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On completely semiprime, semiprime and prime fuzzy ideals in ordered semigroups

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Abstract—In this paper, we first introduce the new concept of completely semiprime fuzzy ideals of an ordered semigroup S, which is an extension of completely semiprime ideals of ordered semigroup S, and investigate some its related properties. Especially, we characterize an ordered semigroup that is a semilattice of simple ordered semigroups in terms of completely semiprime fuzzy ideals of ordered semigroups. Furthermore, we introduce the notion of semiprime fuzzy ideals of ordered semigroup S and establish the relations between completely semiprime fuzzy ideals and semiprime fuzzy ideals of S. Finally, we give a characterization of prime fuzzy ideals of an ordered semigroup S and show that a nonconstant fuzzy ideal f of an ordered semigroup S is prime if and only if f is two-valued, and $\max\{f(a), f(b)\} = \inf f((aSb]), \forall a, b \in S$.

Keywords—Ordered fuzzy point, fuzzy left (right) ideal of an ordered semigroup, completely semiprime fuzzy ideal, semiprime fuzzy ideal, prime fuzzy ideal.

I. Introduction

Let S be a nonempty set. A fuzzy subset of S is, by definition, an arbitrary mapping $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. The important concept of a fuzzy set put forth by Zadeh in 1965 [1] has opened up keen insights and applications in a wide range of scientific fields. A theory of fuzzy sets on ordered semigroups has been recently developed [2-6]. Following the terminology given by Zadeh, if S is an ordered semigroup, fuzzy sets in ordered semigroups S have been first considered by Kehayopulu and Tsingelis in [2], then they defined " fuzzy" analogous for several notations, which have proven useful in the theory of ordered semigroups. Moreover, they proved that each ordered groupoid can be embedded into an ordered groupoid having the greatest element (poe-groupoid) in terms of fuzzy sets [3]. The concept of ordered fuzzy points of an ordered semigroup S has been first introduced by Xie and Tang [15], and prime fuzzy ideals of an ordered semigroup S were studied in [16]. Authors also introduced the concepts of weakly prime fuzzy ideals, completely prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of an ordered semigroup S, and established the relations among five types ideals. Furthermore, we characterize weakly prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of S by their level ideals [15].

As we know, fuzzy ideals (fuzzy left, right ideals) with special properties of ordered semigroups always play an important role in the study of ordered semigroups structure. The ordered fuzzy points of an ordered semigroup S are key tools to describe the algebraic subsystems of S. Motivated by the study of prime fuzzy ideals in rings, semigroups and ordered semigroups, and also motivated by Kehayopulu and Tsingelis's works in ordered semigroups in terms of fuzzy subsets, in this paper we attempt to introduce and give a detailed investigation of completely semiprime, semiprime and prime fuzzy ideals of an ordered semigroup S. We first introduce the new concept of completely semiprime fuzzy ideals of an ordered semigroup S, which is an extension of completely semiprime (also called semiprime in [7]) ideals of ordered semigroup S. Although our definition is equivalent to the definition of completely semiprime fuzzy ideals of ordered semigroups in [15], it is simpler to use. Moreover, we investigate some its related properties. Especially, we characterize an ordered semigroup that is a semilattice of simple ordered semigroups in terms of completely semiprime fuzzy ideals of ordered semigroups. As an application, we obtain that an ordered semigroup S is simple if and only if it is intra-regular and archimedean. Furthermore, we introduce the notion of semiprime fuzzy ideals of ordered semigroup S and establish the relations between completely semiprime fuzzy ideals and semiprime fuzzy ideals of S. Finally, we give a characterization of prime fuzzy ideals of an ordered semigroup S and show that a nonconstant fuzzy ideal f of an ordered semigroup S is prime if and only if f is two-valued, and $\max\{f(a), f(b)\} = \inf f((aSb]), \forall a, b \in S$. In this paper, we illustrate that one can pass from the theory of ordered semigroups to the theory of "fuzzy" ordered semigroups. As an application of the results of this paper, the corresponding results of semigroup (without order) are also obtained by moderate modifications.

