Fuzzy Ideals in Near-subtraction Semigroups

D.R Prince Williams

Abstract— In this paper, we introduce a notion of fuzzy ideals in near-subtraction semigroups and study their related properties.

Keywords — subtraction algebra, subtraction semigroup, an ideal, near—subtraction semigroup, fuzzy level set, fuzzy ideal, fuzzy homomorphism.

I. INTRODUCTION

The systems of the form $\Phi$, where $(\Phi; o, \setminus)$, considered by B. M. Schein [7], is a set of functions closed under the composition “$\circ$” of functions (and hence $(\Phi; o)$ is a function semigroup) and the set theoretic subtraction “$\setminus$” (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B.Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Near-ring theory has been developed by Pilz[6]. Based on near-ring theory, Dheena et al. [2], introduced the near-subtraction semigroups and strongly regular near-subtraction semigroups.

The concept of fuzzy subset was introduced by L.A.Zadeh [8]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. K.J. Lee and C.H. Park[5] introduced the notion of a fuzzy ideal in subtraction algebras, and give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebra. In this paper, we introduce the notion of fuzzy ideal in near-subtraction semigroup and have studied their related properties.

II. PRELIMINARIES

Definition 2.1: A non-empty set $X$ together with the binary operation “$\cdot$” is said to be a subtraction algebra if it satisfies the following:

$\forall x, y, z \in X$,

(1) $x - (y - x) = x$,

(2) $x - (x - y) = y - (y - x)$,

(3) $(x - y) - z = (x - z) - y$, for all $x, y, z \in X$.

Example 2.2: Let $X = \{0, a, b, 1\}$ in which “$-\cdot$” is defined by

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Then $(X, -\cdot)$ is a subtraction algebra.

In a subtraction algebra the following holds:

(P1) $x - 0 = x$ and $0 - x = 0$.

(P2) $(x - y) - x = 0$.

(P3) $(x - y) - y = x - y$.

(P4) $(x - y) - (y - x) = x - y$, where $0 = x - x$ is an element that does not depend on the choice of $x \in X$.

Following [9], we have the following definition of subtraction semigroup.

Definition 2.3: A non-empty set $X$ together with the binary operations “$-\cdot$” and “$\cdot$” is said to be a subtraction semigroup if it satisfies the following:

(SS1) $(X; -\cdot)$ is a subtraction algebra.

(SS2) $(X; \cdot)$ is a semigroup.

(SS3) $(x - y) - z = xy - xz$ and $(x - y)z = xz - yz$, for all $x, y, z \in X$.

Example 2.4: [2] Let $X = \{0, a, b, 1\}$ in which “$-\cdot$” and “$\cdot$” are defined by

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Then $(X, -\cdot)$ is a subtraction semigroup.

Now we have the following definition of near-subtraction semigroup.

Definition 2.5: A non-empty set $X$ together with the binary operations “$-\cdot$” and “$\cdot$” is said to be a near-subtraction semigroup if it satisfies the following:

(NS1) $(X; -\cdot)$ is a subtraction algebra.

(NS2) $(X; \cdot)$ is a semigroup.

(NS3) $(x - y)z = xz - yz$, for all $x, y, z \in X$.

It is clear that $0x = 0$, for all $x \in X$. Similarly we can define a near-subtraction semigroup (left). Hereafter a near-subtraction semigroup means it is a near-subtraction semigroup (right) only.

Example 2.6: [2] Let $X = \{0, a, b, 1\}$ in which “$-\cdot$” and “$\cdot$” are defined by

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Then $(X, -\cdot)$ is a near-subtraction semigroup.
Definition 2.7: A near-subtraction semigroup X is said to be zero-symmetric if \( x0 = 0 \) for every \( x \in X \).

Definition 2.8: A near-subtraction semigroup X is said to have an identity if there exists an element \( 1 \in X \) such that \( 1.x = x = 1.x \) for every \( x \in X \).

Definition 2.9: A non-empty subset \( S \) of a subtraction algebra \( X \) is said to be a subalgebra of \( X \), if \( x - y \in S \), whenever \( x, y \in S \).

Definition 2.10: Let \( (X,\ldots) \) be a near-subtraction semigroup. A non-empty subset \( I \) of \( X \) is called
\[(11) \text{ a left ideal if } I \text{ is a subalgebra of } (X,\ldots) \text{ and } x \in \{x \in I | y \in I \} \text{ for all } x, y \in I \text{ and } i \in I, \]
\[(12) \text{ a right ideal if } I \text{ is a subalgebra of } (X,\ldots) \text{ and } Ix \subseteq I \text{ for all } x \in I \text{ and } \]
\[(13) \text{ an ideal if } I \text{ is both a left and right ideal. } \]

Remark 2.11: (i) Suppose if \( X \) is a subtraction semigroup and \( I \) is a left ideal of \( X \), then \( x - (y - i) = x - y + x - i \in I \) \text{ by Property 1 of subtraction algebra}. Thus we have \( X \subseteq I \).

(ii) If \( X \) is a zero symmetric near-subtraction semigroup, then \( x - x(0 - i) = x + x - i \in X \).

For the sake of completeness, now we study some concepts of fuzzy theory.

