A Fuzzy Mixed Integer Multi-Scenario Portfolio Optimization Model

M. S. Osman, A. A. Tharwat, I. A. El-Khodary, and A. G. Chalabi

Abstract—In this paper, we propose a multiple objective optimization model with respect to portfolio selection problem for investors looking forward to diversify their equity investments in a number of equity markets. Based on Markowitz’s M-V model we developed a Fuzzy Mixed Integer Multi-Objective Nonlinear Programming Problem (FMIMONLP) to maximize the investors’ future gains on equity markets, reach the optimal proportion of the budget to be invested in different equities. A numerical example with a comprehensive analysis on artificial data from several equity markets is presented in order to illustrate the proposed model and its solution method. The model performed well compared with the deterministic version of the model.

Keywords—Equity Markets, Future Scenarios, Portfolio Selection, Multiple Criteria Fuzzy Optimization

I. INTRODUCTION

Portfolio Selection is how to configure a variety of equities positions to best meet the decision makers (DMs) risk and return trade-off. In 1952, Markowitz the founder of modern portfolio theory assumed DMs are risk averse, and variance is a measure for investment risk; a Mean-Variance portfolio selection MV-Model is established by Markowitz [1] who assumed it is necessary to calculate the covariance between the risky assets, giving the model to calculate the actual operation has brought difficulties.

DMs can efficiently allocate their capitals through potential portfolio diversification to include a number of multi-national risky equities that have several fuzzy returns or short-term holding periods. The fuzzy returns result in conflicting future alternatives. If equity returns related to one scenario are one of those conflicting alternatives, then a multi criteria mathematical portfolio program can be developed which considers different scenarios in order to maximize the portfolio future returns and arrive at the net capital gain. This is to be achieved through an equity portfolio aiming to reach ultimate goal of preserving and generating wealth from a number of equity markets that the DM selects from.

II. PROBLEM DEFINITION

After the financial crisis, all equity markets collapsed during the year 2008 and rebounded in 2009. However some of the equity markets’ investors are still a very far from understanding the logic of the markets, if markets commonly have bubbles, then investors can efficiently allocate their capitals through potential portfolio diversification to include a number of multi-national risky equities that have several fuzzy returns. Moreover, to mitigate the risks of equity allocation within a portfolio, they shall diversify their holdings in several markets.

Possibility portfolio models were initially proposed in Tanaka and Guo [2],[3] where portfolio models are based on exponential possibility distributions, rather than the mean-variance form in Markowitz’s model that regards the portfolio selection as a probability phenomenon, possibility portfolio models integrate the past equities and experts’ judgments to catch variations to equity markets more plausibly [8], those researchers’ effort has been developed by others, such as: Zhang [4]; Lacagnina [5]; Takashi, Ishii [6],[7]; Chen, G., [8],[9], and others [6],[10], and [11].

On the other hand, a Mixed Integer Multi Objective Non Linear Programming (MIMONLP) is an efficient technique to model and solve decision problems in which several conflicting and incommensurable objectives are to be optimized simultaneously subject to specified constraints [12]. MONLP model with fuzzy parameters in its objective functions and/or constraints is called a Fuzzy MONEP problem. For the reason that the values of the parameters in a MONEP model are often imprecisely or ambiguously understood to the experts it may be more appropriate to interpret the experts’ understanding of these parameters as fuzzy values. Moreover, this could be more appropriate than modeling the MONEP problem where there are random values in the model parameters [5].

In [2],[13] the authors supposed a few models considering the future scenario with fuzzy returns and multi-objective programming problem or even portfolio multi-objective optimization, while according to our comprehensive survey there are no one develops a fuzzy model regarding portfolio considering several equity markets’ diversification. Through a survey, to a considerable extend, we could not conceive any research in multi objective models considering multi markets.

However, considering the psychology of DMs to diversify capital in several markets, and the uncertainty of given information, since it is difficult to predict important factors either decision parameters [e.g. Return on Equities, Maximum Portfolio Tolerance, Return associated to particular Scenario, number of shares per equity (Quantities or Volume), and others] or decision variables (i.e. The proportion of the total investment devoted to equity bought/sold by DM onto a particular scenario) Hence, the future return, and other parameters of future scenario can be adopted. Here, the problem is to maximize the fuzzy returns in the future after the financial crisis.
scenarios which are often in conflict with each other, where the total risk is considered fuzzy, considering several equities from different equity markets. The portfolio problem has to be converted into a Mixed Integer Nonlinear Multi-objective Programming Problem Model.

