cap\nu)(ax-a(b-x))\\ =\ &min\{\mu(ax-a(b-x), \nu(ax-a(b-x)\}\\ \geq\ &min\{\mu(x), \nu(x)\}\\ =\ &(\mu\cap\nu)(x).\end{aligned}$$

(iii) For all $x, y \in X$,we have

$$\begin{aligned}(\mu\cap\nu)(xy)) &= min\{\mu(xy), \nu(xy)\}\\ &\geq min\{\mu(y), \nu(y)\}\\ &= (\mu\cap\nu)(y).\end{aligned}$$

Hence,$\mu\cap\nu$ is a fuzzy ideal of X. ■

Theorem 4.2: Let μ be a fuzzy subset in X and $Im(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_k\}$,where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{A_n | n = 0, 1, ..., k\}$ be a family of ideals of X such that
(i) $A_0 \subseteq A_1 \subseteq ... \subseteq A_k = X$,
(ii) $\mu(A^*) = \alpha_n$,where $A_n^* = A_n \setminus A_{n-1}, A_{-1} = \phi$ for all $n = 0, 1, ..., k$.
Then μ is a fuzzy ideal of X.

Proof: Suppose $\{A_n | n = 0, 1, ..., k\}$ be a family of ideals of X.

(i) For all $x, y \in X$,Then we discuss the following cases:If $x \in A_n$ and $y \in A_n$ such that $x-y \in A_n$,since A_n is an ideal of X.thus

$$\mu(x-y) \geq \alpha_n = min\{\mu(x), \mu(y)\}.$$

If $x \notin A_n^*$ and $y \notin A_n^*$,then the following four cases arise:

1) $x \in X \setminus A_n$ and $y \in X \setminus A_n$
2) $x \in A_{n-1}$ and $y \in A_{n-1}$
3) $x \in X \setminus A_n$ and $y \in A_{n-1}$
4) $x \in A_{n-1}$ and $y \in R \setminus A_n$

But,in either cases,we know that

$$\mu(x-y) \geq min\{\mu(x), \mu(y)\}.$$

If $x \in X \setminus A_n^*$ and $y \notin A_n^*$ then either $y \in A_{n-1}$ or $y \in X \setminus A_n$. It follows that either $x \in A_n$ or $x \in X \setminus A_n$.Thus

$$\mu(x-y) \geq min\{\mu(x), \mu(y)\}.$$

If $x \notin X \setminus A_n^*$ and $y \in A_n^*$ then by similar process we have

$$\mu(x-y) \geq min\{\mu(x), \mu(y)\}.$$

(ii)If $a, b \in X$ and $x \in A_n$ then $ax-a(b-x) \in A_n$.Then

$$\mu(ax-a(b-x)) \geq min\{\mu(a), \mu(b)\}.$$

If $a, b \in X$ and $x \notin A_n$ then,we have

$$\mu(ax-a(b-x)) \geq \alpha_n = \mu(x).$$

(iii) Similarly, for $x, y \in X$,we have

$$\mu(xy) \geq \mu(y).$$

Hence μ is a fuzzy ideal of X. ■

Theorem 4.3: Let $\{A_n | n \in \mathbb{N}\}$ be a family of ideals of X which is nested,that is,$X = A_1 \supset A_2 \supset$ ....Let μ be a fuzzy subset in X defined by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & if\ x \in A_n \setminus A_{n+1}, n = 1, 2, 3..., \\ 1 & if\ x \in \bigcap_{n=1}^{\infty} A_n\,. \end{cases}$$

for all$x \in X$.Then μ is a fuzzy ideal of X.

Proof: Let $x, y \in X$.

(i)Suppose that $x \in A_k \setminus A_{k+1}$ and $y \in A_r \setminus A_{r+1}$ for $k = 1, 2, ...; r = 1, 2, ...$ .Without loss of generality,we may assume that $k \leq r$.Then $x-y \in A_k$ and so

$$\mu(x-y) \geq \frac{k}{k+1} = min\{\mu(x), \mu(y)\}$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$ then $x-y \in \bigcap_{n=1}^{\infty} A_n$ and thus

$$\mu(x-y) = 1 = min\{\mu(x), \mu(y)\}$$

If $x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$,then there exists $i \in \mathbb{N}$ such that $y \in A_i \setminus A_{i+1}$.It follows that $x-y \in A_i$ so that

$$\mu(x-y) \geq \frac{i}{i+1} = \min\{\mu(x), \mu(y)\}$$

Similarly,we can prove that

$$\mu(x-y) \geq \min(\mu(x), \mu(y))$$

for all $x \notin \bigcap_{n=1}^{\infty} A_n$ then $y \in \bigcap_{n=1}^{\infty} A_n$.

