Rate of convergence for generalized Baskakov-Durrmeyer Operators

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Abstract—In the present paper, we consider the generalized form of Baskakov-Durrmeyer operators to study the rate of convergence, in simultaneous approximation for functions having derivatives of bounded variation.

Keywords—Bounded variation, Baskakov-Durrmeyer operators, Simultaneous approximation, Rate of convergence.

I. INTRODUCTION

In the year 2005, Finta [2] considered a new type of Baskakov-Durrmeyer operator by taking the weight functions of Baskakov basis functions and established sufficient conditions for obtaining strong converse inequality. After that Govil and Gupta [3] studied some approximation properties for these operators and estimated local results in terms of modulus of continuity. Also, further properties like pointwise convergence, asymptotic formula and inverse result in simultaneous approximation have been established in [5]. Very recently, Verma et al. [7] Stancu type generalization of the operators $D_n(f, x)$ and studied the direct error estimates and Voronovskaja type asymptotic formula. To approximate the Lebesgue integrable functions on the interval $[0, \infty)$, we introduce Baskakov-Durrmeyer operators in generalized form as:

$$V_{n,r}(f, x) = \frac{(n + r - 1)!}{n!(n - 1)!} \sum_{k=0}^{\infty} p_{n, r, k}(x) \times \int_0^\infty b_{n, r, k} (t) f(t) dt$$

where

$$p_{n, k}(x) = \frac{(n + k - 1)!}{k!} \frac{x^k}{(1 + x)^{n+k}},$$
$$b_{n, k}(x) = \frac{1}{B(k, n + 1)} \frac{x^k}{(1 + x)^{n+k+1}}.$$  

In the present paper we study the rate of convergence for the operators $V_{n,r}$ for functions having the derivatives of bounded variation, we also mention a corollary which provide the result in simultaneous approximation.

II. AUXILIARY RESULTS

In the sequel we shall need the following lemmas.

Lemma 1: If we define the central moments as

$$\mu_{n,r,m}(x) = \sum_{k=0}^{\infty} p_{n, r, k}(x) \int_0^\infty b_{n, r, k} (t) (t - x)^m dt$$

Then, $\mu_{n,r,0}(x) = 1$, $\mu_{n,r,1}(x) = \frac{r(2x+c)}{n+r+1}$ and for $n > m$ we have the following recurrence relation:

$$(n - r - m) \mu_{n,r,m+1}(x) = x(1 + x) [\mu'_{n,r,m}(x) + 2m \mu_{n,r,m-1}(x)] + (m + r)(1 + 2x).$$

Proof: Taking derivative of above

$$\mu'_{n,r,m}(x) = -m \sum_{k=0}^{\infty} p_{n, r, k}(x)$$
$$\times \int_0^\infty b_{n, r, k} (t) (t - x)^{m-1} dt$$
$$+ \sum_{k=0}^{\infty} p_{n, r, k}(x) \int_0^\infty b_{n, r, k} (t) (t - x)^{m} dt$$
$$= -m \mu_{n,r,m-1}(x) + \sum_{k=0}^{\infty} p_{n, r, k}(x)$$
$$\times \int_0^\infty b_{n, r, k} (t) (t - x)^{m} dt$$
$$x(1 + x) [\mu'_{n,r,m}(x) + 2m \mu_{n,r,m-1}(x)]$$
$$= \sum_{k=0}^{\infty} x(1 + x) p_{n, r, k}(x) \int_0^\infty b_{n, r, k} (t) (t - x)^{m} dt$$

observed that for all $f \in DB_q(0, \infty)$, we can have the representation

$$f(x) = f(c) + \int_c^x \psi(t) dt, \quad x \geq c > 0.$$  

In the recent years, the rate of convergence for the functions having the derivatives of bounded variation is an interesting area of research, several researchers have studied in this direction we refer some of important paper in this area as [6, 8-10]. Also, Bai et al. [1] worked in this direction and estimated the rate of convergence for the several operators. Gupta [4] estimated the rate of convergence for functions of bounded variation on certain Baskakov-Durrmeyer type operators.