In the rest of this paper, a fuzzy multi-objective portfolio optimization model is established, and a solution method shows the utilization of fuzzy mathematics into the multi-objective fuzzy nonlinear portfolio program; finally, we gave a numerical example with comparative analysis.

III. THE FRAMEWORK OF THE PORTFOLIO PROBLEM

The Dimensional Space for the Portfolio Equity Market (DSPEM), consists of 
\[(x_{111}, ..., x_{1smce}, ..., x_{smsmce}),\]
\[m = 1, ..., NMRK, e = 1, ..., NCAT_m, s = 1, ..., S \in \mathbb{R}^{m \times s \times e}, \text{ and } x_{smce} \text{ is the } e^{th} \text{ equity in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in scenario } (s), \]
\[\text{Where } (S) \text{ is the maximum number of scenarios, } NMRK, \text{ is the maximum number of markets, } NCAT_m, \text{ is the maximum number of industrial categories per market } (m), \text{ and the maximum number of equities per category } (e) \text{ at market } (m) \text{ is } NEQT_m. \]

As in Fig. 1, for example, we have DSPEM for two scenarios, two markets, with different structure of industrial categories.

\[\text{Fig. 1 A diagram shows DSPEM for two scenarios, two markets, with different structure of industrial categories.}\]

\[x^{-}_{smce}: \text{The proportion of the total investment in future scenario devoted to the } e^{th} \text{ equity bought in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in scenario } (s), \]
\[x^{+} = (x_{111}, ..., x_{smsmce}), \text{ is the decision vector consist of } s \cdot m \cdot e \cdot e \text{ decision variables corresponding to equities bought.}\]

\[\text{A. List of Decision Variables:}\]

We describe four types of decision variables as follows:
\[
x^{-}_{smce}: \text{The proportion of the total investment in future scenario devoted to the } e^{th} \text{ equity bought in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in scenario } (s), \]
\[x^{+} = (x_{111}, ..., x_{smsmce}), \text{ is the decision vector consist of } s \cdot m \cdot e \cdot e \text{ decision variables corresponding to equities bought.}\]

\[
\text{B. The List of calculated –Decision Variables:}\]

\[x^{+}_{smce}, x^{-}_{smce}: \text{The proportion of investment devoted to the } e^{th} \text{ equity bought/sold respectively in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in the whole portfolio, where } x^{+}_{smce} = \sum_{s=1}^{S} x^{+}_{smce}, \text{ and } x^{-}_{smce} = \sum_{s=1}^{S} x^{-}_{smce}.\]

\[\text{C. List of Decision Parameters:}\]

\[\alpha_{s}: \text{The Weights expressing the probability of each scenario } (s) \text{ to be occurred in the portfolio, } 0 \leq \alpha_{s} \leq 1, \text{ Where } \sum_{s=1}^{S} \alpha_{s} = 1 \forall s = 1, 2, ..., S,\]
\[k_{smce}: \text{The Transaction Cost } ^{2}\text{ per the } e^{th} \text{ equity bought/sold in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in scenario } (s), \text{ and Transaction Costs has to be paid for both bought/sold Transactions on the equity. However, transaction costs for any market are non-fuzzy numbers.}\]
\[k_{me}: \text{The Transaction Cost per the } e^{th} \text{ equity bought in the } c^{th} \text{ industrial category in the } m^{th} \text{ market,}\]
\[\beta^{+}_{smce}: \text{ The price of proportion of total investment devoted to the } e^{th} \text{ equity bought in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in scenario } (s),\]
\[\beta^{-}_{smce}: \text{ The price of proportion of total investment devoted to the } e^{th} \text{ equity sold in the } c^{th} \text{ industrial category in the } m^{th} \text{ market in scenario } (s),\]
\[\tilde{B}: \text{ The Capital Budget for investments devoted to scenario } (s) \text{ including Trans. Costs,}\]
\[\tilde{\tilde{B}}: \text{ The Capital Budget including transaction costs for the}\]

\[\text{2 Transaction costs are fixed for the long term per market, according to the Markets’ Capital Market Association rules.}\]
whole portfolio investments, and should satisfy the following condition \( \sum_{s=1}^{S} \bar{B}_s \leq \bar{B} \),

e_s: The minimum price can be used to buy a one share from the equities have been traded on in the scenario \( s \).