A mapping \( \mu : X \rightarrow [0,1] \) is called a fuzzy set of \( X \) and the complement of a fuzzy set \( \mu \), denoted by \( \mu' \), is the fuzzy set \( X \) given by \( \mu'(x) = 1 - \mu(x) \) if \( x \in X \). The level set of a fuzzy set \( \mu \) is defined as \( U(\mu, t) = \{ x \in X | \mu(x) \geq t \} \) for all \( 0 \leq t \leq 1 \).

III. Fuzzy Ideals

In what follows, let \( X \) denote a near-subtraction semigroup, unless otherwise specified.

Definition 3.1: A fuzzy set \( \mu \) in \( X \) is called a fuzzy ideal of \( X \) if it satisfies the following conditions:
\[(F1) \mu(x - y) \geq \min \{ \mu(x), \mu(y) \} \text{ for all } x, y \in X, \]
\[(F2) \mu(ax - a(b - x)) \geq \mu(x) \text{ for all } a, b, x \in X \text{ and} \]
\[(F3) \mu(xy) \geq \mu(x), \text{for all } x, y \in X. \]

Note that \( \mu \) is a fuzzy left ideal of \( X \) if it satisfies \((F1)\) and \((F2)\), and \( \mu \) is a fuzzy right ideal of \( X \) if it satisfies \((F1)\) and \((F3)\).

Example 3.2: Let \( X = \{0, a, b, 1\} \) in which “−” and “+” are defined by
\[
\begin{array}{ccc|ccc}
0 & a & b & 0 & 0 & 0 \\
0 & a & 0 & a & 0 & a \\
b & b & 0 & b & a & 0 \\
b & b & 0 & b & a & 0 \\
\end{array}
\]

Then \( (X,\ldots) \) is a near-subtraction semigroup. Let \( \mu \) be a fuzzy set on \( X \) defined by \( \mu(0) = 0.8, \mu(a) = 0.5 \) and \( \mu(b) = 0.3 \). Then by routine calculation, it is easy to prove that \( \mu \) is a fuzzy ideal of \( X \).

Theorem 3.3: Let \( \mu \) be a fuzzy left (resp. right) of \( X \). Then the set
\[X_\mu = \{ x \in X | \mu(x) = \mu(0) \}\]
is a left (resp. right) ideal of \( X \).

Proof: Suppose \( \mu \) is a fuzzy left ideal of \( X \) and let \( x, y \in X_\mu \), then
\[\mu(x - y) \geq \min \{ \mu(x), \mu(y) \} = \mu(0). \]
Thus \( x - y \in X_\mu \).

For every \( a, b \in X \) and \( x \in X_\mu \), we have
\[\mu(ax - a(b - x)) \geq \mu(x) = \mu(0). \]
Thus \( ax - a(b - x) \in X_\mu \). Hence, \( X_\mu \) is a left ideal of \( X \). Similarly, we have the desired result for the right case.

Theorem 3.4: Let \( A \) be a non-empty subset of \( X \) and \( \mu_A \) be a fuzzy set in \( X \) defined by
\[\mu_A(x) = \left\{ \begin{array}{ll}
 s & \text{if } x \in A, \\
t & \text{otherwise.}
\end{array} \right. \]
for all \( x \in X \) and \( s, t \in [0,1] \) with \( s > t \). Then \( \mu_A \) is a fuzzy ideal of \( X \) if and only if \( A \) is an ideal of \( X \). Moreover \( X_{\mu_A} = A \).

Proof: Suppose \( \mu_A \) is a fuzzy ideal of \( X \). Let \( x, y \in A \). Then
\[\mu(x - y) \geq \min \{ \mu(x), \mu(y) \} = s. \]
Thus \( x - y \in X_\mu \).

For every \( a, b \in X \) and \( x \in A \), we have
\[\mu(ax - a(b - x)) \geq \mu(x) = s. \]
Thus \( ax - a(b - x) \in A \).

For all \( x, y \in A \) then
\[\mu(xy) \geq \mu(x) = s. \]
Thus \( xy \in A \). Hence, \( \mu_A \) is an ideal of \( X \).

Conversely, suppose that \( A \) is an ideal of \( X \). Let \( x, y \in X \). If at least one of \( x \) and \( y \) does not belong to \( A \), then
\[\mu_A(x - y) \geq t = \min \{ \mu_A(x), \mu_A(y) \}. \]
If \( x, y \in A \) then \( x - y \notin A \), we have
\[\mu_A(x - y) \geq s = \min \{ \mu_A(x), \mu_A(y) \}. \]
Let \( a, b, x \in X \) and if \( x \in A \) such that \( ax - a(b - x) \in A \), we have
\[\mu_A(ax - a(b - x)) \geq s = \mu_A(x). \]
If \( x \notin A \) such that \( ax - a(b - x) \notin A \), we have
\[\mu_A(ax - a(b - x)) \geq t = \mu_A(x). \]
For all \( x, y \in A \) then \( xy \in A \), we have
\[\mu_A(xy) \geq s = \mu(x). \]
Suppose \( x \notin A \) then
\[\mu_A(xy) \geq t = \mu(x). \]
Hence \( \mu_A \) is a fuzzy ideal of \( X \). Moreover
\[X_{\mu_A} = \{ x \in X | \mu_A(x) = \mu_A(0) \} = \{ x \in X | \mu_A(x) = s \} = \{ x \in X | x \in A \} = A. \]
only if $A$ is a left(resp. right) ideal.

