Rate of convergence for generalized Baskakov-Durrmeyer Operators
Durvesh Kumar Verma and P. N. Agrawal

Abstract—In the present paper, we consider the generalized form of Baskakov Durrmeyer operators to study the rate of convergence, in simultaneous approximation for functions having derivatives of bounded variation.

Keywords—Bounded variation, Baskakov-Durrmeyer operators. Simultaneous approximation. Rate of convergence.

I. INTRODUCTION
In the year 2005, Finta [2] considered a new type of Baskakov-Durrmeyer operator by taking the weight functions of Baskakov basis functions and established sufficient conditions for obtaining strong converse inequality. After that Govil and Gupta [3] studied some approximation properties for these operators and estimated local results in terms of modulus of continuity. Also, further properties like pointwise convergence, asymptotic formula and inverse result in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators and estimations in simultaneous approximation have been established in [5].

II. AUXILIARY RESULTS
In the sequel we shall need the following lemmas.

Lemma 1: If we define the central moments as
\[ \mu_{n,r,m}(x) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^1 b_{n-r,k+r}(t) (t - x)^m dt \]
Then, \( \mu_{n,r,0}(x) = 1 \), \( \mu_{n,r,1}(x) = \frac{r(2x+1)}{n+r} \) and for \( n > m \) we have the following recurrence relation:
\[ (n - r - m)\mu_{n,r,m+1}(x) = x(1+x)[\mu_{n,r,m}(x) + 2m\mu_{n,r,m-1}(x)] + (m+r)(1+2x). \]

Proof: Taking derivative of above
\[ \mu'_{n,r,m}(x) = -m \sum_{k=0}^{\infty} p_{n+r,k}(x) \times \int_0^1 b_{n-r,k+r}(t) (t - x)^{m-1} dt + \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^1 b_{n-r,k+r}(t) (t - x)^m dt \]
\[ = -m \mu_{n,r,m-1}(x) + \sum_{k=0}^{\infty} p_{n+r,k}(x) \times \int_0^1 b_{n-r,k+r}(t) (t - x)^m dt \]
\[ x(1+x)[\mu'_{n,r,m}(x) + m\mu_{n,r,m-1}(x)] = \sum_{k=0}^{\infty} x(1+x)p_{n+r,k}(x) \int_0^1 b_{n-r,k+r}(t) (t - x)^m dt \]
using \(x(1 + x)p_{n+r,k}(x) = [k - (n + r)]x[p_{n+r,k}(x)]\), we get
\[
x(1 + x)[p_{n,r,m}(x) + mp_{n,r,m-1}(x)] = \sum_{k=0}^{\infty} [k - (n + r)]x[p_{n+r,k}(x)]
\times \int_{0}^{\infty} b_{n-r,k+r}(t) (t-x)^m dt
= \sum_{k=0}^{\infty} kp_{n+r,k}(x) \int_{0}^{\infty} b_{n-r,k+r}(t) (t-x)^m dt
\]
\[= n + (n + r) x \mu_{n,r,m}(x).
\]
Combining (1)-(6), we get the desired result.

**Remark 1:** Let \(x \in (0, \infty)\) and \(C > 2\), then for \(n\) sufficiently large, Lemma 1, yields that
\[
\mu_{n,r,2}(x, r) \leq \frac{C x(1 + x)}{(n + r - 1)}.
\]

**Lemma 2:** Let \(x \in (0, \infty)\) and \(C > 2\), then for \(n\) sufficiently large, we have
\[
\lambda_{n,r}(x, y) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{y} b_{n-r,k+r}(t) dt 
\leq \frac{C x(1 + x)}{(n + r - 1)(y-x)^2}, \quad 0 \leq y < x,
\]
\[
1 - \lambda_{n,r}(x, z) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{z}^{\infty} b_{n-r,k+r}(t) dt 
\leq \frac{C x(1 + x)}{(n + r - 1)(z-x)^2}, \quad x < z < \infty.
\]

**Proof:** The proof of the lemma follows exactly by Remark 1. For instance, for the first inequality for \(n\) sufficiently large and \(0 \leq y < x\), we have
\[
\lambda_{n,r}(x, y) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{y} b_{n-r,k+r}(t) dt 
\leq \frac{C x(1 + x)}{(n + r - 1)(y-x)^2}.
\]

Next, to estimate \(I_1\), by using the equality \([(k + r - 1) - (n - r + 2)t)]b_{n-r,k+r}(t) = (t+1)br_{n-r,k+r}(t)\), we have
\[
I_1 = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} b_{n-r,k+r}(t) dt 
\leq \frac{C x(1 + x)}{(n + r - 1)(y-x)^2}.
\]

The proof of the second inequality follows along the similar lines.