\( \bar{\omega} \): The Maximum Available Risk accepted by the DM,

\( \overline{NR_{sm}} \): The minimum Net Return Ratio for the scenario \( s \),

that is calculated with respect to the budget assigned to the market \( m \) in scenario \( s \),

\( \overline{NR}_s \): The minimum Net Return Ratio for scenario \( s \), and

should satisfy the following condition: \( \sum_{m=1}^{NCAT_m} \alpha_s \cdot \overline{NR}_{sm} \leq \overline{NR}_s, \ \forall \ 1 \leq s \leq S \).

\( \rho \): The minimal total revenue for portfolio that is satisfied by the DM, it is a proportion from invested budget, and should satisfy the following condition \( \sum_{s=1}^{S} \alpha_s \cdot \overline{NR}_s + B \leq \rho \),

\( C_s(x^+, x^-) \): The number of total Transaction Costs carried by DM for all Equities included in scenario \( s \),

\( u^+_{smce}, u^-_{smce} \): The upper pounds of \( x^+_{smce}, x^-_{smce} \) corresponding to proportion of Investment devoted to the \( e^th \) equity bought / sold respectively, in the \( e^th \) industrial category in the \( m^th \) market in scenario \( s \),

\( l^+_{smce}, l^-_{smce} \): The lower pounds of \( x^+_{smce}, x^-_{smce} \) corresponding to proportion of Investment devoted to the \( e^th \) equity bought or sold respectively, in the \( e^th \) industrial category in the \( m^th \) market in scenario \( s \),

\( \overline{\theta}_{smce} \): The maximum available proportion of the total investment devoted to \( x^+_{smce}, x^-_{smce} \), equity bought by DM through scenario \( s \) to be sold by DM,

D. Pre-Preparations Calculations:

\( k^+_{mce} \): The weighted sum of all the transaction costs for the \( e^th \) equity (bought/sold) in the \( e^th \) industrial category in the \( m^th \) market at scenario \( s \), \( k^+_{mce} = \sum_{s=1}^{S} \alpha_s \cdot k^+_{smce} \),

\( k^-_{sm} \): The Transaction Cost for market \( m \) at scenario \( s \), \( k^-_{sm} = \alpha_s \cdot \sum_{e=1}^{NEQT_m} k^+_{smce} \), \( 1 \leq s \leq S \) & \( m = 1, ..., NCAT_m \),

\( k^+ \): The Transaction Cost associated to scenario \( s \), \( k^+ = \sum_{m=1}^{NCAT_m} k^-_{sm}, and \ 1 \leq s \leq S \).

\( \overline{\alpha}_{smce} \): The total fund for investments devoted to the \( e^th \) equity bought in the \( e^th \) industrial category in the \( m^th \) market in scenario \( s \), and can be expressed as the sum of the price devoted to the \( e^th \) equity bought in the \( e^th \) industrial category in the \( m^th \) market in scenario \( s \) plus the transaction cost per scenario \( s \), that is \( \overline{\alpha}_{smce} = (1 + k^+_{smce}) \cdot \overline{\alpha}_{smce} \),

\( \overline{\alpha}_{smce} \): The Total Fund has to be returned from investments devoted to the \( e^th \) equity sold in the \( e^th \) industrial category in the \( m^th \) market in scenario \( s \), it can be expressed in the same manner as \( \overline{\alpha}_{smce} \) has been expressed, that is \( \overline{\alpha}_{smce} = (1 + k^-_{sm}) \cdot \overline{\alpha}_{smce} \).

\( \bar{r}_{smce} \): The Rate of Return for the \( e^th \) equity in the \( e^th \) industrial category in the \( m^th \) market in scenario \( s \),

Where \( \bar{r}_{smce} = \frac{\overline{\alpha}_{smce} - \overline{\alpha}_{mce}}{\overline{\alpha}_{mce}} \), assuming no Dividend Yield.

\( \bar{r}_{mce} \): The rate of return for the \( e^th \) equity in the \( e^th \) industrial category in the \( m^th \) market for the equity that \( x^+_{mce} \) has been calculated on, and it can be expressed as the weighted sum of all the fuzzy rate of returns for the \( e^th \) equity in the \( e^th \) industrial category in the \( m^th \) market for scenario \( s \), \( \bar{r}_{mce} = \sum_{s=1}^{S} \alpha_s \cdot \bar{r}_{smce} \) \( \forall m = 1, ..., NCAT_m \), \( c = 1, 2, ..., NEQT_m \),

\( \bar{r}_s \): The Return associated to scenario \( s \) at market \( m \), \( \bar{r}_s = \alpha_s \cdot \sum_{e=1}^{NEQT_m} r_{smce}, 1 \leq s \leq S \) & \( m = 1, ..., NCAT_m \).