II. Preliminaries and some notations

Throughout this paper, we denote by Z^+ the set of all positive integers. In sequel we denote, by S, an ordered semigroup, that is, a semigroup S with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$ (for example, see [8]). For convenience we use the notation $S^1 := S \cup \{1\}$, where $1a = a1 := 1$ for all $a \in S$ and $11 = 1$. A function f from S to the real closed interval [0,1] is called a *fuzzy subset* of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) \equiv 1$ for all $x \in S$ (the fuzzy subset S is also denoted by 1 [13]). Let A be a nonempty subset of S. We denote by f_A the characteristic mapping of A, that is, the mapping of S into $[0, 1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

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Then f_A is a fuzzy subset of S. Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by

$$f(x) \leq g(x)$$

for all $x \in S$, and $f \cap g$ and $f \cup g$ are defined by

$$(f \cap g)(x) = \min(f(x), g(x)) = f(x) \wedge g(x),$$

$$(f \cup g)(x) = \max(f(x), g(x)) = f(x) \vee g(x),$$

for all $x \in S$. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice with the maximum element S and the minimum element 0, which is a mapping from S into $[0, 1]$ defined by

$$0: \ S \rightarrow [0,1], \ x \mapsto 0(x) := 0, \ \forall x \in S.$$

Throughout this paper, $\inf(\sup) f(X)$ means the greatest lower bound (least upper bound) of the set $\{f(t) | t \in X\}$ for any fuzzy subset f of S and $X \subseteq S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $x \in S$, we define $A_x := \{(y, z) \in S \times S | \ x \leq yz\}$. The product $f \circ g$ of f and g is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \min\{f(y), g(z)\}, & \text{if } A_x \neq \emptyset \ , \\ 0 & , \text{ if } A_x = \emptyset, \end{cases}$$

for all $x \in S$. It is well known (cf. [3, Theorem]) that this operation "$\circ$" is associative and $(F(S), \circ, \subseteq)$ is a *poe*-semigroup.

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \ \text{ for some } \ h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

Lemma 2.1 ([8]): Let S be an ordered semigroup. Then, we have:

(1) $A \subseteq (A], \forall A \subseteq S$.
(2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.
(3) $(A](B] \subseteq (AB], \forall A, B \subseteq S$.
(4) $(A] = ((A]], \forall A \subseteq S$.
(5) For every left (resp. right) ideal or ideal T of S, $(T] = T$.
(6) $((A](B]] = (AB], \ \forall A, B \subseteq S$.
(7) $(Sa], (aS]$ and $(SaS]$ is a left ideal, a right ideal and an ideal of S, respectively, $\forall a \in S$.

A non-empty subset A of an ordered semigroup S is called a *left* (resp. *right*) ideal of S if

(1) $SA \subseteq A$ (resp. $AS \subseteq A$), and
(2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$.

If A is both a left and a right ideal of S, then it is called an *(two-sided) ideal* of S [8]. We denote by $L(a)$ (resp. $R(a), I(a)$) the left (resp. right, two-sided) ideal of S generated by a $(a \in S)$. Then $L(a) = (a \cup Sa] = (S^1 a], R(a) = (a \cup aS] = (aS^1]$ and $I(a) = (a \cup Sa \cup aS \cup SaS] = (S^1 a S^1]$ [8]. An ideal I of S is called *prime* (also called *weakly prime* in [7])if for any two ideals A, B of S such that $AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$; I is called *completely semiprime* (also called *semiprime* in [7]) if for any element a of S such that $a^2 \in I$, then $a \in I$.

Definition 2.2 ([11]): An ordered semigroup S is *simple* if for every ideal I of S, we have $I = S$. Equivalently, $(SaS] = S$ for any $a \in S$.

Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xy) \geq f(y), \forall x, y \in S$. Equivalently, $S \circ f \subseteq f$.

A fuzzy subset f of S is called a *fuzzy right ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xy) \geq f(x), \forall x, y \in S$. Equivalently, $f \circ S \subseteq f$.

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S [2, 17]. A fuzzy ideal f of S is called *prime* if for any fuzzy ideals g and h of S, $g \circ h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$ [16].