(ii)Now,let $a, b \in X$.If ,$x \in A_r \setminus A_{r+1}$ for some $k = 1, 2, ...$,then $ax-a(b-x) \in A_k$.Thus

$$\mu(ax-a(b-x)) \geq \frac{k}{k+1} = \mu(x)$$

If $x \in \bigcap_{n=1}^{\infty} A_n$ then $ax-a(b-x) \in \bigcap_{n=1}^{\infty} A_n$ for all $a, b \in X$.Thus

$$\mu(ax-a(b-x)) = 1 = \mu(x)\,.$$

Assume that $a \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, ...$,and $b \in \bigcap_{n=1}^{\infty} A_n$(or , $a \in \bigcap_{n=1}^{\infty} A_n$ and $b \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3...$).Then $x \in A_r$ and so

$$\mu(ax-a(b-x)) \geq \frac{r}{r+1} = \mu(x)$$

(iii) Now,if $x, y \in A_k \setminus A_{k+1}$ for some $r = 1, 2, 3...$, then $y \in A_r$ as A_r is a ideal of X.Thus

$$\mu(xy) \geq \frac{r}{r+1} = \mu(y).$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$ then $y \in \bigcap_{n=1}^{\infty} A_n$ and so

$$\mu(xy) = 1 = \mu(y).$$

Hence, μ is a fuzzy ideal of X.

∎

Let $\mu : X \longrightarrow [0,1]$ be a fuzzy subset of X.The smallest fuzzy ideal containing μ is called the fuzzy ideal generated by μ, and μ is said to be n-valued if $\mu(X)$ is a finite set of n elements.When no specific n is intended, we call μ a finite-valued fuzzy subset.

Theorem 4.4: A fuzzy ideal ν of X is finite valued if and only if a finite-valued fuzzy subset μ of X is generated by ν.

Proof: If $\nu : X \longrightarrow [0,1]$ is a finite-valued fuzzy ideal of X,then one may choose $\mu = \nu$.Consequently, assume that $\mu : X \longrightarrow [0,1]$ is a n-valued fuzzy subset with n distinct values $t_1, t_2, ..., t_n$,where $t_1 > t_2 > ... > t_n$.Let G^i be the inverse image of t_i under μ, that is,$G^i = \mu^{-1}(t_i)$.Obviously, $\bigcup_{i=1}^{j} G^i \subseteq \bigcup_{i=1}^{r} G^i$ when $j < r$.We denote by A^j the ideal of X generated by the set $\bigcup_{i=1}^{j} G^i$.Then we have the following chain of ideals:

$$A^1 \subseteq A^2 \subseteq ... \subseteq A^n = X.$$

Define a fuzzy $\nu : X \longrightarrow [0,1]$ by

$$\nu(x) = \begin{cases} t_n & if \in A^n, \\ t_j & if \ \in A^j \backslash A^{j-1}; j = 1, 2, ..., n-1 \end{cases}.$$

We claim that ν is a fuzzy ideal of X and μ is generated by ν.Let $x, y \in X$ and let i and j be the smallest integer such that $x \in A^i$ and $y \in A^j$.we may assume that $i > j$ without loss of generality.Then $x - y \in A^i$ and $xy \in A^i$ and so

$$\nu(x-y) \geq t_j = \min\{t_i, t_j\} = \min\{\nu(x), \nu(y)\}$$

and

$$\nu(xy) \geq t_j = \nu(y).$$

Now,let $a, b \in X$ .If $x \in A^j$ for some $i < j$,then $x \in A^i$ as A^i is a ideal of X.Thus

$$\nu(ax - a(b-x)) \geq t_j = \nu(x).$$

Hence, μ is a fuzzy ideal of X.
If $x \in X$ and $\mu(x) = t_j$,then $x \in G^j$ and so $x \in A^j$.But we get $\nu(x) \geq t_j = \mu(x)$.Consequently, $\mu \subseteq \nu$.Let γ be any fuzzy ideal of X which is a subset of μ.Then,$\bigcup_{i=1}^{j} G^i = U(\mu; t_j) \subseteq U(\gamma; t_j)$, and thus $A^j \subseteq U(\gamma; t_j)$.Hence,$\gamma \subseteq \mu$ and μ is generated by ν.Note that $\mid Im\mu \mid = n = \mid Im\nu \mid$.This completes the proof. ∎

