The Proof of Two Conjectures Related to Pell’s Equation $x^2 - Dy^2 = \pm 4$

Armend Sh. Shabani

Abstract—Let $D \neq 1$ be a positive non-square integer. In this paper are given the proofs for two conjectures related to Pell’s equation $x^2 - Dy^2 = \pm 4$, proposed by A. Tekcan.

Keywords—Pell’s equation, solutions of Pell’s equation.

I. INTRODUCTION

Let $D \neq 1$ be any positive non-square integer and $N$ be any fixed integer. The equation

$$x^2 - Dy^2 = \pm N$$

(1)

is known as Pell’s equation. It is named mistakenly after John Pell (1611-1685) who was a mathematician who in fact did not contribute for solving it (see [2]).

For $N = 1$, the Pell’s equation $x^2 - Dy^2 = \pm 1$ is known as classical Pell’s Equation and it has infinitely many solutions $(x_n, y_n)$ for $n \geq 1$. There are different methods for finding the first non-trivial solution $(x_1, y_1)$ called the fundamental solution from which all others are easily computed (see [3], and [8]).

There are many papers in which are considered different types of Pell’s equation (see [4], [5], [6], [7]).

In these notes we will be focused on paper [1] in which A. Tekcan considered the equations:

$$x^2 - D^2 = \pm 4$$

(2)

and among other results he obtained the following:

Theorem 1.1. If $(x_1, y_1)$ is the fundamental solution of the Pell’s equation $x^2 - Dy^2 = 4$ then

$$x_n = \frac{x_n x_{n-1} + Dy_{n-1} y_{n-1}}{2}; y_n = \frac{x_n y_{n-1} + x_{n-1} y_n}{2}$$

(3)

for $n \geq 2$.

Theorem 1.2. If $(x_1, y_1)$ is the fundamental solution of the Pell’s equation $x^2 - Dy^2 = -4$ then:

$$x_{n+1} = (x_1^2 + Dy_1^2) x_{n-1} + 2Dx_1 y_1 y_{n-1}$$

$$y_{n+1} = 2x_1 y_1 x_{n-1} + (x_1^2 + Dy_1^2) y_{n-1}$$

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Theorem 1.2. If $(x_1, y_1)$ is the fundamental solution of the Pell’s equation $x^2 - Dy^2 = -4$ then:

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$$y_{n+1} = 2x_1 y_1 x_{n-1} + (x_1^2 + Dy_1^2) y_{n-1}$$

for $n \geq 1$.

Also the following conjectures were proposed:

Conjecture 1.3. If $(x_1, y_1)$ is the fundamental solution of the Pell’s equation $x^2 - Dy^2 = 4$ then $(x_n, y_n)$ satisfy the following recurrence relations

$$x_n = (x_1 - 1)(x_{n-1} + x_{n-2}) - x_{n-3}$$

$$y_n = (x_1 - 1)(y_{n-1} + y_{n-2}) - y_{n-3}$$

(5)

for $n \geq 4$.

Conjecture 1.4. If $(x_1, y_1)$ is the fundamental solution of the Pell’s equation $x^2 - Dy^2 = -4$ then $(x_{2n}, y_{2n})$ satisfy the following recurrence relations

$$x_{2n} = (x_1^2 + 1)(x_{2n-2} + x_{2n-3}) - x_{2n-5}$$

$$y_{2n} = (x_1^2 + 1)(y_{2n-2} + y_{2n-3}) - y_{2n-5}$$

(6)

for $n \geq 3$.

II. MAIN RESULTS. PROOF OF CONJECTURES

We will prove above mentioned conjectures using the method of mathematical induction.

