An Application of Differential Subordination to Analytic Functions

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Abstract—In the present paper, using the technique of differential subordination, we obtain certain results for analytic functions defined by a multiplier transformation in the open unit disc \( E = \{ z : |z| < 1 \} \). We claim that our results extend and generalize the existing results in this particular direction.

Keywords—Analytic function, Differential subordination, Multiplier transformation.

I. INTRODUCTION

Let \( A_p \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\},
\]

which are analytic in the open unit disc \( E = \{ z : |z| < 1 \} \).

We write \( A_1 = A \).

A function \( f \in A \) is said to be starlike of order \( 0 \leq \alpha < 1 \) if it satisfies the condition

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > 1, \quad z \in E.
\]

Let \( S^* \) denote the class of starlike functions of order \( \alpha \). We write \( S^*(0) = S^* \), therefore, \( S^* \) is the class of starlike functions (w.r.t. origin).

For \( f \in A_p \), we define the multiplier transformation \( l_p(n, \alpha) \) as

\[
l_p(n, \alpha)[f](z) = z^p + \sum_{k=p+1}^{\infty} (k + \alpha)^n a_k z^k, \quad (\alpha > 0, n \in \mathbb{Z}).
\]

The operator \( l_p(n, \alpha) \) has been recently studied by Aghalary et al. [9]. Earlier, the operator \( l_1(n, \alpha) \) was investigated by Cho and Srivastava [7] and Cho and Kim [8], whereas the operator \( l_1(n, 1) \) was studied by Uralgaddi and Somanatha [1].

\( l_1(n, 0) \) is the well-known Såhlöw ([5]) derivative operator \( D^n \), defined as:

\[
D^n[f](z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,
\]

where \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \} \) and \( f \in A \).

A function \( f \in A \) is said to belong to the class \( S_n(\alpha) \) if it satisfies the condition

\[
\Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > 1, \quad z \in E.
\]

In 1989, the class \( S_n(\alpha) \) has been studied by Owa, Shen and Obradović [10].

Uralgaddi [2] proved if \( f(z) = z + \alpha_m z^{m+1} + \alpha_{m+2} z^{m+2} + \cdots \in S_n(\alpha) \) for some \( m, n \in \mathbb{N} \), then

\[
\Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > 1, \quad z \in E.
\]

Recently, Li and Owa [6], proved the following results:

Theorem 1.1: Let \( f(z) = z + \alpha_m z^{m+1} + \alpha_{m+2} z^{m+2} + \cdots \) be analytic in \( E \) and satisfy the condition

\[
\Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > \frac{2 - m(n+1)}{2}, \quad z \in E
\]

for some \( m \in \mathbb{N} \) and \( n \in \mathbb{N}_0 \). Then

\[
\Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > \frac{2 - m(n+1)}{2}, \quad z \in E
\]

Theorem 1.2: If \( f(z) = z + \alpha_m z^{m+1} + \alpha_{m+2} z^{m+2} + \cdots \in S_n(\alpha) \) for some \( 0 \leq \alpha < 1, n \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \), then for any \( 0 < \leq \frac{2(1 - \alpha)}{2(1 - \alpha)m}, \) the sharp estimate is

\[
\Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > \frac{2 - m(n-1)}{m}, \quad z \in E.
\]

The main objective of the present paper is to generalize certain existing results stated above using differential subordination and find the corresponding generalized results for multiplier transformation \( l_p(n, \alpha) \) in the subordination form.

II. PRELIMINARIES

We shall need the following definitions and lemmas to prove our results.

Definition 2.1: Let \( f \) and \( g \) be analytic in \( E \). We say that \( f \) is subordinate to \( g \) in \( E \), written as \( f(z) \prec g(z) \) in \( E \), if \( g \) is univalent in \( E \), \( f(0) = g(0) \) and \( f'(z) \in g'(E) \).

Definition 2.2: Let \( : \mathbb{C}^2 \times E \rightarrow \mathbb{C} \) and let \( h \) be univalent in \( E \). If \( p \) is analytic in \( E \) and satisfies the differential subordination

\[
(p(z), zp'(z); h(z)) \prec (p(0), 0; h(0)), \quad (p(0), 0; h(0)) = h(0), \quad (1)
\]
then \( p \) is called a solution of the differential subordination (1). The univalent function \( q \) is called a dominant of the differential subordination (1) if \( p < q \) for all \( p \) satisfying (1). A dominant \( \hat{q} \) that satisfies \( \hat{q} < q \) for all dominants \( q \) of (1), is said to be the best dominant of (1).

Definition 2.3: A function \( L(z, t) \), \( z \in \mathbb{E} \) and \( t \geq 0 \) is said to be a subordination chain if \( L(z, t) \) is analytic and univalent in \( \mathbb{E} \) for all \( t \geq 0 \), \( L(z, t) \) is continuously differentiable on \([0, \infty)\) for all \( z \in \mathbb{E} \) and \( L(z, t_1) < L(z, t_2) \) for all \( 0 \leq t_1 < t_2 \).