A near-subtraction semigroup X is a said to be *Noetherian* (see [9]) if it satisfies the ascending chain condition on ideals of X.

Theorem 4.5: If X is a Noetherian near-subtraction semigroup , then every fuzzy ideal of X is finite valued.

Proof: Let $\mu : X \longrightarrow [0,1]$ be a fuzzy ideal of X which is not finite valued.Then,there exists sequence of distinct numbers $\mu(0) = t_1 > t_2 > ... > t_n$,where $t_i = \mu(x_i)$ for some $x_i \in R$. This sequence induces an infinite sequence of distinct ideals of X:

$$U(\mu; t_1) \subset U(\mu; t_2) \subset ... \subset U(\mu; t_n) \subset ... \ .$$

This is a contradiction.

∎

Combining Theorem 4.4 and Theorem 4.5,we have the following corollary.

Corollary 4.6: If X is a Noetherian near-subtraction semigroup, then every fuzzy ideal of X is generated by a finite fuzzy subset in X.

V. Normal fuzzy ideals

Definition 5.1: A fuzzy ideal μ of X is said to be *normal* if there exists $a \in X$ such that $\mu(a) = 1$.

We note that if μ is a normal fuzzy ideal μ of X is normal if and only if $\mu(1) = 1$.Let $\mathbb{F}_N(X)$ denote the set of all normal fuzzy ideal of X.

Theorem 5.2: Let μ be a fuzzy ideal of X and let μ^+ be a fuzzy set in X given by $\mu^+(x) = \mu(x) + 1 - \mu(1)$,for all $x \in X$.Then $\mu^+ \in \mathbb{F}_N(X)$ and $\mu \subseteq \mu^+$.

Proof: For any $x, y, z \in X$ we have $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1 \geq \mu^+(x)$ and
(i)For all $x, y \in X$,we have

$$\begin{aligned} \mu^+(x-y) &= \mu(x-y) + 1 - \mu(1) \\ &\geq min\{\mu(x), \mu(y)\} + 1 - \mu(1) \\ &= min\{\mu(x) + 1 - \mu(1), \mu(y) + 1 - \mu(1)\} \\ &= min\{\mu^+(x), \mu^+(y)\}. \end{aligned}$$

(ii)For all $x, a, b \in X$,we have

$$\begin{aligned} \mu^+(ax - a(b-x)) &= \mu(ax - b(x-a)) + 1 - \mu(1) \\ &\geq \mu(x) + 1 - \mu(1) \\ &= \mu^+(x). \end{aligned}$$

(iii)For all $x, y \in X$,we have

$$\begin{aligned} \mu^+(xy) &= \mu(xy) + 1 - \mu(1) \\ &\geq \mu(y) + 1 - \mu(1) \\ &= \mu^+(y). \end{aligned}$$

Hence $\mu^+ \in \mathbb{F}_N(X)$.Obviously, $\mu \subseteq \mu^+$.

∎

Corollary 5.3: If μ be a fuzzy ideal of X satisfying $\mu^+(a) = 0$ for some $a \in X$, then $\mu(a) = 0$.

It is clear that fuzzy ideal μ of X is normal if and only if $\mu^+ = \mu$,and for any fuzzy ideal μ of X we have $(\mu^+)^+ = \mu^+$.Hence if μ is a normal fuzzy ideal of X,then$(\mu^+)^+ = \mu$

Theorem 5.4: Let μ be a fuzzy ideal of X and let $\phi : [0, \mu(0)] \longrightarrow [0,1]$ be an increasing function.Let μ_ϕ be a fuzzy set in X defined by $\mu_\phi(x) = \phi(\mu(x))$ for all $x \in X$.Then μ_ϕ is a fuzzy ideal of X.Moreover,if $\phi(\mu(0)) = 1$ then $\mu_\phi \in \mathbb{F}_N(X)$,and if $\phi(t) \geq t$ for all $t \in [0,1]$ then $\mu \subseteq \mu_\phi$.