A. Objective Functions:

IV. THE PROPOSED PORTFOLIO SELECTION MODEL

A. Objective Functions:

A several holding periods are the number of scenarios devoted to the portfolio maximum expected return for any scenario \( s \) that is expressed as follows:

\[
\text{Maximize } f(x_s) = \sum_{m=1}^{NCAT_m} \sum_{c=1}^{NEQT_m} \sum_{e=1}^{S} \bar{r}_{smce} \cdot (x^+_{smce} - x^-_{smce}), \quad 1 \leq s \leq S,
\]

Where \( m = 1, ..., NCAT_m \), \( c = 1, ..., NCAT_m \), & \( e = 1, ..., NEQT_m \).

B. Feasible set:

The set of constraints can be divided into two large blocks; global constraints on the portfolio (related to the whole portfolio), and possible temporal scenario constraints (affected by holding periods). Thus Constraints \( C01, C04, C08, \) and \( C09 \) are global constraints on the portfolio, whereas Constraints \( C02, C03, C05, C06, \) and \( C07 \) are classified as temporal scenario constraints. However constraints \( C10, C11, \) and \( C12 \) are boundary restrictions constraints. Next each constraint are clarified.

\( C01 \)- As in Takashi and Ishii [9], Let redefine a constraint that represents the DM’s satisfied Risk in the whole portfolio, this is as follows:

\[
\sum_{s=1}^{S} \alpha_s \cdot \left( \sum_{m=1}^{NCAT_m} \sum_{c=1}^{NEQT_m} \sum_{e=1}^{S} (\bar{r}_{smce} - \bar{r}_{mce}) \cdot \frac{(x^+_{smce} - x^-_{smce})^2}{\overline{\alpha}_{smce}} \right) \leq \bar{\omega}
\]

Where \( m = 1, 2, ..., NCAT_m \), \( c = 1, 2, ..., NEQT_m \), and \( e = 1, 2, ..., NEQT_m \).

\( C02 \)- Also, it is more realistic expressing the minimum Net Return Ratio \( \overline{NR}_{sm} \) regarding the scenario \( s \), that is calculated with respect to the budget assigned to the market \( m \), after ignoring transaction costs \( C_s(x^+, x^-) \), in scenario \( s \), that can be constrained as follows:

\[
\sum_{s=1}^{S} \alpha_s \cdot \left( \sum_{m=1}^{NCAT_m} \sum_{c=1}^{NEQT_m} \sum_{e=1}^{S} (\bar{r}_{smce} - \bar{r}_{mce}) \cdot \frac{(x^+_{smce} - x^-_{smce})^2}{\overline{\alpha}_{smce}} \right) \leq \bar{\omega}
\]
We express that the number of shares for the equities bought traded in each scenario, that can be constrained as follows:

\[ \sum_{s=1}^{S} \sum_{m=1}^{NK} \sum_{c=1}^{NP} \sum_{e=1}^{E} \bar{r}_s \cdot (x_{sme}^+ - x_{sme}^-) \]

\[ \geq R_{s} \quad \forall \quad s = 1, 2, ..., S, \]

(C08) We define that the minimal total revenue for portfolio that is satisfied by the DM, is a proportion from total invested budget, and \( \sum_{e=1}^{E} k_s \cdot (x_{sme}^+ - x_{sme}^-) \geq R_{s} \), should be satisfied, and \( R_{s} \) is the Rate of Return for the \( e \)th equity in the \( m \)th market in scenario \( s \).