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using \( x(1 + x)p'_{n+r,k}(x) = [k - (n + r)]x[p_{n+r,k}(x)] \), we get
\[
x(1 + x)[p'_{n,r,m}(x) + m\mu_{n,r,m}(x)] = \sum_{k=0}^{\infty}[k - (n + r)]x[p_{n+r,k}(x)]
\times \int_{0}^{\infty} b_{n-r,k+r}(t - x)^{m}dt
\]
\[
= \sum_{k=0}^{\infty} k p_{n+r,k}(x) \int_{0}^{\infty} b_{n-r,k+r}(t - x)^{m}dt
- (n + r)x\mu_{n,r,m}(x)
\]
\[
= I - (n + r)x\mu_{n,r,m}(x).
\]  

Combining (1)-(6), we get the desired result.

**Remark 1:** Let \( x \in (0, \infty) \) and \( C > 2 \), then for \( n \) sufficiently large, Lemma 1, yields that
\[
\mu_{n,r,2}(x,c) \leq \frac{C(x + 1)}{(n - r - 1)}
\]

**Lemma 2:** Let \( x \in (0, \infty) \) and \( C > 2 \), then for \( n \) sufficiently large, we have
\[
\lambda_{n,r}(x,y) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{y} b_{n-r,k+r}(t)dt
\]  
\[
\leq \frac{C(x + 1)}{(n - r - 1)(x - y)^{2}}; \quad 0 \leq y < x,
\]  
\[
1 - \lambda_{n,r}(x,z) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{z}^{\infty} b_{n-r,k+r}(t)dt
\]  
\[
\leq \frac{C(x + 1)}{(n - r - 1)(z - x)}; \quad x < z < \infty.
\]

**Proof:** The proof of the lemma follows easily by Remark 1. For instance, for the first inequality for \( n \) sufficiently large and \( 0 \leq y < x \), we have
\[
\lambda_{n,r}(x,y) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{y} b_{n-r,k+r}(t)dt
\]
\[
\leq \frac{C(x + 1)}{(n - r - 1)(x - y)^{2}}; \quad 0 \leq y < x,
\]
\[
1 - \lambda_{n,r}(x,z) = \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{z}^{\infty} b_{n-r,k+r}(t)dt
\]
\[
\leq \frac{C(x + 1)}{(n - r - 1)(z - x)}; \quad x < z < \infty.
\]

**Lemma 3:** Let \( f \) be \( s \) times differentiable on \([0, \infty)\) such that \( f^{(s-1)}(t) = O(t^{q}) \) as \( t \to \infty \) where \( q \) is a positive integer. Then for any \( r, s \in N^{0} \) and \( n > \max\{r, q, s + 1\} \), we have
\[
D^{s}V_{n,r}(f, x) = V_{n,r+s}(D^{s}f, x), \quad \frac{d}{dx} = D
\]

**Proof:** We prove the result by applying the principle of mathematical induction and using the following identity
\[
D p_{n,k}(x) = n[p_{n+1,k}(x) - p_{n+1,k-1}(x)] \quad \text{and} \quad (7)
\]
\[
D b_{n,k}(t) = (n + 1)[b_{n+1,k-1}(t) - b_{n+1,k}(t)] \quad \text{and} \quad (8)
\]

The above identity is true even for the case \( k = 0 \), as we observe that \( p_{n+1, \text{negative}} = 0 \). Using (7), (8) and integrating by parts, we have
Let us suppose for induction, proof of the lemma is completed. Thus, the result is true for $s = m + 1$, hence by mathematical induction, proof of the lemma is completed.

### III. RATE OF CONVERGENCE

In this section we prove our main results.