Lemma 2.3 ([2, Propositions 2, 3]): Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right) ideal of S if and only if the characteristic mapping f_A of A is a fuzzy left (resp. right) ideal of S.

Definition 2.4 ([4]): An ordered semigroup S is called *fuzzy simple* if every fuzzy ideal of S is a constant function, that is, for every fuzzy ideal f of S, we have $f(a) = f(b), \forall a, b \in S$.

Lemma 2.5 ([4]): An ordered semigroup S is simple if and only if it is fuzzy simple.

Lemma 2.6 ([15]): Let f be a fuzzy subset of an ordered semigroup S. Then f is a strongly convex fuzzy subset of S if and only if $x \leq y \Rightarrow f(x) \geq f(y)$, for all $x, y \in S$.

Lemma 2.7 ([15]): Let a_λ, b_μ $(\lambda \neq 0, \mu \neq 0)$ be ordered fuzzy points of S, and f, g fuzzy subsets of S. Then the following statements are true:

(1) $(\forall x \in S)(S \circ a_\lambda \circ S)(x) = \begin{cases} \lambda, & \text{if } x \in (SaS], \\ 0, & \text{if } x \notin (SaS], \end{cases}$ and $S \circ a_\lambda \circ S$ is a fuzzy ideal of S.
(2) $a_\lambda \circ b_\mu = (ab)_{\lambda \wedge \mu}$ for all ordered fuzzy points a_λ and b_μ of S. In particular, $a_\lambda \circ a_\lambda = (a^2)_\lambda$.
(3) If f, g are fuzzy ideals of S, then $f \circ g$, $f \cup g$ are fuzzy ideals of S.
(4) If f is a fuzzy ideal of S, then $f \circ S \subseteq f$ and $S \circ f \subseteq f$.
(5) If $f \subseteq g$, and $h \in F(S)$, then $f \circ h \subseteq g \circ h, \ h \circ f \subseteq h \circ g$.

Definition 2.8: Let f be any function from a set S to a set T and μ any fuzzy fuzzy subset of T. Then $f^{-1}(\mu)$, the pre-image of μ under f, is a fuzzy subset of S, defined by $(f^{-1}(\mu))(x) = \mu(f(x))$ for all $x \in S$.

Definition 2.9 ([18]): Let S, T are two ordered semigroups, a mapping $f: \ S \rightarrow T$ is called *isotone* if $x, y \in S, \ x \leq y$ implies $f(x) \leq f(y)$ in T. f is called a *homomorphism* if it is isotone and satisfies that $f(xy) = f(x)f(y)$ in T, for all $x, y \in S$.

Lemma 2.10 ([15]): Let f be a fuzzy subset of an ordered semigroup S. Then f is a prime fuzzy ideal of S if and only if f satisfies the following conditions:

(1) $|Im(f)| = 2$, which means that range of f consists of exactly two points of [0,1].
(2) $f_1 \neq \emptyset$, and f_1 is a prime ideal of S.

The reader is referred to [15,19,20] for notation and terminology not defined in this paper.

III. Completely semiprime fuzzy ideals of ordered semigroups

Definition 3.1: A fuzzy subset f of an ordered semigroup S is called *completely semiprime* if $f(a) \geq f(a^2)$ for all $a \in S$.

The following theorem shows that the concept of fuzzy completely semiprimality in an ordered semigroup is an extension of completely semiprimality.

Theorem 3.2: Let A be a nonempty subset of an ordered semigroup S. Then the following statements are equivalent:

(1) A is completely semiprime.

(2) The characteristic function f_A of A is completely semiprime.

Proof. $\Longrightarrow$. Let a be any element of S. If $a^2 \in A$, then, since A is completely semiprime, we have $a \in A$. Thus $f_A(a) = 1 = f_A(a^2)$. If $a^2 \notin A$, then we have $f_A(a) \geq 0 = f_A(a^2)$. Therefore we have $f_A(a) \geq f_A(a^2)$ for all $a \in S$, and f_A is a completely semiprime fuzzy sunset of S.