**Theorem 3.6:** Let $\mu$ be a fuzzy subset of $X$. Then $\mu$ is a fuzzy ideal of $X$ if and only if each non-empty level subset $U(\mu; t)$ of $\mu$ is an ideal of $X$.

**Proof:** Assume that $\mu$ is a fuzzy ideal of $X$ and $U(\mu; t)$ is a non-empty level subset of $X$.

(i) Since $U(\mu; t)$ is a non-empty level subset of $\mu$, there exists $x, y \in U(\mu; t)$, $\mu(x - y) \geq \min(\mu(x), \mu(y)) = t$. Thus $x - y \in U(\mu; t)$.

(ii) Let if possible, $x, y \in U(\mu; t)$, we have $a(x - a(b - x)) \geq \mu(x) \geq t$. Thus $ax - a(b - x) \in U(\mu; t)$.

(iii) Let $x, y \in U(\mu; t)$, such that $\mu(xy) \geq \mu(x) \geq t$. Thus $xy \in U(\mu; t)$. Hence, $L(\mu; t)$ is an ideal of $R$.

Conversely, suppose that $U(\mu; t)$ is an ideal of $X$.

(i) Let if possible, $\mu(x_0 - y_0) < \min\{\mu(x_0), \mu(y_0)\}$, for some $x_0, y_0 \in U(\mu; t)$, then by taking

$$t_0 = \frac{1}{2}(\mu(x_0 - y_0) + \min\{\mu(x_0), \mu(y_0)\}),$$

we have $\mu(x_0 - y_0) > t_0$, for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$. Thus $x_0 - y_0 \notin U(\mu; t)$, for some $x_0, y_0 \in U(\mu; t)$. This is a contradiction, and so $\mu(x - y) \geq \min(\mu(x), \mu(y))$, for all $x, y \in U(\mu; t)$.

(ii) Let if possible, $\mu(x_0 - y_0) < \min\{\mu(x_0), \mu(y_0)\}$, for some $x_0, y_0 \in U(\mu; t)$, then by taking

$$t_0 = \frac{1}{2}(\mu(ax_0 - a(b - x_0)) + \mu(x_0)),$$

we have $\mu(ax_0 - a(b - x_0)) > t_0$, for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$. Thus $ax_0 - a(b - x_0) \notin U(\mu; t)$, for some $x_0 \in U(\mu; t)$ and for all $a, b \in X$. This is a contradiction, and so $\mu(ax - a(b - x)) \geq \mu(x)$, for all $x \in U(\mu; t)$ and $a, b \in X$.

(iii) Let if possible, $\mu(x_0 - y_0) < \min\{\mu(x_0), \mu(y_0)\}$, for some $x_0, y_0 \in U(\mu; t)$, then by taking

$$t_0 = \frac{1}{2}(\mu(x_0y_0) + \mu(x_0)),$$

we have $\mu(x_0y_0) > t_0$, for $\mu(x_0) \geq t_0, \mu(y_0) \geq t_0$. Thus $x_0y_0 \notin U(\mu; t)$, for some $x_0, y_0 \in U(\mu; t)$, and for all $x, y \in U(\mu; t)$. Hence $U(\mu; t)$ is a fuzzy ideal of $X$.

**Definition 3.7:** Let $X$ be a near-subtraction semigroup and a family of fuzzy sets $\{\mu_i\}_{i \in I}$ in $X$. Then the intersection

$$\bigwedge_{i \in I} \mu_i$$

of $\{\mu_i\}_{i \in I}$ is defined by

$$\bigwedge_{i \in I} \mu_i(x) = \inf \{\mu_i(x) | i \in I\}$$

**Theorem 3.8:** If $\{\mu_i\}_{i \in I}$ is a family of fuzzy ideal of $X$, then $\bigwedge_{i \in I} \mu_i(x)$ is a fuzzy ideal of $X$.

**Proof:** Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy ideal of $X$.

(i) For all $x, y \in X$, we have

$$\left(\bigwedge_{i \in I} \mu_i\right)(x - y) = \inf \{\mu_i(x - y) | i \in I\} \geq \inf \{\min(\mu_{i}(x), \mu_{i}(y)) | i \in I\}$$

$$= \min \{\inf(\mu_{i}(x)|i \in I), \inf(\mu_{i}(y)|i \in I)\}$$

$$= \min \left\{\left(\bigwedge_{i \in I} \mu_i\right)(x), \left(\bigwedge_{i \in I} \mu_i\right)(y)\right\}$$

(ii) For all $a, b, x \in X$, we have

$$\left(\bigwedge_{i \in I} \mu_i\right)(ax - a(b - x)) = \inf \{\mu_i(ax - a(b - x)) | i \in I\} \geq \inf \{\min(\mu_{i}(x)) | i \in I\}$$

$$= \inf \{\mu_{i}(x)|i \in I\}$$

$$= \left(\bigwedge_{i \in I} \mu_i\right)(x).$$

Hence $\left(\bigwedge_{i \in I} \mu_i\right)$ is a fuzzy ideal of $X$.

**Definition 3.9:** Let $f : X \to X'$ be a mapping, where $X$ and $X'$ are non-empty sets and $\mu$ is a fuzzy subset of $X$. The preimage of $\mu$ under $f$ written $\mu^f$ is a fuzzy subset of $X$ defined by $\mu^f = \mu(f(x))$, for all $x \in X$.