**Theorem 1:** Let $f \in DB_{0}(0, \infty), q > 0$ and $x \in (0, \infty)$. If $V_{n} f(x) - f(x)$

$$
\left[\frac{(n + r - 1)!(n - r)!}{n!(n - 1)!}\right] V_{n,r}(f;x) - f(x)
$$

\begin{align*}
\leq & \frac{C(1 + cr)}{n} \sum_{k=1}^{\infty} \sqrt{n} \frac{x}{x + z/\sqrt{n}} \left((f')_{x}\right) + \frac{x}{\sqrt{n}} \left((f')_{x}\right) \\
+ & \frac{C(1 + cr)}{n} \left(|f(2x) - f(x) - x f'(x)| + |f(x)| + O(n^{-q})ight) \\
+ & \frac{C(1 + x)}{n - r - 1} \frac{|f'(x^+)| + |f'(x^-)| - f'(x^-)}{2} \sqrt{\frac{C x(1 + x)}{n}} \\
+ & \frac{|f(x^+)| + f'(x^-)|}{2} r(1 + 2x)
\end{align*}

where $V_{b,a} f(x)$ denotes the total variation of $f_{x}$ on $[a, b]$, the auxiliary function $f_{x}$ is defined by

$$f_{x}(t) = \begin{cases} 
    f(t) - f(x^-), & 0 \leq t < x, \\
    0, & t = x, \\
    f(t) - f(x^+), & x < t < \infty.
\end{cases}$$

**Proof:** Using the mean value theorem, we have

$$
\left|\frac{(n + r - 1)!(n - r)!}{n!(n - 1)!}\right| V_{n,r}(f;x) - f(x)
$$

$$
\leq \sum_{k=0}^{\infty} p_{n,k}(x) \sum_{k=0}^{\infty} b_{n,k}(t) f(t) \sum_{k=0}^{\infty} p_{n,k}(x) f(t) dt
$$

$$
\leq \int_{0}^{\infty} \left|\sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t) f(t) dt\right| dt
$$
Also, using the identity

\[
  f'(u) = \frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) \\
  + \frac{f'(x^+) - f'(x^-)}{2} \text{sgn}(u - x) \\
  + \left[ f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right] \chi_x(u),
\]

where

\[
  \chi_x(u) = \begin{cases} 
  1, & u = x, \\
  0, & u \neq x.
\end{cases}
\]

Obviously, we have

\[
  \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^\infty \left( \int_x^\infty \left[ f'(x) \\
  - \frac{f'(x^+) + f'(x^-)}{2} \right] \chi_x(u) du \right) p_{n-r,k+r}(t) dt = 0.
\]

Thus, using the above identities, we can write

\[
  \frac{(n + r - 1)!(n - r)!}{n!(n - 1)!} V_{n,r}(f;x) - f(x) \\
  \leq \left| \int_0^\infty \left( \int_x^\infty \sum_{k=0}^{\infty} p_{n+r,k}(x)b_{n-r,k+r}(t) \\
  \times \left( \frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) \right) du \right) dt \\
  + \int_0^\infty \left( \int_x^\infty \sum_{k=0}^{\infty} p_{n+r,k}(x)p_{n-r,k+r}(t) \\
  \times \frac{f'(x^+) - f'(x^-)}{2} \text{sgn}(u - x) du \right) dt \right|.
\]

Also, it can be verified that

\[
  \left| \int_0^\infty \left( \int_x^\infty \frac{f'(x^+) - f'(x^-)}{2} \text{sgn}(u - x) du \right) \\
  \times \sum_{k=0}^{\infty} p_{n+r,k}(x)p_{n-r,k+r}(t) dt \right| \\
  \leq \frac{|f'(x^+) - f'(x^-)|}{2} [\mu_{n,r,2}(x)]^{1/2}
\]

and

\[
  \int_0^\infty \left( \int_x^\infty \frac{f'(x^+) + f'(x^-)}{2} du \right) \\
  \sum_{k=0}^{\infty} p_{n+r,k}(x)p_{n-r,k+r}(t) dt \\
  \leq \frac{|f'(x^+) + f'(x^-)|}{2} \mu_{n,r,1}(x).
\]