Proof: (i)Let $x, y \in X$.Then

$$\begin{aligned} \mu_\phi(x-y) &= \phi(\mu(x-y)) \\ &\geq \phi(min\{\mu(x), \mu(y)\}) \\ &= min\{\phi(\mu(x)), \phi(\mu(y))\} \\ &= min\{\mu_\phi(x), \mu_\phi(y)\}. \end{aligned}$$

(ii)Let $a, b, x \in X$.Then

$$\begin{aligned}\mu_\phi(ax-a(b-x)) &= \phi(\mu(ax-a(b-x)))\\ &\geq \phi(\mu(x))\\ &= \mu_\phi(x).\end{aligned}$$

(iii)Let $x, y \in X$.Then

$$\begin{aligned}\mu_\phi(xy) &= \phi(\mu(xy))\\ &\geq \phi(\mu(y))\\ &= \mu_\phi(y).\end{aligned}$$

Hence μ_ϕ is a fuzzy ideal of X.If $\phi(\mu(0)) = 1$ then obviously μ_ϕ is normal , and so $\mu_\phi \in \mathbb{F}_N(X)$. Assume that $\phi(t) \geq t$ for all $t \in [0, \mu(0)]$.Then $\mu_\phi(x) = \phi(\mu(x)) \geq \mu(x)$ for all $x \in X$,which proves that $\mu \subseteq \mu_\phi$.

■

Theorem 5.5: Let $\mu \in \mathbb{F}_N(X)$ be a non-constant maximal element of the poset $(\mathbb{F}_N(X), \subseteq)$.Then μ takes only the values 0 and 1.

Proof: Since μ is normal,we have $\mu(0) = 1$.Let $\mu(x) \neq 1$ for some $x \in X$.We claim that $\mu(x) = 0$.If not,then there exists $x_0 \in X$ such that $0 < \mu(x_0) < 1$.Define on X a fuzzy set ν putting $\nu(x) = \frac{\mu(x)+\mu(x_0)}{2}$ for all $x \in X$.Then,clearly ν is well-defined.

(i) For all $x, y \in X$,we have

$$\begin{aligned}\nu(x-y) &= \frac{\mu(x-y)+\mu(x_0)}{2}\\ &\geq \frac{min\{\mu(x),\mu(y)\}+\mu(x_0)}{2}\\ &= \frac{min\{\mu(x)+\mu(x_0),\mu(y)+\mu(x_0)\}}{2}\\ &= min\{\frac{\mu(x)+\mu(x_0)}{2},\frac{\mu(y)+\mu(x_0)}{2}\}\\ &= min\{\nu(x),\nu(y)\}.\end{aligned}$$

(ii) For all $a, b, x \in X$,we have

$$\begin{aligned}\nu(ax-a(b-x)) &= \frac{\mu(ax-a(b-x))+\mu(x_0)}{2}\\ &\geq \frac{\mu(x)+\mu(x_0)}{2}\\ &= \nu(x).\end{aligned}$$

(iii) For all $x, y \in X$,we have

$$\begin{aligned}\nu(xy) &= \frac{\mu(xy)+\mu(x_0)}{2}\\ &\geq \frac{\mu(y)+\mu(x_0)}{2}\\ &= \nu(y).\end{aligned}$$

Thus ν is a fuzzy ideal of X.By Theorem 5.2, ν^+ is a maximal fuzzy ideal of X.Note that

$$\begin{aligned}\nu^+(x_0) &= \nu(x_0)+1-\nu(0)\\ &= \frac{\mu(x_0)+\mu(x_0)}{2}+1-\frac{\mu(0)+\mu(x_0)}{2}\\ &= \frac{\mu(x_0)+1}{2}.\end{aligned}$$

and $\nu^+(x_0) < 1 = \frac{\mu(0)+1}{2} = \nu^+(0)$.Hence ν^+ is non-constant, and μ is not a maximal element of $\mathbb{F}_N(X)$.This is a contradiction. ■

Definition 5.6: A fuzzy ideal μ of X is said to be *maximal* if it satisfies:

(M1) μ is non-constant, and
(M2) μ^+ is a maximal element of $(\mathbb{F}_N(X), \subseteq)$.