(C09) We suppose all funds must be invested in equities that are available, that is expressed as:

\[ \sum_{s=1}^{S} \sum_{m=1}^{NK} \sum_{c=1}^{NP} \sum_{e=1}^{E} x_{sme}^+ = 1, \]

(C10) It has been supposed that \( \bar{r}_{sme} \) is the maximum available proportion of equity bought \( (x_{sme}^+) \) to be sold \( (x_{sme}^-) \) by DM, it has been expressed as:

\[ x_{sme}^- \leq \bar{r}_{sme} \cdot x_{sme}^+, \quad 0 \leq \bar{r}_{sme} \leq 1, \quad m = 1, ..., NK, \quad c = 1, ..., NC, \quad e = 1, ..., NEQ, \]

(C11) It has been supposed that \( (u_{sme}^+, l_{sme}^-) \) are the upper and lower bounds expressing the proportion of total investment devoted to equities bought expressed as follows:

\[ l_{sme}^- \leq x_{sme}^- \leq u_{sme}^- \quad \forall \quad 1 \leq s \leq S, \quad m = 1, ..., NK, \quad c = 1, ..., NC, \quad e = 1, ..., NEQ. \]

Then, the Fuzzy Vector Optimization Problem (FVOP) can take the form of Mixed Integer Non-Linear Fuzzy Multi-Objective, Mathematically that can be expressed as follows:

Maximize

\[ f(s) = \sum_{m=1}^{NK} \sum_{c=1}^{NP} \sum_{e=1}^{E} \bar{r}_{sme} \cdot (x_{sme}^+ - x_{sme}^-), \]

Subject To:

\[ \sum_{s=1}^{S} \sum_{m=1}^{NK} \sum_{c=1}^{NP} \sum_{e=1}^{E} k_s \cdot (x_{sme}^+ - x_{sme}^-) \leq R_{s} \]

\[ \forall \quad 1 \leq s \leq S \]

(C05) We express The Budget Requirements devoted to scenario \( s \) as follows:

\[ \sum_{c=1}^{NP} \sum_{e=1}^{E} [(a_{sme}^+ \cdot Q_{sme}^+ + a_{sme}^- \cdot Q_{sme}^-)] \leq B_s \quad \forall \quad 1 \leq s \leq S \]

(C06) It has been supposed that \( (R_s) \) the fraction between budget, and total amount paid for investments regarding the number of shares for the equities bought traded in scenario \( s \), that has to be minimized, expressed as:

\[ B_s = \sum_{m=1}^{NK} \sum_{c=1}^{NP} \sum_{e=1}^{E} (a_{sme}^+ \cdot Q_{sme}^+ + a_{sme}^- \cdot Q_{sme}^-) \leq R_s \quad \forall \quad 1 \leq s \leq S \]

(C07) It has been defined \( (\varepsilon_s) \) that \( (r_s) \) cannot exceed in scenario \( s \) that has to be minimized, expressed as:

\[ r_s \leq \varepsilon_s \quad \forall \quad 1 \leq s \leq S \]

(C08) We express \( (\rho) \) the total Revenue for Portfolio, that is represent a percentage of the total invested Capital Budget; then
A. **Initialization:**

1. Set maximal number of \( S = S \) where \( S \) is a number of objectives, (by the Expert & Analyst),
2. Set the required Weights \( w_s = [0,1] \) \( \forall \ 1 \leq s \leq S \), are the set of pre-defined weight required for each scenarios. (by the Expert ‘Portfolio Manager’),
3. Set \( \alpha_s \in [0,1] \) the probability of each scenario \( s \) to be occurred in the portfolio, \( \sum_{s=1}^{S} \alpha_s = 1 \) (by Expert),
4. Set the number of industrial categories (\( e^{th} \) industrial categories), and the \( m^{th} \) market up to maximum number of markets to be traded on, and the \( e^{th} \) equity in the \( e^{th} \) industrial category in the \( m^{th} \) market in Scenario \( (s) \), shall be a model input parameters. (by the Expert & Analyst),
5. Set the equities expected returns \( \bar{r}_{s} \), and its transaction costs \( k_s \), (by Expert),
6. Set approximated Risk Tolerance (\( \bar{w} \)), Budget \( B,B_s \), Revenue (\( \rho \)). (by Expert),
7. Set the equities prices for every one of the buy/sell deals, \( p_{s} \) & \( \bar{p}_{s} \) & \( \theta_{s} \) (by Expert),
8. Set the upper bounds \( (u^{+}_{s}) \), (\( u^{-}_{s} \)), and lower bounds \( (l^{+}_{s}) \), (\( l^{-}_{s} \)) for investments, set \( l^{-}_{s} \) equal to zero, and \( \varepsilon_{s} \) for all \( 1 \leq s \leq S \), (by Expert).