Combining (9)-(11), we get

\[
  \frac{(n + r - 1)!(n - r)!}{n!(n - 1)!} V_{n,r}(f;x) - f(x) \\
  \leq \left| \int_0^\infty \left( \int_x^\infty (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x)p_{n-r,k+r}(t) dt \\
  + \int_0^\infty \left( \int_x^\infty (f')_x(u) du \right) \sum_{k=0}^{\infty} p_{n+r,k}(x)p_{n-r,k+r}(t) dt \\
  + \frac{|f'(x^+) - f'(x^-)|}{2} [\mu_{n,r,2}(x)]^{1/2} + \frac{|f'(x^+) + f'(x^-)|}{2} \mu_{n,r,1}(x) \\
  = |A_{n,r}(f;x) + B_{n,r}(f;x)| + \frac{|f'(x^+) - f'(x^-)|}{2} [\mu_{n,r,2}(x)]^{1/2} + \frac{|f'(x^+) + f'(x^-)|}{2} \mu_{n,r,1}(x). \\
\]

Applying Remark 1 and Lemma 1 in (12), we have

\[
  \frac{(n + r - 1)!(n - r)!}{n!(n - 1)!} V_{n,r}(f;x) - f(x) \\
  \leq |A_{n,r}(f;x) + B_{n,r}(f;x)| + \frac{|f'(x^+) - f'(x^-)|}{2} \left\{ \frac{C(x(1 + x)}{n + r - 1} + \frac{|f'(x^+) + f'(x^-)|}{2} r(1 + 2x) \right\} \\
  \leq \frac{\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} (f(t) - f(x)) p_{n-r,k+r}(t) dt} \\
  + \frac{\sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} (f'(t) - f'(x)) p_{n-r,k+r}(t) dt} \]

\[
  \leq \int_{2x}^{\infty} (f'(t) - f'(x)) dt \\
  + \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r}(t) dt \]

\[
  \leq |A_{n,r}(f;x)| = |B_{n,r}(f;x)| + \frac{C(1 + x)}{(n - r - 1)x} |f(2x) - f(x) - x f'(x^+)| \\
  + \frac{C(1 + x)}{(n - r - 1)x} \sum_{k=1}^{\infty} \left( \frac{v + \frac{x}{r}}{\sqrt{n}} \right) x^{\frac{x}{r}} (f'(x^+))_x (14)
\]
For estimating the integral
\[ \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r-1}(t)C_{1}t^{2n}dt \]
above, we proceed as follows:

Since \( t \geq 2x \) implies that \( t \leq 2(t-x) \) and it follows from Lemma 1, that
\[ \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r-1}(t)C_{1}t^{2n}dt \]
\[ \leq C_{2}^{2n} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} p_{n-r,k+r-1}(t)C_{1}(t-x)^{2n}dt \]
\[ = C_{2}^{2n} \mu_{n,r,2n}(x) = O(n^{-r}), \text{ as } n \to \infty. \]  
(15)

\[ \frac{|f(x)|}{x^2} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} p_{n-r,k+r-1}(t)(t-x)^2dt \]
\[ = |f(x)| \frac{C(1+x)}{(n-r-1)x}. \]  
(16)

By using the Schwarz inequality and Remark 1, we get the estimate as follows:
\[ |f'(x+)| \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{2x}^{\infty} p_{n-r,k+r-1}(t)dt \leq |f'(x)| \]
\[ \times \left( \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_{0}^{\infty} p_{n-r,2n-1}(t)(t-x)^2dt \right)^{1/2} \]
\[ \left( C_{x}^{2}(1+x) \right)^{1/2} \frac{C_{2}(x+1)x^{n-r-1}}{n-r-1}. \]  
(17)

Collecting the estimates from (14)-(17), we obtain
\[ |A_{n,r}(f,x)| \]
\[ = O(n^{-r}) + |f'(x+)| \sqrt{C_{x}^{2}(1+x) \frac{C_{2}(x+1)x^{n-r-1}}{n-r-1}} \]
\[ + \frac{C(1+x)}{(n-r-1)x} |f(2x) - f(x) - xf'(x)| + |f(x)| \]
\[ + \frac{C(1+x)}{n-r-1} \sum_{k=1}^{\infty} \sqrt{x} \left( (f')_{x+} + \frac{x}{\sqrt{n}} \right) (f')_{x-}. \]  
(18)