$\Longleftarrow$. Let $a^2 \in A$, $a \in S$. Then, since f_A is completely semiprime, we have $f_A(a) \geq f_A(a^2) \geq 1$. Since f_A is a fuzzy subset of S and $f_A(a) \leq 1$ for any $a \in S$, so we have $f_A(a) = 1$, which implies that $a \in A$. It thus follows that A is completely semiprime.

Theorem 3.3: Let f be any fuzzy ideal of an ordered semigroup S. Then the following statements are equivalent:

(1) f is completely semiprime.

(2) $(\forall a \in S)\ f(a) = f(a^2)$.

(3) $(\forall a \in S)\ (\forall n \in Z^+)\ f(a) = f(a^n)$.

Proof. It is clear that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2). Let a be any element of S. Then, since f is a completely semiprime fuzzy ideal of S, we have

$$f(a) \geq f(a^2) \geq f(a),$$

and so we have $f(a) = f(a^2)$.

(2) $\Rightarrow$ (3). We prove this result by induction. Clearly, the result holds for $n = 2$. Let $k \geq 2$ be any positive integer. Let $f(a^n) = f(a)$ holds for $\forall a \in S$ and $\forall n \in Z^+,\ 1 \leq n \leq k$. We claim that $f(a^{k+1}) = f(a)$. Indeed:

Case 1. If k is odd, let $k = 2m + 1$. Then $f(a^{k+1}) = f[(a^{m+1})^2] = f(a^{m+1})$. Since $m + 1 < k$, by the induction hypothesis, $f(a^{m+1}) = f(a)$.

Case 2. If k is even, let $k = 2m$. Then again by the induction hypothesis, we have

$$\begin{aligned} f(a) &\leq f(a^{k+1}) = f(a^{2m+1}) \\ &\leq f(a^{2m+2}) = f[(a^{m+1})^2] \\ &= f(a^{m+1}) = f(a), \end{aligned}$$

which implies that $f(a^{k+1}) = f(a)$. This proves the result.

The following theorem gives a characterization of completely semiprime fuzzy ideals of an ordered semigroup by ordered fuzzy points.

Theorem 3.4: Let f be a fuzzy ideal of an ordered semigroup S. Then f is completely semiprime if and only if for any ordered fuzzy points $a_\lambda \in S$ $(\forall \lambda \in (0,1])$, $a_\lambda^2 \in f$ implies $a_\lambda \in f$.

Proof. Let f be a fuzzy ideal of an ordered semigroup S and $a \in S$. Then $f(a) \geq f(a^2)$. If $a_\lambda^2 \in f,\ \lambda \in (0,1]$, then $f(a^2) \geq \lambda$, and so $f(a) \geq \lambda$, which implies $a_\lambda \in f$.

Conversely, let a be any element of S. Put $\lambda = f(a^2)$. If $\lambda \in (0,1]$, since $a_\lambda^2 \in f$, then, by hypothesis, we have $a_\lambda \in f$. Which implies $f(a) \geq \lambda = f(a^2)$. This completes the proof.

Remark 3.5: By Theorem 3.4, we see that our definition is equivalent to the previous definition (see Definition 6.3 in [15]), but it is simpler to use.

Proposition 3.6: If f is a completely semiprime fuzzy ideal of an ordered semigroup S, then $f(ab) = f(ba)$ for all $a, b \in S$.

Proof. Suppose that f is a completely semiprime fuzzy ideal of an ordered semigroup S and $\forall a, b \in S$. Then, by Theorem 3.3, we have

$$f(ab) = f[(ab)^2] = f(abab) \geq f(ba).$$

Similarly, $f(ba) \geq f(ab)$. It thus follows that $f(ab) = f(ba)$.

Theorem 3.7: Let $f : S \to T$ be a homomorphism of ordered semigroups and μ a completely semiprime fuzzy ideal of T. Then $f^{-1}(\mu)$ is a completely semiprime fuzzy ideal of S.