**Theorem 3.10:** Let $f : X \to X'$ be a homomorphism of near-subtraction semigroups. If $\mu$ is a fuzzy ideal of $X'$, then $\mu^f$ is a fuzzy ideal of $X$.

**Proof:** Suppose $\mu$ is a fuzzy ideal of $X'$, then

(i) For all $x, y \in X$, we have

$$\mu^f(x - y) = \mu(f(x - y)) = \mu(f(x) - f(y))$$

$$\geq \min \{\mu(f(x)), \mu(f(y))\}$$

$$= \min \{\mu^f(x), \mu^f(y)\}.$$
(iii) For all $x, y \in X$, we have
\[
\mu^f(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \geq \mu(f(x)) = \mu^f(y).
\]
Hence $\mu^f$ is a fuzzy ideal of $X$.

**Theorem 3.11:** Let $f : X \to X'$ be a homomorphism of near-subtraction semigroup. If $\mu^f$ is a fuzzy ideal of $X$, then $\mu$ is a fuzzy ideal of $X'$.

**Proof:** Suppose $\mu$ is a fuzzy ideal of $X$, then
(i) Let $x', y' \in X'$, there exists $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$, we have
\[
\mu(x'-y') = \mu(f(x) - f(y)) = \mu(f(x - y)) \geq \mu^f(x - y) \geq \min \{\mu^f(x), \mu^f(y)\} = \mu(x', y').
\]
(ii) Let $a', b', x' \in X'$, there exists $a, b, x \in X$ such that $f(a) = a', f(b) = b'$ and $f(x) = x'$, we have
\[
\mu(a'x' - b'(a' - x')) = \mu((f(a)f(x) - f(b)(f(a) - f(x)))) = \mu(f(ax) - f(b)(a - x)) \geq \mu(f(ax) - f(b(a - x))) = \mu(f(ax - b(a - x))) = \mu^f(x) \geq \mu^f(x).
\]
(iii) Let $x', y' \in X'$, there exists $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$, we have
\[
\mu(x'y') = \mu(f(x)f(y)) = \mu(f(xy)) = \mu^f(xy) \geq \mu^f(x) = \mu(f(x)) = \mu^f(x') = \mu(x').
\]

Hence $\mu$ is a fuzzy ideal of $X'$.

**Definition 3.12:** Let $f$ be a mapping defined on $X$. If $\nu$ is a fuzzy subset of $f(X)$, then the fuzzy subset $\mu = \nu \circ f$ in $X$ (i.e., the fuzzy subset defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called the preimage of $\nu$ under $f$.

**Proposition 3.13:** An onto homomorphic preimage of a fuzzy ideal of $X$ is a fuzzy ideal.

**Proof:** Straightforward.

Let $\mu$ be a fuzzy subset in $X$ and $f$ be a mapping defined on $X$. Then the fuzzy subset $\mu^f$ in $f(X)$ defined by $\mu^f(y) = \sup_{z \in f^{-1}(y)} \mu(z)$ for all $y \in f(X)$ is called the image of $\mu$ under $f$. A fuzzy subset $\mu$ in $X$ is said to have an sup property if for every subset $N \subseteq X$, there exists $n_0 \in N$ such that $\mu(n_0) = \sup_{n \in N} \mu(n)$.

**Proposition 3.14:** An onto homomorphic image of a fuzzy ideal with sup property is fuzzy ideal.

**Proof:** Let $f : X \to X'$ be an onto homomorphism of near-subtraction semigroup and let $\mu$ be a fuzzy ideal of $X$ with the sup property.

(i) Given $x', y' \in X'$, we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that
\[
\mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n), \mu(y_0) = \sup_{n \in f^{-1}(y')} \mu(n)
\]
respectively. Then, we have
\[
\mu^f(x'-y') = \sup_{z \in f^{-1}(x'-y')} \mu(z) \geq \min \left\{ \mu(x_0), \mu(y_0) \right\} = \mu^f(x') = \mu^f(y')
\]
(ii) Given $a', b', x' \in X'$, we let $a_0 \in f^{-1}(a')$, $b_0 \in f^{-1}(b')$, $x_0 \in f^{-1}(x')$ be such that
\[
\mu^f(a'x' - b'(a' - x')) = \sup_{z \in f^{-1}(a'x' - b'(a' - x'))} \mu(z) \geq \mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n) = \mu^f(x')
\]
(iii) Given $x', y' \in X'$, we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that
\[
\mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n), \mu(y_0) = \sup_{n \in f^{-1}(y')} \mu(n)
\]
respectively. Then, we have
\[
\mu^f(x'y') = \sup_{z \in f^{-1}(x'y')} \mu(z) \geq \mu(x_0) = \sup_{n \in f^{-1}(x')} \mu(n) = \mu^f(x')
\]
Hence, $\mu^f$ is a fuzzy ideal of $X'$.

**IV. Chain Conditions**

**Proposition 4.1:** Let $\mu$ and $\nu$ be a fuzzy subset of $X$. If they are fuzzy ideal of $X$, then $\mu \cap \nu$, where $\mu \cap \nu$ is defined by
\[(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}\] for all \(x \in X\).