Lemma 2.4: ([13, page 159]). The function \( L(z, t) = \lim_{t \to \infty} a_1(t)z + \cdots \) with \( a_1(t) \neq 0 \) for all \( t \geq 0 \), and \( \lim_{t \to \infty} a_1(t) = \infty \), is said to be a subordination chain if and only if \( \Re \left[ \frac{z L / \bar{z}}{L / x} \right] > 0 \) for all \( z \in \mathbb{E} \) and \( t \geq 0 \).

Lemma 2.5: ([12]). Let \( F \) be analytic in \( \mathbb{E} \) and let \( G \) be analytic and univalent in \( \mathbb{E} \) except for points \( \sigma \) such that \( \lim_{z \to \sigma} G(z) = \infty \), with \( F(0) = G(0) \). If \( F \neq G \) in \( \mathbb{E} \), then there is a point \( z_0 \in \mathbb{E} \) and \( \sigma \in \mathbb{E} \) (boundary of \( \mathbb{E} \)) such that \( F(z) - G(z) < G(\zeta) \), \( F(z_0) = G(0) \) and \( z_0 F'(z_0) = m \sigma G'(\sigma) \) for some \( m \geq 1 \).

III. MAIN RESULTS

The following result is essentially due to Miller and Mocanu [13, page 76]. For the completeness of our results, we also prove it here with an alternative proof using subordination chain.

Theorem 3.1: Let \( q, q(z) \neq 0, z \in \mathbb{E} \), be a univalent function such that \( \frac{Q'(z)}{Q(z)} = \hat{q}(z) \) is starlike in \( \mathbb{E} \). If an analytic function \( P, P(z) \neq 0, z \in \mathbb{E} \), satisfies the differential subordination

\[
\frac{z P'(z)}{P(z)} < \frac{Q'(z)}{Q(z)} = h(z), \tag{2}
\]

then

\[ P < q = \exp \left[ \int_0^z \frac{h(t)}{t} dt \right]. \]

and \( q \) is the best dominant.

Proof: Let us define \( h \) as

\[ h(z) = \frac{Q'(z)}{Q(z)}, \quad z \in \mathbb{E}. \tag{3} \]

Since \( h \) is starlike and hence univalent in \( \mathbb{E} \). The subordination in (2) is, therefore, well-defined in \( \mathbb{E} \).

We need to show that \( P < q \). Suppose to the contrary that \( P \neq q \) in \( \mathbb{E} \). Then by Lemma 2.5, there exist points \( z_0 \in \mathbb{E} \) and \( \sigma \in \mathbb{E} \) such that \( P(z_0) = q(\sigma) \) and \( \sigma P'(z_0) = m \sigma q'(\sigma), m \geq 1 \). Then

\[
\frac{z_0 P'(z_0)}{P(z_0)} = \frac{m \sigma q'(\sigma)}{q(\sigma)}, \quad z \in \mathbb{E}. \tag{4}
\]

Consider a function

\[ L(z, t) = (1 + t) \frac{Q'(z)}{Q(z)}, \quad z \in \mathbb{E}. \tag{5} \]

The function \( L(z, t) \) is analytic in \( \mathbb{E} \) for all \( t > 0 \) and is continuously differentiable on \([0, \infty)\) for all \( z \in \mathbb{E} \). Now,

\[ a_1(t) = \frac{(L(z, t))}{z} \right|_{t_0} = (1 + t) \frac{q'(0)}{q(0)}. \]

As \( q \) is univalent in \( \mathbb{E} \), so \( q(0) \neq 0 \). Therefore, it follows that \( a_1(t) \neq 0 \) and \( \lim_{t \to \infty} |a_1(t)| = \infty \). A simple calculation yields

\[
\frac{z L / \bar{z}}{L / t} = (1 + t) \frac{z Q'(z)}{Q(z)}, \quad z \in \mathbb{E},
\]

where \( Q(z) = \frac{z Q'(z)}{Q(z)} \). Since \( Q \) is starlike in \( \mathbb{E} \) and \( t \geq 0 \). Therefore, we obtain

\[
\Re \left[ \frac{z L / \bar{z}}{L / t} \right] > 0, \quad z \in \mathbb{E}.
\]

Hence, in view of Lemma 2.4, \( L(z, t) \) is a subordination chain. Therefore, \( L(z, t_1) < L(z, t_2) \) for \( 0 \leq t_1 \leq t_2 \). From (5), we have \( L(z, 0) = h(z) \), thus we deduce that \( L(0, t) \neq h(\mathbb{E}) \) for \( |z| = 1 \) and \( t \geq 0 \). In view of (4) and (5), we can write

\[
\frac{z_0 P'(z_0)}{P(z_0)} = L(0, m - 1) \neq h(\mathbb{E}),
\]

where \( z_0 \in \mathbb{E}, |z_0| = 1 \) and \( m = 1 \), which is a contradiction to (2).

Hence,

\[ P < q = \exp \left[ \int_0^z \frac{h(t)}{t} dt \right]. \]

This completes the proof of the theorem.

