Existence and uniqueness of periodic solution for a discrete-time SIR epidemic model with time delays and impulses

Ling Liu and Yuan Ye

Abstract—In this paper, a discrete-time SIR epidemic model with nonlinear incidence rate, time delays and impulses is investigated. Sufficient conditions for the existence and uniqueness of periodic solutions are obtained by using contraction theorem and inequality techniques. An example is employed to illustrate our results.

Keywords—Discrete-time SIR epidemic model, Time delay, Nonlinear incidence rate, Impulse.

I. INTRODUCTION

EPIDEMICS play a key role in the development and survival of biological species, not in the least of our own human species. Their study has thus attracted interest since ancient times. In 1927, Kermack and McKendrick [1] constructed a famous SIR epidemic model to study the law of the black plague. In 1932, an SIS mathematical model was constructed, and the "threshold theory" was established [2]. From then on, the theory and application of epidemic dynamics was developed greatly.

The basic and important concern for mathematical models in epidemiology is qualitative analysis; the persistence, permanence, asymptotic stability, and the existence and uniqueness of solutions for the model. Many influential results related in these topics have been established and can be found in many articles and books (see [3-15]).

Incidence rate plays an important role in the modelling of epidemic dynamics. In many epidemic dynamic models, the bilinear incidence rate and the standard incidence rate are frequently used. It has been suggested by several authors that the disease transmission process may have a nonlinear incidence rate (see [11, 13-15]).

Recently, much attention has been given to the persistence and global stability of epidemic model with time delays ([4-8, 14-17]).

However, to the best of our knowledge, discrete-time models are seldom considered. In a previous work we studied such a SIR discrete-time model. The advantages of a discrete-time approach are multiple. First, difference models are more realistic than differential ones since the epidemic statistics are time approach are multiple. First, difference models are more realistic than differential ones since the epidemic statistics are time delays and impulses are seldom considered. In a previous work we studied such a SIR discrete-time model. The advantages of a discrete-time SIR epidemic model. The advantages of a discrete-time approach are multiple. First, difference models are more realistic than differential ones since the epidemic statistics are time delays and impulses are seldom considered. In a previous work we studied such a SIR discrete-time model.
(H1) $\tau(n) \geq 0$ is an $\omega$-periodic function, i.e., $\tau(n + \omega) = \tau(n)$, and $(\{b_k\}, \{n_k\})$ real $\omega$-periodic sequence, where $\omega$ is an integer such that $\omega > 1$, i.e., there exists a $q \in N$ such that $n_{k+q} = n_k + \omega$, $b_{k(k+q)} = b_k, b_k > -1, 0 < n_1 < n_2 < \ldots < n_k < n_{k+1} \ldots = 1, 2, 3, \ldots$.

(H2) $\prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq n_k < \omega} (1 - \mu_1) \neq 1,$

(H3) There exist two positive numbers $r, L$ such that

$$0 < r = \max\{r_1, r_2, r_3\}, 0 < L = \max\{r, r_4\} < 1,$$

where

\[ r_1 = \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_2) \prod_{0 \leq n_k < \omega} (1 + b^*_0)\beta}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1)\alpha}, \]

\[ r_2 = \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k})\beta}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1)\alpha}, \]

\[ r_3 = \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b^*_0)\gamma}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_3)\gamma}, \]

\[ r_4 = \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b^*_0)\beta}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_3)\beta}. \]

\[ M = \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b^*_0)\beta}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1)}, \]

The remaining parts of this paper are organized as follows. In Section 2, we shall derive sufficient conditions for the existence of unique periodic solution for system (1) by using contraction theorem. We then, in Section 3, give an example to illustrate the new results of this paper.