**Proof:** (i) For all \(x, y \in X\), we have

\[
(\mu \cap \nu)(x - y) = \min\{\mu(x - y), \nu(x - y)\} \geq \min\{\mu(x), \nu(x)\}.
\]

\[
= \min\{\mu(x), \nu(y)\} = (\mu \cap \nu)(y).
\]

(ii) For all \(x, y \in X\), we have

\[
(\mu \cap \nu)(ax - a(b - x)) = \min\{\mu(ax - a(b - x)), \nu(ax - a(b - x))\} \geq \min\{\mu(x), \nu(x)\}.
\]

\[
= (\mu \cap \nu)(x).
\]

(iii) For all \(x, y \in X\), we have

\[
(\mu \cap \nu)(xy) = \min\{\mu(xy), \nu(xy)\} \geq \min\{\mu(y), \nu(y)\}.
\]

\[
= (\mu \cap \nu)(y).
\]

Hence, \(\mu \cap \nu\) is a fuzzy ideal of \(X\).

**Theorem 4.2:** Let \(\mu\) be a fuzzy subset in \(X\) and \(IM(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}\), where \(\alpha_i < \alpha_j\) whenever \(i > j\). Let \(\{A_n | n = 0, 1, \ldots, k\}\) be a family of ideals of \(X\) such that

(i) \(A_0 \subseteq A_1 \subseteq \ldots \subseteq A_k = X\).

(ii) \(\mu(A^*) = \alpha_n\), where \(A^*_n = A_n \setminus A_{n-1}, A_{-1} = \phi\) for all \(n = 0, 1, \ldots, k\).

Then \(\mu\) is a fuzzy ideal of \(X\).

**Proof:** Suppose \(\{A_n | n = 0, 1, \ldots, k\}\) be a family of ideals of \(X\).

(i) For all \(x, y \in X\), Then we discuss the following cases:

1. \(x \in A_n\) and \(y \in X \setminus A_n\)
2. \(x \in A_{n-1}\) and \(y \in A_{n-1}\)
3. \(x \in X \setminus A_n\) and \(y \in A_{n-1}\)
4. \(x \in A_{n-1}\) and \(y \in R \setminus A_n\)

But, in either case, we know that

\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.
\]

If \(x \in X \setminus A^*_n\) and \(y \notin A^*_n\), then either \(y \in A_{n-1}\) or \(y \in X \setminus A_n\). It follows that either \(x \in A_n\) or \(x \in X \setminus A_n\). Thus

\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.
\]

If \(x \notin X \setminus A^*_n\) and \(y \in A^*_n\), then by similar process we have

\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.
\]

(ii) If \(a, b \in X\) and \(x \in A_n\), then \(ax - a(b - x) \in A_n\). Then

\[
\mu(ax - a(b - x)) \geq \min\{\mu(a), \mu(b)\}.
\]

If \(a, b \in X\) and \(x \notin A_n\), then we have

\[
\mu(ax - a(b - x)) \geq \alpha_n = \mu(x).
\]

(iii) Similarly, for \(x, y \in X\), we have

\[
\mu(xy) \geq \mu(y).
\]

Hence \(\mu\) is a fuzzy ideal of \(X\).

**Theorem 4.3:** Let \(\{A_n | n \in N\}\) be a family of ideals of \(X\) which is nested, that is \(X = A_1 \supset A_2 \supset \ldots\) Let \(\mu\) be a fuzzy subset in \(X\) defined by

\[
\mu(x) = \begin{cases} 
\frac{n}{n + 1} & \text{if } x \in A_n, A_{n+1}, n = 1, 2, 3, \\
1 & \text{if } x \in \bigcap_{n=1}^{\infty} A_n.
\end{cases}
\]

for all \(x \in X\). Then \(\mu\) is a fuzzy ideal of \(X\).

**Proof:** Let \(x, y \in X\).

(i) Suppose that \(x \in A_k \setminus A_{k+1}\) and \(y \in A_r \setminus A_{r+1}\) for \(k = 1, 2, \ldots; r = 1, 2, \ldots\). Without loss of generality, we may assume that \(k \leq r\). Then \(x - y \in A_k\) and so

\[
\mu(x - y) \geq \frac{k}{k + 1} = \min\{\mu(x), \mu(y)\}.
\]

If \(x, y \in \bigcap_{n=1}^{\infty} A_n\) then \(x - y \in \bigcap_{n=1}^{\infty} A_n\) and thus

\[
\mu(x - y) = 1 = \min\{\mu(x), \mu(y)\}.
\]

Similarly, we can prove that

\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\}
\]

for all \(x \notin \bigcap_{n=1}^{\infty} A_n\) then \(y \notin \bigcap_{n=1}^{\infty} A_n\).