Proof. First we show that $f^{-1}(\mu)$ is a fuzzy ideal of ordered semigroup S. Indeed: Let $x, y \in S$ and $x \leq y$. Then, since f is a homomorphism of ordered semigroups from S to T, we have $f(x) \leq f(y)$. Since μ is a fuzzy ideal of ordered semigroup T, and so $\mu(f(x)) \geq \mu(f(y))$, i.e., $f^{-1}(\mu)(x) \geq f^{-1}(\mu)(y)$. Furthermore, for any $x, y \in S$, we have

$$\begin{aligned} f^{-1}(\mu)(xy) &= \mu[f(xy)] = \mu[f(x)f(y)] \\ &\geq \mu(f(x)) \vee \mu(f(y)),\ \text{since } \mu \text{ is a fuzzy} \\ &\quad \text{ideal of } T. \\ &= f^{-1}(\mu)(x) \vee f^{-1}(\mu)(y). \end{aligned}$$

Moreover, $f^{-1}(\mu)$ is completely semiprime. Indeed: For any $a \in S$, we have

$$\begin{aligned} f^{-1}(\mu)(a^2) &= \mu[f(a^2)] = \mu[(f(a))^2] \\ &= \mu[f(a)],\ \text{since } \mu \text{ is completely semiprime.} \\ &= f^{-1}(\mu)(a). \end{aligned}$$

Therefore, by Theorem 3.3, $f^{-1}(\mu)$ is a completely semiprime fuzzy ideal of S.

An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if, for each element a of S, there exists $x, y \in S$ such that $a \leq xa^2y$.

Equivalent definition: $a \in (Sa^2S], \forall a \in S$ [9].

Proposition 3.8: An ordered semigroup S is intra-regular if and only if

$$(\forall a \in S)\ f(a) = f(a^2)$$

for every fuzzy ideal f of S.

Proof. $\Longrightarrow$. Let f be a fuzzy ideal of S and $a \in S$. Then, by hypothesis, there exist $x, y \in S$ such that $a \leq xa^2y$, and

$$f(a) \geq f(xa^2y) \geq f(a^2y) \geq f(a^2) \geq f(a),$$

which implies that $f(a) = f(a^2)$.

$\Longleftarrow$. By Lemma 2.3, $f_{I(a^2)}$ is a fuzzy ideal of S. By hypothesis, we have $f_{I(a^2)}(a) = f_{I(a^2)}(a^2) = 1$, so $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$. Thus $a \leq t$ for some $t \in a^2 \cup Sa^2 \cup a^2S \cup Sa^2S$. If $t = a^2$, then $a \leq a^2 \leq a^4 \in Sa^2S$, that is $a \in (Sa^2S]$. If $t = xa^2$ for some $x \in S$, then $a \leq xa^2 \leq x(xa^2)a = x^2a^2a \in Sa^2S$. If $t = a^2y$

for some $y \in S$, then $a \leq a^2y \leq a(a^2y)y = aa^2y^2 \in Sa^2S$. If $t \in Sa^2S$, then $a \in (Sa^2S]$. Thus S is intra-regular.

Theorem 3.9: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is intra-regular.
(2) S is a semilattice of simple semigroups.
(3) Every ideal of S is completely semiprime.
(4) Every fuzzy ideal of S is completely semiprime.

Proof. The equivalence of (1), (2) and (3) is due to Remark 2 in [12], and of (1) and (4) is due to Theorem 3.3 and Proposition 3.8.

An ordered semigroup S is called *archimedean* if for any $a, b \in S$, there exists $m \in Z^+$ such that $b^m \in (S^1aS^1]$. Equivalently, for any $a, b \in S$, there exists $m \in Z^+$ such that $b^m \leq xay$ for some $x, y \in S^1$ [14].

Theorem 3.10: Let S be an archimedean ordered semigroup. Then every completely semiprime fuzzy ideal of S is a constant function.

Proof. Let f be any completely semiprime fuzzy ideal of S and $\forall a, b \in S$. Then, since S is archimedean, there exist $x, y \in S^1$ such that $b^m \leq xay$ for some $m \in Z^+$. Then we have

$$f(b) = f(b^m) \geq f(xay) \geq f(a).$$

In a similar way, we have $f(a) \geq f(b)$. Thus, $f(a) = f(b)$. Since a, b are any elements of S, this means that f is a constant function.