Theorem 5.7: If a fuzzy ideal of X is maximal,then
(i) μ is normal,
(ii) μ takes only the values 0 and 1,
(iii) $\chi_{\mu^0} = \mu$,where $\mu^0 = \{x \in X | \mu(0) = 1\}$,
(iv) μ^0 is a maximal ideal of X.

Proof: Let μ be a maximal fuzzy ideal of X.Then μ^+ is a non-constant maximal element of the poset $(\mathbb{F}_N(X), \subseteq)$.It follows from the Theorem 5.5 that μ^+ takes only two values 0 and 1.Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$,and $\mu^+(0) = 0$ if and only if $\mu(x) = \mu(0) - 1$.By corollary 5.3,we have $\mu(x) = 0$ and so $\mu(0) = 1$.Hence μ is normal and $\mu^+ = \mu$.This proves (i) and (ii).
(iii) Obvious.
(iv) It is clear that μ^0 is a proper ideal of X.Obviously $\mu^0 \neq X$ because μ takes two values.Let A be an ideal containing μ^0.Then $\mu_{\mu^0} \subseteq \mu_A$,and consequence,$\mu = \mu^0_\mu \subseteq \mu_A$.Since μ is normal,μ_A also is normal and takes only two values 0 and 1.But,by the assumption,μ is maximal,so $\mu = \mu_A$ or $\mu = \phi$,where $\phi(x) = 1$ for all $x \in X$.In the last case $\mu^0 = X$,which is impossible.So,$\mu = \mu_A$.i.e.$\mu_A = \chi_A$.Hence $\mu^0 = A$

■

Definition 5.8: A fuzzy ideal μ of X is said to be *complete* if it is normal and there exists $z \in X$ such that $\mu(z) = 0$.

Theorem 5.9: Let μ be a fuzzy ideal of X and let w be a fixed element of X such that $\mu(1) = \mu(w)$.Define a fuzzy set μ^* in X by $\mu^*(x) = \frac{\mu(x)-\mu(w)}{\mu(1)-\mu(w)}$ for all $x \in X$.Then μ^* is a complete fuzzy ideal of X.

Proof: (i)For any $x, y \in X$,we have

$$\begin{aligned}\mu^*(x-y) &= \frac{\mu(x-y)-\mu(w)}{\mu(1)-\mu(w)}\\ &\geq \frac{min\{\mu(x),\mu(y)\}-\mu(w)}{\mu(1)-\mu(w)}\\ &= \min\left\{\frac{\mu(x)-\mu(w)}{\mu(1)-\mu(w)},\frac{\mu(y)-\mu(w)}{\mu(1)-\mu(w)}\right\}\\ &= min\{\mu^*(x),\mu^*(y)\}.\end{aligned}$$

(ii)For any $x, y \in X$,we have

$$\begin{aligned}\mu^*(ax-a(b-x)) &= \frac{\mu(ax-a(b-x))-\mu(w)}{\mu(1)-\mu(w)}\\ &\geq \frac{\mu(x)-\mu(w)}{\mu(1)-\mu(w)}\\ &= \mu^*(x).\end{aligned}$$

(iii)For any $x, y \in X$,we have

$$\begin{aligned}\mu^*(xy) &= \frac{\mu(xy)-\mu(w)}{\mu(1)-\mu(w)}\\ &\geq \frac{\mu(y)-\mu(w)}{\mu(1)-\mu(w)}\\ &= \mu^*(y).\end{aligned}$$

Hence $\mu^* \in \mathbb{F}_N(S)$.Since $\mu^*(w) = 0$,thus μ^* is a complete fuzzy ideal of X. ■

Theorem 5.10: Every maximal fuzzy ideal of X is completely normal.

Proof: Let μ be a maximal fuzzy ideal of X.Then by Theorem 5.7 ,μ is a normal and $\mu = \mu^+$ takes only two values 0 and 1.Since μ is non-constant,it follows that $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in X$.Hence μ is completely normal. ■

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