(ii) Now let \(a, b \in X\). If \(x \in A_k \setminus A_{k+1}\) for some \(k = 1, 2, \ldots\), then \(ax - a(b - x) \in A_k\). Thus

\[
\mu(ax - a(b - x)) \geq \frac{k}{k + 1} = \mu(x).
\]

If \(x \in \bigcap_{n=1}^{\infty} A_n\) then \(ax - a(b - x) \in \bigcap_{n=1}^{\infty} A_n\) for all \(a, b \in X\). Thus

\[
\mu(ax - a(b - x)) = \frac{r}{r + 1} = \mu(x).
\]

Assume that \(a \in A_r \setminus A_{r+1}\) for some \(r = 1, 2, 3, \ldots\) and \(b \in \bigcap_{n=1}^{\infty} A_n\) or \(a \in \bigcap_{n=1}^{\infty} A_n\) and \(b \in A_r \setminus A_{r+1}\) for some \(r = 1, 2, 3, \ldots\). Then \(x \in A_{r+1}\) and so

\[
\mu(ax - a(b - x)) \geq \frac{r}{r + 1} = \mu(x).
\]

(iii) Now let \(x, y \in A_k \setminus A_{k+1}\) for some \(r = 1, 2, 3, \ldots\), then \(y \in A_r\) as \(A_r\) is an ideal of \(X\). Thus

\[
\mu(xy) \geq \frac{r}{r + 1} = \mu(y).
\]

If \(x, y \in \bigcap_{n=1}^{\infty} A_n\) then \(y \in \bigcap_{n=1}^{\infty} A_n\) and so

\[
\mu(xy) = 1 = \mu(y).
\]
Hence, \( \mu \) is a fuzzy ideal of \( X \).

Let \( \mu : X \rightarrow [0, 1] \) be a fuzzy subset of \( X \). The smallest fuzzy ideal containing \( \mu \) is called the fuzzy ideal generated by \( \mu \), and \( \mu \) is said to be \( n \)-valued if \( \mu(X) \) is a finite set of \( n \) elements. When no specific \( n \) is intended, we call \( \mu \) a finite-valued fuzzy subset.

**Theorem 4.4:** A fuzzy ideal \( \nu \) of \( X \) is finite valued if and only if a finite-valued fuzzy subset \( \mu \) of \( X \) is generated by \( \nu \).

**Proof:** If \( \nu : X \rightarrow [0, 1] \) is a finite-valued fuzzy ideal of \( X \), then one may choose \( \mu = \nu \). Consequently, assume that \( \mu : X \rightarrow [0, 1] \) is an \( n \)-valued fuzzy subset with \( n \) distinct values \( t_1, t_2, \ldots, t_n \), where \( t_1 > t_2 > \ldots > t_n \). Let \( G^i \) be the inverse image of \( t_i \) under \( \mu \), that is, \( G^i = \mu^{-1}(t_i) \). Obviously, \( \bigcup_{i=1}^{r} G^i \subseteq t_j \) for \( j < r \). We denote by \( A^i \) the ideal of \( X \) generated by the set \( \bigcup_{i=1}^{j} G^i \). Then we have the following chain of ideals:

\[
A^1 \subseteq A^2 \subseteq \ldots \subseteq A^n = X.
\]

Define a fuzzy \( \nu : X \rightarrow [0, 1] \) by

\[
\nu(x) = \begin{cases} 
0 & \text{if } x \in \bigcap_{i=1}^{r} A^i, \\
1 & \text{if } x \in \bigcup_{i=1}^{r} A^i \\
\nu(x) & \text{otherwise}
\end{cases}
\]

We claim that \( \nu \) is a fuzzy ideal of \( X \) and \( \mu \) is generated by \( \nu \). Let \( x, y \in X \) and let \( i \) and \( j \) be the smallest integer such that \( x \in A^i \) and \( y \in A^j \). We may assume that \( i > j \) without loss of generality. Then \( x - y \) generates a finite-valued fuzzy subset of \( X \). Thus

\[
\nu(x - y) \geq t_j = \min \{ t_i, t_j \} = \min \{ \nu(x), \nu(y) \}
\]

and

\[
\nu(xy) \geq t_j = \nu(y).
\]

Now, let \( \mu \) be any fuzzy ideal of \( X \) and let \( \gamma \) be any fuzzy ideal of \( X \) which is a subset of \( \mu \). Then, \( \bigcup_{i=1}^{r} G^i = U(\mu; t_j) \subseteq U(\gamma; t_j) \), and thus \( A^i \subseteq A^j \). Hence, \( \gamma \subseteq \mu \) and \( \mu \) is generated by \( \nu \). Note that \( |\text{Im} \gamma| = n = |\text{Im} \nu| \). This completes the proof.

A near-subtraction semigroup \( X \) is said to be *Noetherian* (see [9]) if it satisfies the ascending chain condition on ideals of \( X \).

**Theorem 4.5:** If \( X \) is a Noetherian near-subtraction semigroup, then every fuzzy ideal of \( X \) is finite valued.

**Proof:** Let \( \mu : X \rightarrow [0, 1] \) be a fuzzy ideal of \( X \) which is not finite valued. Then, there exists sequence of distinct numbers \( \mu(0) = t_1 > t_2 > \ldots > t_n \), where \( t_i = \mu(x_i) \) for some \( x_i \in R \). This sequence induces an infinite sequence of distinct ideals of \( X \):

\[
U(\mu; t_1) \subseteq U(\mu; t_2) \subseteq \ldots \subseteq U(\mu; t_n) \subseteq \ldots
\]

This is a contradiction.

Combining Theorem 4.4 and Theorem 4.5, we have the following corollary.

**Corollary 4.6:** If \( X \) is a Noetherian near-subtraction semigroup, then every fuzzy ideal of \( X \) is generated by a finite fuzzy subset in \( X \).