Corollary 3.11: Let S be an archimedean ordered semigroup. Then it contains no proper completely semiprime ideal.

Proof. Suppose that A be any completely semiprime ideal of an archimedean ordered semigroup S. Then, by Theorem 3.2, the characteristic function f_A of A is a completely semiprime fuzzy ideal of S. Since S is archimedean, so we have, by Theorem 3.10, that f_A is a constant function, which means $f_A(x) = 1$ for any $x \in S$. It thus follows that $A = S$.

Theorem 3.12: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is simple.
(2) S is intra-regular and archimedean.

Proof. Suppose first that S is simple. Then it follows, by Lemma 2.5, that every fuzzy ideal of S is a constant function. Therefore, we have

$$(\forall a \in S)\ f(a) = f(a^2).$$

Then it follows, by Theorems 3.3 and 3.8, that S is intra-regular. In order to prove that S is archimedean, let a, b be any elements of S. Then it follows, by Definition 2.2, that there exist $x, y \in S$ such that $a^1 = a \leq xby$, which means that S is archimedean.

Conversely, assume that (2) holds. Let f be any fuzzy ideal of S. Then it follows, by Theorem 3.9, that f is completely semiprime. And then, since S is archimedean, it follows, by Theorem 3.10, that f is a constant function. This means that S is fuzzy simple. It thus follows, by Lemma 2.5, that S is simple.

IV. Semiprime and prime fuzzy ideals of ordered semigroups

Definition 4.1: A fuzzy subset f of an ordered semigroup S is called *semiprime* if f is not a constant function and for any fuzzy ideal g of S, $g^2 \subseteq f$ implies $g \subseteq f$.

Theorem 4.2: Let f be a fuzzy ideal of an ordered semigroup S. Then f is semiprime if and only if for any fuzzy ideal g of S, $g^n \subseteq f,\ n \in Z^+$ implies that $g \subseteq f$.

Proof. $\Longleftarrow$. This is obvious.

$\Longrightarrow$. Let f be a semiprime fuzzy ideal of an ordered semigroup S. Here we prove the result by induction. Clearly the result holds for $n = 2$. Let $k \geq 2$ be any positive integer and let the result holds for each positive integer $n,\ 1 \leq n \leq k$. We claim that $g^{k+1} \subseteq f\ \Rightarrow g \subseteq f$. Indeed:

Case 1. If k is odd, let $k = 2m+1$. Then $g^{k+1} = g^{2(m+1)} = (g^{m+1})^2$.

Case 2. If k is even, let $k = 2m$. Then, by Lemma 2.7, we have

$$g^{k+1} = g^{2m+1} \supseteq g^{2m+2} = (g^{m+1})^2.$$

Thus in either case, if $g^{k+1} \subseteq f$, then $g^{m+1} \subseteq f$. Since $m + 1 \leq k$, the induction hypothesis insures that $g \subseteq f$.

In Definition 4.2, we have defined semiprime fuzzy ideals of an ordered semigroup S. The definition, however, make no reference to the grade of membership of an element of S. The purpose of following Theorem 4.3 is to characterize semiprime fuzzy ideal in terms of its effect on the elements of S. We shall see that the following theorem is simpler to use.

Theorem 4.3: Let S be an ordered semigroup. Then a fuzzy ideal f of S is semiprime if and only if $f(a) = \inf f((aSa])$ for all $a \in S$.

Proof. $\Longleftarrow$. Assume $f(a) = \inf f((aSa]),\ \forall a \in S$. Let g be any fuzzy ideal of S such that $g^2 \subseteq f$. If $g \not\subseteq f$, then there exists $a \in S$ such that $f(a) < g(a)$. Since $f(a) = \inf f((aSa])$, there exists $t \in S$ such that $b \leq ata,\ b \in S$, and $f(ata) \leq f(b) = f(a) < g(a)$. From $g \circ g = g^2 \subseteq f$, we have

$$\begin{aligned} g(a) &> f(ata) \geq (g \circ g)(ata) \\ &= \bigvee_{(x,y)\in A_{ata}} [\min\{g(x), g(y)\}] \\ &\geq \min\{g(at), g(a)\} = g(a). \end{aligned}$$

But this is a contradiction.