B. **Pre-Preparational Calculations:**

9. Set return for each scenario, and return for the scenario in each market, as well as the transaction costs for each scenario. (by Expert),
10. Set \( k_{s} = \sum_{s=1}^{S} \alpha_s \cdot k_{s} \), and \( r_{s} = \sum_{s=1}^{S} \alpha_s \cdot r_{s} \), (by Analyst),
11. Set \( r_{s}, k_{s} \) for all \( 1 \leq s \leq S \ & m = 1,..,NMRK \) (by Expert),
12. Set \( \bar{r}_{s}, \bar{k}_{s} \), and \( \bar{r}_{s} = \sum_{m=1}^{NMARK} r_{s} \), \( \bar{k}_{s} = \sum_{m=1}^{NMARK} k_{s} \), \( \forall s = 1,2,..,S \), (by Analyst),
13. Set \( \bar{p}_{s} = (1 + k_{s}) \cdot \bar{p}_{s} \), \( \bar{a}_{s} = (1 + k_{s}) \cdot \bar{a}_{s} \)

C. **Determining the Fuzzy Membership Functions:**

14. Set \( \alpha = -\bar{a}^{*} \), Apply the increasing half-trapezoidal membership function for returns, and a decreasing function for risk; (by Analyst);
15. For representing the expected return on each equity existed in the scenarios of the portfolio, this can be written by next equation of increasing half-trapezoidal membership function. Fig. 2 shows the membership function for each equity return,

\[
\mu_{r}(x) = \begin{cases} 
1 & \text{if } r(x) \leq r_{i}^{0} \\
1 + \frac{r_{i}^{0} - r_{i}}{\Delta r_{i}} & \text{if } r_{i}^{0} - \Delta r_{i} \leq r_{i} \leq r_{i}^{0} \\
0 & \text{otherwise.}
\end{cases}
\]

![Fig. 2 Membership function for each equity return](image-url)

16. For representing the maximum tolerance risk for the portfolio, this can be written by next equation of decreasing membership function. Fig. 3 shows the membership function for the portfolio risk;
Fig. 3 The Membership function for the portfolio risk

D. De-fuzzyfication for the Model:

17- Solve the fuzzy MINL-VOP problem using the weighting method of VOP, and determine the sensitivity analysis, for comparative analysis. If satisfied solution, stop.

E. Solving the Model:

18- Ask the DM if the solution is satisfied, if yes Stop, and view results. If solution were not satisfied, set new Weights... go to (step 2).

19- End.

A Flowchart for the proposed model solution is illustrated in Fig. 4

VI. Numerical Example

In this section we give an example to illustrate the model for portfolio selection proposed in this paper. We suppose that one investor chooses eight different types of equities related to different number of industrial categories in two Stock Exchanges for his/her investments, assuming there are two scenarios, the first devoted to the one day settlement, and the second scenario devoted to the two days settlement given that the probabilities ($\alpha_k$) of scenario one and two to occur are 40% and 60%, respectively; the budget for each scenario is 1,000,000 EGP; Whereas investor’s required revenue at least 100,000 EGP, and for risk tolerance intervals its estimated to be between 0.035, and 0.07.

We present the given prices, expected rate of returns, and Theta’s intervals (see Table I), whereas the expected rate of returns intervals for equities in the whole portfolio are shown in Table II.

\[
\mu_\omega(p) = \begin{cases} 
1 & \text{if } \omega(p) \leq \omega_0^p \\
1 - \frac{\omega_p - \omega_0^p}{\Delta \omega_p} & \text{if } \omega_0^p \leq \omega_1 \leq \omega_0^p + \Delta \omega_p \\
0 & \text{otherwise}.
\end{cases}
\]

\[
\begin{array}{c}
\omega_0^p \\
\omega_0^p + \Delta \omega_p \\
\omega_p
\end{array}
\]

Fig. 4 Flowchart for the proposed model solution

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<th>Market #</th>
<th>Ind. Cat. #</th>
<th>Equity #</th>
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<td>3</td>
<td>0.250</td>
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<td>0.250</td>
</tr>
</tbody>
</table>

Table I: The Prices, the Expected Rate of Returns, and Theta’s Intervals

Table II: The Expected Rate of Returns Intervals for Equities in the Whole Portfolio
The summation of all decision variables related to the proportions of total equities are having the summation of one, whereas the summation of proportions of total equities sold is not exceeding One. After we run the proposed model deterministically once and fuzzed once again we found that output described in table III.