**Lemma 3:** Let \(f\) be \(s\) times differentiable on \([0, \infty)\) such that \(f^{(s-1)}(t) = O(t^q)\) as \(t \to \infty\) where \(q\) is a positive integer. Then for any \(r, s \in N^0\) and \(n > \max\{q, r + s + 1\}\), we have
\[
D^s V_{n,r}(f, x) = V_{n,r+s}(D^s f, x), \quad D = \frac{d}{dx}.
\]

**Proof:** We prove the result by applying the principle of mathematical induction and using the following identity
\[
D p_{n,k}(x) = n[p_{n+1,k-1}(x) - p_{n+1,k}(x)] \quad \text{and} \quad (7)
\]
\[
D b_{n,k}(t) = (a + 1)[b_{n+1,k-1}(t) - b_{n+1,k}(t)]. \quad (8)
\]

The above identity is true even for the case \(k = 0\), as we observe that \(p_{n+1, negative} = 0\). Using (7), (8) and integrating by parts, we have
\( D_{n,r}(f, x) = \frac{(n+r-1)!}{n!(n-1)!} \sum_{k=0}^{\infty} D_{n+r,k}(x) \int_0^\infty b_{n-r,k+r}(t)f(t) \, dt \)

\[
\times \sum_{k=0}^{\infty} D_{n+r,m}(x) \int_0^\infty b_{n-r,m+k+r+m}(t) D^m f(t) \, dt
\]

Integrating by parts the last integral, we have

\[
D_{n,r}^m(f, x) = \frac{(n+r)!}{n!(n-1)!} \sum_{k=0}^{\infty} p_{n+r,m+k}(x) \]

\[
\int_0^\infty b_{n-r,m+k+r+m}(t) D^m f(t) \, dt.
\]

which shows that the result holds for \( s = 1 \). Let us suppose that the result holds for \( s = m \) i.e.

\[
D_{n,r}^m V_{n,r}(f, x) = V_{n,r+m}(D_{n,r}^m f, x)
\]

\[
= \frac{(n+r+m-1)!}{n!(n-r-1)!} \sum_{k=0}^{\infty} p_{n+r,m+k}(x) \int_0^\infty b_{n-r,m+k+r+m}(t) D^m f(t) \, dt
\]

Now,

\[
D_{n,r}^{m+1} V_{n,r}(f, x) = \frac{(n+r+m)!}{n!(n-1)!} \sum_{k=0}^{\infty} p_{n+r,m+k-1}(x) \int_0^\infty b_{n-r,m+k+r+m}(t) D^m f(t) \, dt
\]

\[
- p_{n+r,m+k}(x) \int_0^\infty b_{n-r,m+k+r+m}(t) D^m f(t) \, dt
\]

\[
= \frac{(n+r+m)!}{n!(n-1)!} \sum_{k=0}^{\infty} p_{n+r,m+k+1}(x) \int_0^\infty b_{n-r,m+k+r+m+1}(t) D^m f(t) \, dt
\]

\[
- b_{n-r,m+k+r+m}(t) D^m f(t) \, dt.
\]

Again integrating by parts the last integral, we have

\[
D_{n,r}^{m+1} V_{n,r}(f, x) = \frac{(n+r+m)!}{n!(n-1)!} \sum_{k=0}^{\infty} p_{n+r,m+k+1}(x) \int_0^\infty b_{n-r,m+k+r+m+1}(t) D^m f(t) \, dt
\]

Thus, the result is true for \( s = m+1 \), hence by mathematical induction, proof of the lemma is completed.

### III. RATE OF CONVERGENCE

In this section we prove our main results.

**Theorem 1:** Let \( f \in DB_q(0, \infty), q > 0 \) and \( x \in (0, \infty) \). The for \( C > 2 \) and \( n \) sufficiently large, we have

\[
\left| \frac{(n+r)!}{n!(n-1)!} V_{n,r}(f; x) - f(x) \right| \leq C \frac{(1+cx)^{x/2}}{n^{(x^2)/2}} \, \left( \int_0^x f(t)^2 \, dt \right)^{\frac{1}{2}} + O(\frac{1}{n^{x^2}})
\]

Proof: Using the mean value theorem, we have

\[
\left| \frac{(n+r)!}{n!(n-1)!} V_{n,r}(f; x) - f(x) \right| \leq \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^\infty b_{n-r,k+r}(t) |f(t) - f(x)| \, dt
\]