On other hand, to estimate \( B_{n,r}(f,x) \), applying the Lemma 2 with \( y = x - \frac{t}{n} \) and integration by parts, we have
\[ |B_{n,r}(f,x)| \]
\[ = \left| \int_{0}^{x} \int_{x}^{t} (f')_{x+}(u)d_{x}(\lambda_{n,r}(x,t)) \right| \]
\[ = \int_{0}^{y} (f')_{x+}(t)\lambda_{n,r}(x,t)dt \]
\[ \leq \left( \frac{1}{y} + \frac{C_{2}(1+y)}{x} \left( (f')_{x+}(t)\lambda_{n,r}(x,t)dt \right) \right) \]
\[ \leq C_{x}(1+x) \frac{1}{(n-r-1)} \left( \frac{1}{(x-t)^{2}} + \frac{x}{\sqrt{n}} \right) \int_{0}^{y} \int_{x}^{t} (f')_{x+}(t)d_{x}(\lambda_{n,r}(x,t)) dt \]
\[ \leq C_{x}(1+x) \frac{1}{(n-r-1)} \left( \frac{1}{(x-t)^{2}} + \frac{x}{\sqrt{n}} \right) \int_{0}^{y} \int_{x}^{t} (f')_{x+}(t)d_{x}(\lambda_{n,r}(x,t)) dt \]
\[ = C_{x}(1+x) \frac{1}{(n-r-1)} \left( \frac{1}{(x-t)^{2}} + \frac{x}{\sqrt{n}} \right) \int_{0}^{y} \int_{x}^{t} (f')_{x+}(t)d_{x}(\lambda_{n,r}(x,t)) dt \]
\[ \leq C_{x}(1+x) \frac{1}{(n-r-1)} \sum_{k=1}^{\infty} \sqrt{x} \left( (f')_{x+} + \frac{x}{\sqrt{n}} \right) (f')_{x-}, \]  
(19)

where \( u = \frac{x-t}{n} \).

Combining (12), (18) and (19) we get the desired result.

As a consequence of Lemma 3, we can easily prove the following corollary for the derivatives of the operators \( V_{n,r} \).

**Corollary 1:** Let \( f^{(s)} \in DB_{q}(0,\infty) \), \( q > 0 \) and \( x \in (0, \infty) \). Then for \( C > 2 \) and \( n \) sufficiently large, we have
\[ \frac{|n+r-1|!}{n!} D^{r}V_{n,r}(f;x) - f^{(s)}(x) \leq \]
\[ \frac{C(1+x)}{(n-r-1)x} \left( \sum_{k=1}^{\infty} \left( (D^{n}f)_{x+} + \frac{x}{\sqrt{n}} \right) (f')_{x+} \right) \]
\[ + \frac{C(1+x)}{n-r-1} \left( D^{n}f(2x) - D^{n}f(x) - D^{n}f(x^{+}) \right) \]
\[ + \frac{C(1+x)}{n-r-1} \left( D^{n}f(2x) - D^{n}f(x) - D^{n}f(x^{+}) \right) \]
\[ + \frac{D^{n}f(2x) - D^{n}f(x) - D^{n}f(x^{+})}{r(1+2x)} \]
\[ \frac{2}{n-r-1}, \]
where \( (f')_{x+} \) denotes the total variation of \( f_{x} \) on \([a,b]\), the auxiliary function \( f_{x} \) is defined by
\[ D^{n+1}f_{x}(t) \]
\[ = \left\{ \begin{array}{ll}
D^{n+1}f(t) - D^{n+1}f(x^{-}), & 0 \leq t < x, \\
0, & t = x,
\end{array} \right. \]
\[ D^{n+1}f(t) - D^{n+1}f(x^{+}), & x < t < \infty. \]

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**References**