II. Existence of unique periodic solution

Consider the following non-impulsive equation:

\[
\begin{align*}
S(n) &= \tilde{S}(n) + \sum_{0 \leq n_k < n} \prod_{0 \leq n_k < \omega} (1 + b_{1k})^{-1} \left[ B \sum_{0 \leq n_k < n} \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq n_k < \omega} (1 - \mu_1) \right], \\
\tilde{S}(n+1) - \tilde{S}(n) &= -\mu_1 \tilde{S}(n) + \sum_{0 \leq n_k < n} \prod_{0 \leq n_k < \omega} (1 + b_{2k})^{-1} \left[ B \sum_{0 \leq n_k < n} \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{0 \leq n_k < \omega} (1 - \mu_3) \right], \\
\tilde{S}(n+1) - \tilde{S}(n) &= -\mu_2 \tilde{I}(n) + \sum_{0 \leq n_k < n} \prod_{0 \leq n_k < \omega} (1 + b_{3k})^{-1} \left[ B \sum_{0 \leq n_k < n} \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{0 \leq n_k < \omega} (1 - \mu_3) \right],
\end{align*}
\]

\[ \text{(2)} \]

Lemma 1. Assume that (H1) holds. Then

(i) if $\tilde{x}(n) = (\tilde{S}(n), \tilde{I}(n), \tilde{R}(n))^T$ is a solution of (2), then

\[ x(n) = \left( \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}), \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}), \tilde{R}(n) \prod_{0 \leq n_k < n} (1 + b_{3k}) \right)^T \]

is a solution of (1);

(ii) if $x(n) = (S(n), I(n), R(n))^T$ is a solution of (1), then

\[ \tilde{x}(n) = \left( S(n) \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1}, I(n) \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1}, R(n) \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \right)^T \]

is a solution of (2).

Proof: First, we prove (i). When $n = 0, 1, 2, \ldots$, we have

\[ S(n+1) - (1 - \mu_1)S(n) - B + \frac{\beta S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \]

\[ = \tilde{S}(n+1) \prod_{0 \leq n_k < n+1} (1 + b_{1k}) - (1 - \mu_1)\tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) - B \]
\[
\begin{align*}
\beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \tilde{I}(n - \tau(n)) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})^{- \frac{1}{2}} \\
= \prod_{0 \leq n_k < n} (1 + b_{1k}) \left\{ \tilde{S}(n + 1) - (1 - \mu_1) \tilde{S}(n) \\
- \beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})^{- \frac{1}{2}} \right\} \\
= 0, \\
\end{align*}
\]

\[I(n + 1) = (1 - \mu_2 - \gamma) I(n) - \frac{\beta S(n)}{I(n) - \tau(n)} I(n - \tau(n)) \]

\[
\tilde{I}(n + 1) = \tilde{I}(n + 1) \prod_{0 \leq n_k < n + 1} (1 + b_{2k}) \\
- (1 - \mu_2 - \gamma) \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \\
- \left( \beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})^{- \frac{1}{2}} \right) \\
= \prod_{0 \leq n_k < n} (1 + b_{2k}) \left[ \tilde{I}(n + 1) - (1 - \mu_2 - \gamma) \tilde{I}(n) \\
- \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1} \right] \\
- \left( \beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right) \\
\times \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})^{- \frac{1}{2}} \right] \\
= 0, \\
\end{align*}
\]

\[
\tilde{R}(n + 1) = \tilde{R}(n + 1) \prod_{0 \leq n_k < n + 1} (1 + b_{3k}) \\
- (1 - \mu_3) \tilde{R}(n) \prod_{0 \leq n_k < n} (1 + b_{3k}) \\
- \gamma \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \\
= \prod_{0 \leq n_k < n} (1 + b_{3k}) \left[ \tilde{R}(n + 1) - (1 - \mu_3) \tilde{R}(n) \\
- \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \gamma \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \right] \\
= 0. \\
\]
\[ R(n+1) = R(n) \prod_{0 \leq n_s < n_k+1} (1 + b_{3k})^{-1} \]

\[ = (1 + b_{3k})R(n_k) \prod_{0 \leq n_s < n_k+1} (1 + b_{3k})^{-1} \]

\[ = R(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{3k})^{-1} \]

\[ = \tilde{R}(n_k). \]