V. NORMAL FUZZY IDEALS

**Definition 5.1:** A fuzzy ideal \( \mu \) of \( X \) is said to be *normal* if \( \mu(a) = 1 \) for all \( a \in X \).

We note that if \( \mu \) is a normal fuzzy ideal of \( X \), then \( \mu(1) = 1 \) if and only if \( \mu(X) = 1 \).

**Theorem 5.2:** Let \( \mu \) be a fuzzy ideal of \( X \) and let \( \mu^+ \) be a fuzzy set in \( X \) generated by \( \mu(x) + 1 - \mu(1) \) for all \( x \in X \). Then \( \mu^+ \) is a fuzzy ideal of \( X \).

**Proof:** Let \( x, y, z \in X \) we \( \mu^+(x) = \mu(x) + 1 - \mu(1) \) and \( \mu^+(1) = 1 \).

(i) For all \( x, y, z \in X \), we have

\[
\mu^+(x - y) = \mu(x - y) + 1 - \mu(1) \geq \mu(x) + 1 - \mu(1) = \mu^+(x).
\]

(ii) For all \( a, b, c \in X \), we have

\[
\mu^+(a - b + c) = \mu(a - b + c) + 1 - \mu(1) \geq \mu(a) + 1 - \mu(1) = \mu^+(a).
\]

(iii) For all \( x, y \in X \), we have

\[
\mu^+(xy) = \mu(xy) + 1 - \mu(1) \geq \mu(y) + 1 - \mu(1) = \mu^+(y).
\]

Hence \( \mu^+ \) is a fuzzy ideal of \( X \). Obviously, \( \mu \subseteq \mu^+ \).

**Corollary 5.3:** If \( \mu \) be a fuzzy ideal of \( X \) satisfying \( \mu^+(a) = 0 \) for some \( a \in X \), then \( \mu(a) = 0 \).

It is clear that fuzzy ideal \( \mu \) of \( X \) is normal if and only if \( \mu = \mu^+ \). Hence if \( \mu \) is a normal fuzzy ideal of \( X \), then \( \mu^+ = \mu \).

**Theorem 5.4:** Let \( \mu \) be a fuzzy ideal of \( X \) and let \( \phi : [0, \mu(0)] \rightarrow [0, 1] \) be an increasing function. Let \( \mu_\phi \) be a fuzzy set in \( X \) defined by \( \mu_\phi(x) = \phi(\mu(x)) \) for all \( x \in X \). Then \( \mu_\phi \) is a fuzzy ideal of \( X \).

**Proof:** (i) Let \( x, y \in X \). Then

\[
\mu_\phi(x - y) = \phi(\mu(x - y)) \geq \phi(\min(\mu(x), \mu(y))) = \min(\phi(\mu(x)), \phi(\mu(y))) = \min(\mu_\phi(x), \mu_\phi(y)).
\]
(ii) Let $a, b, x \in X$. Then
\[
\mu_{\phi}(ax - a(b - x)) = \phi(\mu(ax - a(b - x))) \\
\geq \phi(\mu(x)) \\
= \mu_{\phi}(x).
\]

(iii) Let $x, y \in X$. Then
\[
\mu_{\phi}(xy) = \phi(\mu(xy)) \\
\geq \phi(\mu(y)) \\
= \mu_{\phi}(y).
\]

Hence $\mu_{\phi}$ is a fuzzy ideal of $X$. If $\phi(\mu(0)) = 1$ then obviously $\mu_{\phi}$ is normal, and so $\mu_{\phi} \in \mathbb{F}_N(X)$. Assume that $\phi(t) \geq t$ for all $t \in [0, \mu(0)]$. Then $\mu_{\phi}(x) = \phi(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that $\mu \subseteq \mu_{\phi}$.

Theorem 5.5: Let $\mu \in \mathbb{F}_N(X)$ be a non-constant maximal element of the set $(\mathbb{F}_N(X), \subseteq)$. Then $\mu$ takes only the values 0 and 1.

Proof: Since $\mu$ is normal, we have $\mu(0) = 1$. Let $\mu(x) \neq 1$ for some $x \in X$. We claim that $\mu(x) = 0$ if not, then there exists $x_0 \in X$ such that $0 < \mu(x_0) < 1$. Define on $X$ a fuzzy set $\nu$ putting $\nu(x) = \frac{\mu(x) + \nu(x_0)}{2}$ for all $x \in X$. Then, clearly $\nu$ is well-defined.

(i) For all $x, y \in X$, we have
\[
\nu(x - y) = \frac{\mu(x - y) + \mu(x_0)}{2} \\
\geq \frac{\min\{\mu(x), \mu(y)\} + \mu(x_0)}{2} \\
= \frac{\mu(x) + \mu(x_0)}{2}. \\
\]

(ii) For all $a, b, x \in X$, we have
\[
\nu(ax - a(b - x)) = \frac{\mu(ax - a(b - x)) + \mu(x_0)}{2} \\
\geq \frac{\min\{\mu(x), \mu(y)\} + \mu(x_0)}{2} \\
= \nu(x).
\]

Thus $\nu$ is a fuzzy ideal of $X$. By Theorem 5.2, $\nu^+$ is a maximal fuzzy ideal of $X$. Note that
\[
\nu^+(x_0) = \frac{\nu(x_0) + 1 - \nu(0)}{2} \\
= \frac{\mu(x_0) + \mu(x_0)}{2} + 1 - \frac{\mu(0) + \mu(x_0)}{2} \\
= \frac{\mu(x_0) + 1}{2}.
\]

and $\nu^+(x_0) < 1 = \frac{\mu(0)+1}{2} = \nu^+(0)$. Hence $\nu^+$ is non-constant, and $\mu$ is not a maximal element of $\mathbb{F}_N(X)$. This is a contradiction.