### Table III

<table>
<thead>
<tr>
<th>Scenario #</th>
<th>Market #</th>
<th>Ind. Cat. #</th>
<th>Equity #</th>
<th>Type of solution</th>
<th>Deterministic</th>
<th>Fuzzy</th>
<th>D. Variable</th>
</tr>
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<tbody>
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<td>0.011</td>
<td>0.010</td>
<td>0.000</td>
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<tr>
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<td>1</td>
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<td></td>
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<td>0.010</td>
<td>0.000</td>
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<td>0.011</td>
<td>0.010</td>
<td>0.000</td>
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<td>0.011</td>
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<td>0.000</td>
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<td>0.011</td>
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<td>0.000</td>
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</tbody>
</table>

### Table IV

<table>
<thead>
<tr>
<th>Scenario No</th>
<th>Equity</th>
<th>Vol. for bought (1)</th>
<th>Lower of total amount paid for bought (2)</th>
<th>Total budget utilized (1 X 2)</th>
<th>Upper of total amount paid for bought (3)</th>
<th>Total budget utilized (1 X 3)</th>
</tr>
</thead>
<tbody>
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<td>90</td>
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<td>250</td>
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<td>75</td>
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<td>150</td>
</tr>
</tbody>
</table>

### Table V

<table>
<thead>
<tr>
<th>Scenario #</th>
<th>Equity</th>
<th>Volumes for bought (1)</th>
<th>Total lower amount paid for bought (2)</th>
<th>Total upper amount paid for bought (3)</th>
<th>Average total amount paid for bought (4)=(2+3)/2</th>
<th>Fuzzy budget in Avg. (5)=4*1</th>
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</thead>
<tbody>
<tr>
<td>s=1</td>
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<td>75</td>
<td>100</td>
</tr>
<tr>
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<td>50</td>
<td>25</td>
<td>75</td>
<td>50</td>
<td>100</td>
</tr>
</tbody>
</table>

Table VI shows the proportion of investment devoted to the $e^{th}$ equity bought/sold respectively in the $c^{th}$ industrial category in the $m^{th}$ market in the whole portfolio, with comparison between the deterministic and the fuzzy solutions.

### Table VI

<table>
<thead>
<tr>
<th>Market #</th>
<th>Ind. Cat. #</th>
<th>Equity</th>
<th>D. Variables</th>
<th>Type of solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.010</td>
<td>Deterministic</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0.010</td>
<td>Fuzzy</td>
</tr>
</tbody>
</table>

Objective function values for the deterministic and fuzzy solutions are 0.053 and 0.060, respectively, whereas for both deterministic and fuzzy solutions $\tau_1$ decision variable scenario 1= 6 l.e., and $\tau_2$ for scenario 2= 7 l.e.. Fig 5. Describes the set of portfolios that has the maximum rate of return for every given level of risk, on other words the minimum risk for every potential rate of return, [1]. The fuzzy multi-objective portfolio optimization model performed well compared with its deterministic.
VII. CONCLUSIONS AND FUTURE RESEARCH

This paper researches the portfolio selection theory using fuzzy mathematics theory. We have proposed a maximization model of fuzzy returns in future scenarios, the fuzzy extension of multi-objective mean-variance portfolio selection problem considering equity markets’ future scenarios about net returns have been considered, and it has been proposed its solution method, with an example. The parameters of investment return and target risk are fuzzed. Then, these are described by linear half trapezoidal membership function. By comparative analyses, we get some following conclusions.

(1) In aspect of the model design, the portfolio selection model based on linear half trapezoidal membership function includes not only historical data, but also DMs’ expectation. That’s in accord with human psychology and fact gives more reliable solution when compare with the deterministic.

(2) The portfolio future return and risk aren’t only one value, but several fuzzy values can be considered through the concept of future scenarios. However the fuzzy model is able to represent the expert knowledge as well DMs’ subjective expectation.

(3) Comparing Markowitz’s programming model [1], [14], and [15] the calculation process of our model is more practical.

Certainly, there are many other aspects which should be studied in the field of fuzzy multi-Scenario portfolio optimization with multi-markets. Some of these aspects are:

(a)-A parametric analysis on the solution for the proposed model on a life data from different Equity markets.

(b)-Developing the model in the context of short selling.

(c)-Adapting Heuristics’ Search Techniques, either with increasing complexity as the number of markets becomes larger or adding non-smooth constraints to this model.

REFERENCES