Proof of conjecture 1.3

First, we show that relations (5) are true for $n = 4$, so we have to show:

$$x_n = (x_1 - 1)(x_{n-1} + x_{n-2}) - x_{n-3}$$

$$y_n = (x_1 - 1)(y_{n-1} + y_{n-2}) - y_{n-3}$$

(7)

Using (3) we obtain the following:

$$x_{n+1} = \frac{x_n + Dy_n}{2} = \frac{x_1^2 + x_1^4 - 4}{2} = x_1^2 - 2$$

$$y_{n+1} = x_n y_n$$

(8)
Then by (3) and (8) we get:

\[
\begin{align*}
x_i &= \frac{x_{x,x} + D_{y,y} y_i}{2} = \frac{x_i(x_i^2 - 2) + D_{y,y} x_i}{2} \\
y_i &= \frac{y_{x,x} + x_i y_i}{2} = \frac{y_i(x_i^2 - 2) + x_i^2 y_i}{2}.
\end{align*}
\]

Next by (3) and (9) we find \(x_i\) and \(y_i\):

\[
x_i = \frac{x_{x,x} + D_{y,y} y_i}{2} = \frac{x_i(x_i^2 - 3) + D_{y,y} (x_i^2 - 1)}{2} = \frac{x_i^3 - 3 x_i}{2} + \frac{4 x_i^3 - 4 (x_i^2 - 1)}{2} = x_i^4 - 4 x_i^2 + 2
\]

\[
y_i = \frac{y_{x,x} + x_i y_i}{2} = \frac{y_i(x_i^2 - 3) + x_i y_i}{2} = x_i y_i (x_i^2 - 2).
\]

So we obtained:

\[
\begin{align*}
x_i &= x_i^4 - 4 x_i^2 + 2 \\
y_i &= x_i y_i (x_i^2 - 2).
\end{align*}
\]

Now, replacing (8) and (9) in (7) one obtains:

\[
x_i = (x_i - 1)(x_i^3 - 3 x_i^2 - 2) - x_i
\]

\[
y_i = (x_i - 1)(x_i^3 - x_i^2 - 3 x_i - 2) - x_i = x_i^4 - 4 x_i^2 + 2
\]

and

\[
y_i = (x_i - 1)(y_i (x_i^2 - 1) + x_i y_i) - y_i
\]

\[
y_i((x_i - 1)(x_i^2 - 1) + x_i^2 - 2) = x_i y_i (x_i^2 - 2)
\]

which are the same formulas as in (10).

It means for \(n = 4\), the recurrence relations (5) hold.

Next, we assume that (5) holds for \(n\) and we show that it holds for \(n + 1\).

Indeed, by (3) and by hypothesis we have:

\[
x_{x,i+1} = \frac{x_i((x_i - 1)(x_i + x_i) - x_{x,i})}{2} + \frac{D_{y,y} ((x_i - 1)(y_{x,i} + y_{y,i}) - y_{x,i})}{2} = (x_i - 1) \left( \frac{x_i(x_{x,i} + x_{y,i}) + D_{y,y} (y_{x,i} + y_{y,i})}{2} - x_{x,i} + D_{y,y} y_{x,i} \right) = (x_i - 1) \left( \frac{x_i(x_{x,i} + x_{y,i}) + D_{y,y} y_{x,i}}{2} + \frac{x_{x,i} + D_{y,y} y_{x,i}}{2} - x_{x,i} \right) = (x_i - 1)(x_i + x_{x,i}) - x_{x,i} = x_i(x_i^4 + 5 x_i^2 + 5)
\]

completing the proof.

**Proof of conjecture 1.4**

First, we show that relations (6) are true for \(n = 3\), so we show that:

\[
\begin{align*}
x_i &= (x_i^2 + 1)(x_i + x_i) - x_i \\
y_i &= (x_i^2 + 1)(y_i + y_i) - y_i
\end{align*}
\]

Using (4) we obtain the following:

\[
x_i = \frac{(x_i^2 + D_{y,y} y_i)x + 2D_{x,y} y_i}{4} = x_i(x_i^2 + 3D_{y,y} y_i)
\]