\[
\leq \int_0^\infty \left| \int_0^x \sum_{k=0}^{\infty} p_{n+r,k}(x) b_{n-r,k+r}(t) f'(u) \, du \right| \, dt
\]
Also, using the identity
\[ f'(u) = \frac{f'(x^+)}{2} + \frac{f'(x^-)}{2} + (f')_x(u) \]
\[ + \frac{f'(x^+)}{2} - \frac{f'(x^-)}{2} \operatorname{sgn}(u - x) \]
\[ + \left[ f'(x) - \frac{f'(x^+)}{2} + \frac{f'(x^-)}{2} \right] \chi_x(u), \]
where
\[ \chi_x(u) = \begin{cases} 1, & u = x, \\ 0, & u \neq x. \end{cases} \]
Obviously, we have
\[ \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} \left( \int_x^{\infty} \frac{f'(x)}{2} \right) \chi_x(u) \, du \right) \, p_{n-r,k+r}(t) \, dt = 0. \]
Thus, using the above identities, we can write
\[ \left| \frac{(n + r - 1)(n - r)!}{n!(n - 1)!} V_{n,r}(f; x) - f(x) \right| \leq \left| \int_0^{\infty} \left( \int_x^{\infty} \frac{f'(x)}{2} \right) \chi_x(u) \, du \right) \, p_{n-r,k+r}(t) \, dt \]
\[ + \left| \sum_{k=0}^{\infty} p_{n+r,k}(x) \left| f'(x^+) - f'(x^-) \right| \mu_{n,r,2}(x) \right|^{1/2}. \]
Also, it can be verified that
\[ \int_0^{\infty} \left( \int_x^{\infty} \frac{f'(x^+) - f'(x^-)}{2} \operatorname{sgn}(u - x) \, du \right) \left| p_{n-r,k+r}(t) \right| dt \leq \left| \frac{f'(x^+) - f'(x^-)}{2} \right| \mu_{n,r,1}(x). \]
Combining (9)-(11), we get
\[ \frac{(n + r - 1)(n - r)!}{n!(n - 1)!} V_{n,r}(f; x) - f(x) \]
\[ \leq \left| \int_0^{\infty} \left( \int_x^{\infty} \frac{f'(x)}{2} \right) \chi_x(u) \, du \right) \, p_{n-r,k+r}(t) \, dt \]
\[ + \left| \sum_{k=0}^{\infty} p_{n+r,k}(x) \left| f'(x^+) - f'(x^-) \right| \mu_{n,r,2}(x) \right|^{1/2}. \]
For estimating the integral
\[ \sum_{k=0}^{\infty} p_{n+k,r}(x) \int_{2x}^{\infty} p_{n-r,k+r-1}(t) C_1 t^2dt \]
above, we proceed as follows:
Since \( t \geq 2x \) implies that \( t \leq 2(t - x) \) and it follows from Lemma 1, that
\[ \sum_{k=0}^{\infty} p_{n+k,r}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t) C_1 t^2dt \leq C_1 2^{2n} \sum_{k=0}^{\infty} p_{n+k,r}(x) \int_{0}^{\infty} p_{n-r,k+r}(t) C_1 (t - x)^{2n}dt \]

\[ = C_1 2^{2n} \mu_{n,2r}(x) = O(n^{-\gamma}), \text{ as } n \to \infty. \quad \text{(15)} \]

By using the Schwarz inequality and Remark 1, we get the estimate as follows:
\[ |f'(x^+)| \sum_{k=0}^{\infty} p_{n+k,r}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t) |t - x| dt \leq |f'(x^+)| \times \left( \sum_{k=0}^{\infty} p_{n+k,r}(x) \int_{0}^{\infty} p_{n-r,k+r}(t) (t - x)^2 dt \right)^{1/2} \]

\[ = |f'(x^+)| \left( \frac{C(x)(1 + x)}{n - r - 1} \right)^{1/2}. \quad \text{(16)} \]

Collecting the estimates from (14)-(17), we obtain
\[ |A_{n,r}(f, x)| = O(n^{-\gamma}) + |f'(x^+)| \left( \frac{C(x)(1 + x)}{n - r - 1} \right)^{1/2} + \left( \int_{0}^{\infty} \left( f'(x) - f(x) - xf'(x^+) \right) \right) \frac{C(1 + x)}{n - r - 1} \sum_{k=1}^{\infty} x^{k - 1} \left( \int_{0}^{\infty} f(x) \right) \left( \frac{x^{k - 1}}{k!} \right)^{1/2} \]

\[ \leq C(x)(1 + x) \int_{0}^{\infty} \left( f(x) \right) \left( \frac{x^{k - 1}}{k!} \right)^{1/2} f(x^+) \frac{1}{(x - t)^2} dt + \int_{0}^{\infty} \left( f(x) \right) \left( \frac{x^{k - 1}}{k!} \right)^{1/2} f(x^+) \frac{1}{(x - t)^2} dt. \]

\[ \text{As a consequence of Lemma 3, we can easily prove the following corollary for the derivatives of the operators } V_{n,r}. \]

**Corollary 1:** Let \( f^{(q)} \in DB_2(0, \infty), q > 0 \) and \( x \in (0, \infty) \). The for \( C > 2 \) and \( n \) sufficiently large, we have
\[ \left| \frac{n + r - 1}{n!} f(x^+)(x) \right| \]

\[ \leq \frac{C(x)(1 + x) \int_{0}^{\infty} \left( f(x) \right) \left( \frac{x^{k - 1}}{k!} \right)^{1/2} f(x^+) \frac{1}{(x - t)^2} dt + \int_{0}^{\infty} \left( f(x) \right) \left( \frac{x^{k - 1}}{k!} \right)^{1/2} f(x^+) \frac{1}{(x - t)^2} dt. \]

\[ \text{where } u = \frac{x}{n}. \]

Combining (12), (18) and (19) we get the desired result.

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