Hence,

\[ \tilde{S}(n_k + 1) - \tilde{S}(n_k) \]

\[ = -\tilde{\mu}_3 \tilde{S}(n_k) + \prod_{0 \leq n_s < n_k} (1 + b_{1k})^{-1} \left[ \tilde{B} \right. \]

\[ \times \prod_{0 \leq n_s < n_k} (1 + b_{2k}) \left( 1 + \alpha \tilde{I}(n_k - \tau(n_k)) \right) \]

\[ \times \prod_{0 \leq n_s < n_k} (1 + b_{2k})^{-\frac{1}{\gamma}} \],

\[ \tilde{I}(n_k + 1) - \tilde{I}(n_k) \]

\[ = -\tilde{\mu}_3 \tilde{R}(n_k) + \prod_{0 \leq n_s < n_k} (1 + b_{3k})^{-1} \gamma \tilde{I}(n_k) \]

\[ \times \prod_{0 \leq n_s < n_k} (1 + b_{2k}). \]

When \( n \geq 0, n \neq n_k, k = 1, 2, \ldots \) it is easy to verify that \((\tilde{S}(n), \tilde{I}(n), \tilde{R}(n))^T\), also satisfies (2). Therefore, (4) is the solution of (2). This completes the proof.

**Lemma 2.** Suppose that \((H_1)\) and \((H_2)\) hold. Let \( \tilde{x}(n) = (\tilde{S}(n), \tilde{I}(n), \tilde{R}(n))^T \) be an \( \omega \)-periodic solution of (1). Then

\[ S(n) = \sum_{s=0}^{n+\omega-1} H_1 \prod_{s \leq n_s < n_k+\omega} (1 + b_{1k}) \]

\[ \times \left[ \tilde{B} - \frac{\tilde{\beta} S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right], \]

\[ I(n) = \sum_{s=0}^{n+\omega-1} H_2 \prod_{s \leq n_s < n_k+\omega} (1 + b_{2k}) \]

\[ \times \frac{\tilde{\beta} S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))}. \]

\[ R(n) = \sum_{s=0}^{n+\omega-1} H_3 \prod_{s \leq n_s < n_k+\omega} (1 + b_{3k}) \gamma I(s), \]

where

\[ H_1 = \frac{1}{\gamma} \sum_{0 \leq n_k < \omega + 1} (1 - \tilde{\mu}_1), \]

\[ H_2 = \frac{1}{\gamma} \sum_{0 \leq n_k < \omega + 1} (1 - \tilde{\mu}_2 - \gamma), \]

\[ H_3 = \frac{1}{\gamma} \sum_{0 \leq n_k < \omega + 1} (1 - \tilde{\mu}_3). \]

**Proof:** By Lemma 1, that \( \bar{x}(n) = (\bar{S}, \bar{I}, \bar{R})^T \) is a solution of (2) is equivalent to that (3) is a solution of (1). From (2) we have

\[ \tilde{S}(n + 1) - (1 - \tilde{\mu}_3) \tilde{S}(n) \]

\[ = \prod_{0 \leq n_s < n_k} (1 + b_{1k})^{-1} \left[ \tilde{B} - \frac{\tilde{\beta} S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \right]. \]

\[ \tilde{I}(n + 1) - (1 - \tilde{\mu}_2 - \gamma) \tilde{I}(n) \]

\[ = \prod_{0 \leq n_s < n_k} (1 + b_{2k})^{-1} \frac{\tilde{\beta} S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))}. \]

\[ \tilde{R}(n + 1) - (1 - \tilde{\mu}_3) \tilde{R}(n) \]

\[ = \prod_{0 \leq n_s < n_k} (1 + b_{3k})^{-1} \gamma \tilde{I}(n). \]

Multiplying both sides of the above equation (14) by \( \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_1)^{-1} \), the equation (15) by \( \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_2 - \gamma)^{-1} \), the equations (16) by \( \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_3)^{-1} \), respectively, we get

\[ S(n + 1) \prod_{0 \leq n_s < n + 1} (1 + b_{1k})^{-1} \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_1)^{-1} \]

\[ - S(n) \prod_{0 \leq n_s < n + 1} (1 + b_{1k})^{-1} \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_1)^{-1} \]

\[ = \prod_{0 \leq n_s < n + 1} (1 + b_{1k})^{-1} \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_1)^{-1} \]

\[ \times \left[ \tilde{B} - \frac{\tilde{\beta} S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \right]. \]