Definition 5.6: A fuzzy ideal $\mu$ of $X$ is said to be maximal if it satisfies:

(M1) $\mu$ is non-constant, and
(M2) $\mu^+$ is a maximal element of $(\mathbb{F}_N(X), \subseteq)$.

Theorem 5.7: If a fuzzy ideal of $X$ is maximal, then

(i) $\mu$ is normal,
(ii) $\mu$ takes only the values 0 and 1,
(iii) $X_{\mu^0} = \mu$, where $\mu^0 = \{x \in X | \mu(0) = 1\}$,
(iv) $\mu^0$ is a maximal ideal of $X$.

Proof: Let $\mu$ be a maximal fuzzy ideal of $X$. Then $\mu^+$ is a non-constant maximal element of the poset $(\mathbb{F}_N(X), \subseteq)$. It follows from Theorem 5.5 that $\mu^*$ takes only two values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = 0$, and $\mu^+(0) = 0$ if and only if $\mu(x) = 0$. By Corollary 5.3, we have $\mu(x) = 0$ and so $\mu(0) = 1$. Hence $\mu$ is normal and $\mu^+ = \mu$. This proves (i) and (ii).

(iii) Obvious.
(iv) It is clear that $\mu^*$ is a proper ideal of $X$. Obviously $\mu^0 \neq X$ because $\mu$ takes two values. Let $A$ be an ideal containing $\mu^0$. Then $\mu_{\phi^0} \subseteq \mu_A$, and consequently, $\mu = \mu_A^0 \subseteq \mu_A$. Since $\mu$ is normal, $\mu_A$ is also normal and takes only two values 0 and 1. But, by the assumption, $\mu$ is maximal, so $\mu = \mu_A$ or $\mu = \phi$, where $\phi(x) = 1$ for all $x \in X$. In the last case $\mu^0 = X$, which is impossible. So, $\mu = \mu_A$, i.e., $\mu_A = \chi_X$. Hence $\mu^0 = A$.

Definition 5.8: A fuzzy ideal $\mu$ of $X$ is said to be complete if it is normal and there exists $z \in X$ such that $\mu(z) = 0$.

Theorem 5.9: Let $\mu$ be a fuzzy ideal of $X$ and let $w$ be a fixed element of $X$ such that $\mu(1) = \mu(w)$. Define a fuzzy set $\mu^*$ in $X$ by $\mu^*(x) = \frac{\mu(x) + \mu(w)}{\mu(1) + \mu(w)}$ for all $x \in X$. Then $\mu^*$ is a complete fuzzy ideal of $X$.

Proof: (i) For any $x, y \in X$, we have
\[
\mu^*(x - y) = \frac{\mu(x - y) - \mu(w)}{\mu(1) - \mu(w)} \\
\geq \frac{\min\{\mu(x), \mu(y)\} - \mu(w)}{\mu(1) - \mu(w)} \\
= \min\{\mu^*(x), \mu^*(y)\}.
\]

(ii) For any $x, y \in X$, we have
\[
\mu^*(ax - a(b - x)) = \frac{\mu(ax - a(b - x)) - \mu(w)}{\mu(1) - \mu(w)} \\
\geq \frac{\min\{\mu(x), \mu(y)\} - \mu(w)}{\mu(1) - \mu(w)} \\
= \mu^*(x).
\]

(iii) For any $x, y \in X$, we have
\[
\mu^*(xy) = \frac{\mu(xy) - \mu(w)}{\mu(1) - \mu(w)} \\
\geq \frac{\min\{\mu(x), \mu(y)\} - \mu(w)}{\mu(1) - \mu(w)} \\
= \mu^*(y).
\]
Hence $\mu^* \in \mathbb{F}_N(S)$. Since $\mu^*(w) = 0$, thus $\mu^*$ is a complete fuzzy ideal of $X$.

**Theorem 5.10:** Every maximal fuzzy ideal of $X$ is completely normal.

**Proof:** Let $\mu$ be a maximal fuzzy ideal of $X$. Then by Theorem 5.7, $\mu$ is a normal and $\mu = \mu^+$ takes only two values 0 and 1. Since $\mu$ is non-constant, it follows that $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in X$. Hence $\mu$ is completely normal.

**REFERENCES**


**D.R Prince Williams** received his Masters degree in Mathematics (1991) and Master of Philosophy in Mathematics (1992) from Pachyappa’s College, University of Madras, Chennai, India. He received his Ph.D degree (1999-2004) from Department of Mathematics, Anna University, Chennai, India. From 1992 to 2004 he worked as Mathematics Faculty in various Engineering Colleges in Chennai, India. From 2005 to 2007, he worked as teaching faculty in Department of Information Technology, Salalah College of Technology, Salalah, Sultanate of Oman. Presently he is working as teaching faculty in Department of Information Technology, Sohar College of Applied Sciences, Sohar, Sultanate of Oman. He has published many papers in national and international journals. His research interests are in the areas of Fuzzy Algebraic Structures, Software Reliability and Mathematical Modelling.