Theorem 3.2: Let \( h \) be starlike univalent in \( \mathbb{E} \) with \( h(0) = 0 \). Let \( F \in \mathcal{A}_p \) satisfy

\[
\frac{I_p(n \cdot, f(z))}{I_p(n \cdot, f(z))} - 1 < h(z), \quad z \in \mathbb{E}, \tag{6}
\]

then

\[
\left( \frac{I_p(n \cdot, f(z))}{z^p} \right)^\beta < q(z) = \exp \left[ \left( 1 - \frac{h(t)}{t} \right) dt \right],
\]

for \( z \in \mathbb{E}, p > 0 \). The function \( q \) is the best dominant.

Proof: Let us write

\[
\left( \frac{I_p(n \cdot, f(z))}{z^p} \right)^\beta = r(z), \quad z \in \mathbb{E}. \tag{7}
\]

Differentiate (7) logarithmically, we obtain

\[
\frac{z I_p(n \cdot, f(z))}{I_p(n \cdot, f(z))} - p = \frac{z r'(z)}{r(z)}, \quad z \in \mathbb{E}. \tag{8}
\]

A little calculation yields the following equality

\[
I_p(n \cdot, f(z)) = (p + 1) I_p(n \cdot, f(z)) - I_p(n \cdot, f(z)). \tag{9}
\]

By making use of (9), from (6) and (8), we have

\[
\frac{I_p(n \cdot, f(z))}{I_p(n \cdot, f(z))} - 1 = \frac{1}{p + 1} \frac{z r'(z)}{r(z)} < h(z).
\]
As \( p \in \mathbb{N} \), \( \rho > 0 \) and by our assumption, \( (\alpha + p) > 0 \). Now in view of Theorem 3.1, we obtain
\[
\left( \frac{l_p(n)}{z^p} \right)^{\alpha} r(z) < q(z), \quad z \in \mathbb{E},
\]
where \( q(z) = \exp \left[ (\alpha + p) \int_0^z \frac{h(t)}{t} \, dt \right], \) it completes the proof.

IV. APPLICATIONS TO ANALYTIC FUNCTIONS

For \( h(z) = \frac{2(1 - |z|^2)z}{1 - z} \), where \( \alpha \neq 1 \), is a real number. It is easy to check that \( h \) is starlike in \( \mathbb{E} \). When we make this selection of \( h \) in Theorem 3.2, we get the following result.

**Corollary 4.1.** If \( f \in A_p \) satisfies
\[
\frac{l_p(n+1)}{l_p(n)} \frac{|f(z)|}{|f|} < \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{E},
\]
then
\[
\left( \frac{l_p(n)}{z^p} \right)^{\alpha} < (1 - z)^{2(\alpha - 1)(\alpha + p)}, \quad z \in \mathbb{E},
\]
where \( \alpha \neq 1, \quad > 0 \), are real numbers.

If we put \( p = 1, \quad = 0 \) in Corollary 4.1, we have the following result.

**Corollary 4.2.** If \( f \in A \) satisfies
\[
\frac{D^{m+1}[f]}{D^m[f]} \frac{|f(z)|}{|f(z)|} < \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{E},
\]
then
\[
\left( \frac{D^{m}[f]}{z} \right)^{\alpha} < (1 - z)^{2(\alpha - 1)/\alpha - 1}, \quad z \in \mathbb{E},
\]
where \( \alpha \neq 1, \quad > 0 \), are real numbers.

**Remark 4.3.** The result in Corollary 4.2, is a generalization of the above stated Theorem 1.2, for \( m = 1 \), due to Li and Owa [6].

**Remark 4.4.** For \( \alpha = \frac{1}{n+1} \) and \( \alpha = \frac{1}{n+1} \), in Corollary 4.2, we obtain the above stated Theorem 1.1, for \( m = 1 \), of Li and Owa [6] in subordination form which is more general than its existing form.

When we select \( \alpha = \frac{1}{n+1} \), in Corollary 4.2, we obtain the following result.

**Corollary 4.5.** If \( f \in S_n(\cdot) \), then
\[
\left( \frac{D^{m-1}[f]}{z} \right)^{\frac{1}{m}} < (1 - z)^{2(\alpha - 1)/\alpha + 1}, \quad z \in \mathbb{E},
\]
where \( \alpha \neq 1 \), is real number.

**Remark 4.6.** The result in Corollary 4.5, sharpens the result of Uralegaddi [2] and generalizes the result of Li and Owa [6]. For \( = 0 \), in Corollary 4.5, we obtain the Corollary 1, due to Li and Owa [6] for \( m = 1 \), in subordination form which is more general than its existing form.

If we select, \( n = 0 \) in Corollary 4.2, we have the following result.

**Corollary 4.7.** If \( f \in A \) satisfies
\[
\frac{Z'(z)}{Z(z)} < \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{E},
\]
then
\[
\left( \frac{f(z)}{z} \right)^{\alpha} < (1 - z)^{2\alpha(\alpha - 1)}, \quad z \in \mathbb{E},
\]
where \( \alpha \neq 1, \quad > 0 \), are real numbers.

**Remark 4.8.** The result in Corollary 4.7, is more general than the result due Miller and Mocanu [11], Golusin [4] and Li and Owa [6], which can be obtained by selecting \( = 0 \) and \( = \frac{1}{2} \).

**REFERENCES**