\[ I(n + 1) \prod_{0 \leq n_s < n + 1} (1 + b_{2k})^{-1} \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_2 - \gamma)^{-1} \]

\[ - I(n) \prod_{0 \leq n_s < n + 1} (1 + b_{2k})^{-1} \prod_{0 \leq s < n + 1} (1 - \tilde{\mu}_2 - \gamma)^{-1} \]

\[ = \prod_{0 \leq n_s < n} (1 + b_{2k})^{-1}. \]
Suppose that a unique solution of system (1) exists. It is easy to know the fact that the existence of \( \Phi: X \to X \) is a closed convex subset of \( X \). According to the definition of the norm of Banach space \( X \), we get

\[
||x|| = \max\{||S||, ||I||, ||R||, ||S|| = \Phi(S(n), I(n + \omega)), R(n + \omega) = R(n)\}.
\]

Then \( X \) is a Banach space with the norm \( ||x|| = \max\{||S||, ||I||, ||R||, ||S|| = \Phi(S(n), I(n + \omega)), R(n + \omega) = R(n)\} \).

Set a mapping \( \Phi: X \to X \) by setting

\[
\Phi x(n) = (\Phi S(n), \Phi I(n), \Phi R(n))^T,
\]

where

\[
\Phi S(n) = \sum_{s=n}^{n+\omega-1} H_1 \prod_{s\leq n_1 < n+\omega} (1 + b_{1k}) \\
\times \left[ B - \frac{\beta S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right],
\]

\[
\Phi I(n) = \sum_{s=n}^{n+\omega-1} H_2 \prod_{s\leq n_1 < n+\omega} (1 + b_{2k}) \frac{\beta S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))},
\]

\[
\Phi R(n) = \sum_{s=n}^{n+\omega-1} H_3 \prod_{s\leq n_1 < n+\omega} (1 + b_{3k}) \gamma I(s).
\]

Proof: It is easy to see that \( X^* \) is a closed convex subset of \( X \). According to the definition of the norm of Banach space \( X \), we get

\[
||x|| = \max\{\max_{0\leq n<\omega} \left| \sum_{s=n}^{n+\omega-1} H_1 \prod_{s\leq n_1 < n+\omega} (1 + b_{1k}) B \right|, 0, 0\},
\]

\[
= \max_{0\leq n<\omega} \left| \sum_{s=n}^{n+\omega-1} H_1 \prod_{s\leq n_1 < n+\omega} (1 + \beta S(s)I(s - \tau(s)) \prod_{s\leq n_1 < n+\omega} (1 - \mu_1)) \prod_{s\leq n_1 < n+\omega} (1 - \mu_1) \right| B \}
\]

\[
\omega \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1) \prod_{n_1 \leq n_1 < n+\omega} (1 - b_{1k}) B \}
\]

Therefore, for \( \forall x \in X^* \), we have

\[
||x|| \leq ||x - x_0|| + ||x_0|| \leq \frac{M r}{1 - r} + M = \frac{M r}{1 - r} \quad (20)
\]

Now, we prove that the mapping \( \Phi \) is a self-mapping from \( X^* \) to \( X^* \). In fact, for \( \forall x \in X^* \), noticing (20), we obtain

\[
\Phi S - S_0 = \max_{0\leq n<\omega} \left| \Phi S - S_0 \right|
\]

\[
= \max_{0\leq n<\omega} \left| \prod_{s=n}^{n+\omega-1} H_1 \prod_{s\leq n_1 < n+\omega} (1 + b_{1k}) \frac{\beta S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right|
\]

\[
\omega \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1) \prod_{n_1 \leq n_1 < n+\omega} (1 - b_{1k}) B \}
\]

\[
\omega \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1) \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1)
\]

\[
\frac{M r}{1 - r}
\]

\[
\frac{M r}{1 - r}
\]

\[
\Phi I - I_0
\]

\[
= \max_{0\leq n<\omega} \left| \Phi I - I_0 \right|
\]

\[
= \max_{0\leq n<\omega} \left| \prod_{s=n}^{n+\omega-1} H_2 \prod_{s\leq n_1 < n+\omega} (1 + b_{2k}) \frac{\beta S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right|
\]