Since \((x_i, y_i)\) is the fundamental solution of \(x^2 - D_{y,y} = -4\), then \(x^2 + 3D_{y,y} = 4(D_{y,y}^2 - 1)\) and since \(D_{y,y} = x^2 + 4\) we obtain \(x_i = x_i(x_i^2 + 3)\). Then by (4) we have

\[
y_i = \frac{2x_i y_i x_i + (x_i^2 + D_{y,y} y_i)y_i}{4} = \frac{y_i (3x_i^2 + D_{y,y} y_i)}{4} = \frac{y_i (3x_i^2 + x_i^2 + 4)}{4} = y_i(x_i^2 + 1).
\]

So we have:

\[
\begin{align*}
x_i &= x_i(x_i^2 + 3) \\
y_i &= y_i(x_i^2 + 1).
\end{align*}
\]

Then, using (4) and (12) we obtain:

\[
x_i = \frac{(x_i^2 + D_{y,y} y_i)x(x_i^2 + 3) + 2D_{x,y} y_i (x_i^2 + 1)}{4} = \frac{(2x_i^2 + 4)x(x_i^2 + 3) + 2x_i(x_i^2 + 4)(x_i^2 + 1)}{4} = \frac{x_i ((x_i^2 + 2)(x_i^2 + 3) + (x_i^2 + 4)(x_i^2 + 1))}{4} = x_i(x_i^4 + 5 x_i^2 + 5)
\]

and
So we have.
\[
\begin{align*}
    x &= x(x^2 + 5x^2 + 5) \\
y &= y(x^4 + 3x^2 + 1)
\end{align*}
\]
Finally using (13) we obtain formulas for \(x_i\) and \(y_i\), depending on the fundamental solution \((x_1, y_1)\).
\[
x_i = \frac{(x_i^2 + D_{x_i}^2)x_i + 2Dx_i y_i}{4}
\]
and
\[
y_i = \frac{2x_i y_i + (x_i^2 + D_{y_i}^2)y_i}{4}
\]

So
\[
\begin{align*}
x_i &= x_i(x^2 + 5x^2 + 5) + (x_i^2 + 2)(x^4 + 3x^2 + 1) \\
y_i &= y_i(x^4 + 5x^2 + 6x^3 + 1)
\end{align*}
\]
Now replacing (12) and (13) in (11) we obtain:
\[
x_i = (x_i^2 + 1)(x_i + x_i) - x_i \\
= (x_i^2 + 1)(x_i(x^2 + 5x^2 + 5) + x_i(x^2 + 3)) - x_i \\
= x_i((x_i^2 + 1)(x_i^2 + 6x^3 + 8) - 1) = x_i(x_i^2 + 7x^4 + 14x^2 + 7)
\]
and
\[
y_i = (x_i^2 + 1)(y_i + y_i) - y_i \\
= (x_i^2 + 1)(y_i(x_i^4 + 3x^2 + 1) + y_i(x_i^2 + 1)) - y_i
\]
which proves that for \(n = 3\) the recurrence relations (6) are true.

Now we assume that \((x_{2n-1}, y_{2n-1})\) satisfies (6) and we prove that also \((x_{2n}, y_{2n})\) satisfies (6).

Indeed by (4) and hypothesis we obtain.
\[
x_{2n+1} = \frac{(x_i^2 + D_{x_i}^2)((x_i^2 + 1)(x_{2n} + x_{2n-1}) - x_{2n-1})}{4} \\
+ \frac{2Dx_i y_i((x_i^2 + 1)(y_{2n} + y_{2n-1}) - y_{2n-1})}{4}
\]
and
\[
y_{2n+1} = \frac{(x_i^2 + D_{y_i}^2)(x_{2n} + x_{2n-1}) + 2Dx_i y_i(y_{2n} + y_{2n-1})}{4} \\
- \frac{(x_i^2 + D_{y_i}^2)x_{2n-1} + 2Dx_i y_{2n-1}}{4}
\]
Finally
\[
\begin{align*}
x &= x(x^2 + 5x^2 + 5) \\
y &= y(x^4 + 3x^2 + 1)
\end{align*}
\]
completing the proof.
REFERENCES