\[
\omega \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1) \prod_{n_1 \leq n_1 < n+\omega} (1 - b_{2k}) B \}
\]

\[
\omega \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1) \prod_{n_1 \leq n_1 < n+\omega} (1 - \mu_1)
\]

\[
\frac{M r}{1 - r}
\]

\[
\Phi R - R_0
\]

\[
= \max_{0\leq n<\omega} \left| \Phi R - R_0 \right|
\]

\[
= \max_{0\leq n<\omega} \left| \prod_{s=n}^{n+\omega-1} H_3 \prod_{s\leq n_1 < n+\omega} (1 + b_{3k}) \gamma I(s) \right|
\]
Next, we prove that the mapping $\Phi$ is a contraction mapping of the $X^*$. In fact, for $\forall x = (S_1, I_1, R_1), y = (S_2, I_2, R_2) \in 
\frac{||x - y||}{L} = \max\{|S_1 - S_2|, |I_1 - I_2|, |R_1 - R_2|\}$, we have

$$|\Phi S_1 - \Phi S_2|_0 = \max |\Phi S_1(n) - \Phi S_2(n)|$$

$$|\Phi I_1 - \Phi I_2|_0 = \max |\Phi I_1(n) - \Phi I_2(n)|$$

$$|\Phi R_1 - \Phi R_2|_0 = \max |\Phi R_1(n) - \Phi R_2(n)|$$

where $L$ is the Lipschitz constant of the mapping $\Phi$. Thus, $\Phi$ is a contraction mapping of the $X^*$. If we denote the solution set of the system by $S$, then $S$ is a non-empty, closed, and bounded set in $X^*$. By the contraction mapping theorem, there exists a unique fixed point $x^* \in S$ such that $x^* = \Phi x^*$.
Then
\[ ||\Phi x - \Phi y|| = \max\{|\Phi S_1 - \Phi S_2|_0, |\Phi I_1 - \Phi I_2|_0, |\Phi R_1 - \Phi R_2|_0\} \leq L||x - y||. \]

Notice that \( L < 1 \), it is clear that \( \Phi \) is a contraction. Therefore, the mapping \( \Phi \) possesses a unique fixed point \( x^* \in X^* \) such that \( \Phi x^* = x^* \), which implies (1) has a unique \( \omega \)-periodic solution. This completes the proof.

III. AN EXAMPLE

In this section, we give an example to illustrate that our results are feasible. Consider the following discrete-time SIR epidemic model with time delays and impulses:

\[
\begin{align*}
S(n+1) - S(n) &= 0.05 - 0.06 S(n), \\
I(n+1) - I(n) &= 0.02 S(n) I(n - \sin(\frac{n\pi}{2})) + \frac{1}{1 + I(n - \sin(\frac{n\pi}{2}))}, \\
R(n+1) - R(n) &= -0.06 + 0.02 I(n), \\
S(n_k+1) - S(n_k) &= 0.1 S(n_k), \\
I(n_k+1) - I(n_k) &= 0.1 I(n_k), \\
R(n_k+1) - R(n_k) &= 0.1 R(n_k),
\end{align*}
\]

where \( n_k = 4k + 1 \).

Noticing that \( \omega = 8, \mu_1 = \mu_2 = \mu_3 = 0.06, \gamma = 0.02, B = 0.05, \beta = 0.02, \alpha = 1 \), it is easy to check that
\[
\prod_{0 \leq n_k < 8} (1 + b_{1k}) = \prod_{0 \leq n_k < 8} (1 + b_{2k}) = \prod_{0 \leq n_k < 8} (1 + b_{3k}) = 1.21 \quad \text{and} \quad r_1 = 0.8082, r_2 = 0.4408, r_3 = 0.8082, \]
therefore \( r = \max\{r_1, r_2, r_3\} = 0.8082 \). At the moment \( r_4 = 0.8096 < 1 \), Hence, \( (H_1) - H_3 \) are all satisfied. By Theorem 1, (21) has a unique 8-periodic solution in the region \( X^* = \{x|x \in X, ||x - x_0|| \leq 4.6435\} \).

REFERENCES