$\Longrightarrow$. If $f(a) \neq \inf f((aSa])$ for some $a \in S$, then $f(a) < \inf f((aSa])$. In fact, for any $b \in (aSa]$, there exists $t \in S$ such that $b \leq ata$, and $f(b) \geq f(ata) \geq f(a)$. Let $\inf f((aSa]) = m$. Define a fuzzy subset h of S as follows:

$$h(x) = \begin{cases} m, & \text{if } x \in (SaS], \\ 0\,, & \text{if } x \notin (SaS]. \end{cases}$$

Then, by Lemma 2.7, h is a fuzzy ideal of S. Furthermore, $h^2 \subseteq f$. Indeed: If $(h \circ h)(x) = m,\ x \in S$, then we have $\bigvee_{(y,z)\in A_x} [\min\{h(y), h(z)\}] = m$. This means there exist $u, v \in (SaS]$ such that $x \leq uv$. Put $u \leq sat,\ v \leq paq$ for some $s, t, p, q \in S$. Then $x \leq uv \leq satpaq$. Thus, since f is a fuzzy ideal of S, we have

$$\begin{aligned} f(x) &\geq f(uv) \geq f(satpaq) \geq f(atpa) \\ &\geq \inf f((aSa]) = m = (h \circ h)(x). \end{aligned}$$

Hence, by hypothesis, $h \subseteq f$. Again define a fuzzy subset g of S as follows:

$$g(x) = \begin{cases} m, & \text{if } x \in (S^1 a S^1], \\ 0, & \text{if } x \notin (S^1 a S^1]. \end{cases}$$

Then, by Lemma 2.7, g is a fuzzy ideal of S. Moreover, $g^4 \subseteq h$. Indeed: Since

$$g^4(x) = \bigvee_{x \leq x_1 x_2 x_3 x_4} [\min\{g(x_1), g(x_2), g(x_3), g(x_4)\}] = m$$

only if $x_1, x_2, x_3, x_4 \in (S^1 a S^1]$. Put $x_i \leq s_i a t_i,\ s_i, t_i \in S^1\ (i = 1, 2, 3, 4)$. Then

$$x \leq x_1 x_2 x_3 x_4 \leq s_1 a t_1 s_2 a t_2 s_3 a t_3 s_4 a t_4 \in SaS,$$

which implies that $x \in (SaS]$. Thus $g^4(x) = m$ implies $h(x) = m$. Hence, $g^4 \subseteq h \subseteq f$. Since f is semiprime, and so, by Lemma 2.7(3), we have $g^2 \subseteq f$ and $g \subseteq f$. But $m = g(a) \leq f(a)$ is a contradiction to $f(a) < m$.

Corollary 4.4: If f is a completely semiprime fuzzy ideal of an ordered semigroup S, then f is semiprime.

Proof. Let f be a completely semiprime fuzzy ideal of an ordered semigroup S and $a \in S$. Then, by Theorem 3.3, we have $f(a) = f(a^2) = f(a^4)$. Since $a^4 \in (aSa]$, and so $\inf f((aSa]) \leq f(a^4) = f(a)$. On the other hand, since f is a fuzzy ideal of S, and so

$$\begin{aligned} \inf f((aSa]) &= \inf\{f(b) \mid b \in (aSa]\} \\ &= \inf\{f(b) \mid b \leq ata,\ t \in S\} \\ &\geq \inf\{f(ata) \mid t \in S\} \geq f(a). \end{aligned}$$

Which means that $f(a) = \inf f((aSa])$. It thus follows, by Theorem 4.3, that f is a semiprime fuzzy ideal of S.

The following Corollary 4.5 shows that the converse of above Corollary 4.4 also holds on commutative ordered semigroup.

Corollary 4.5: Let S be a commutative ordered semigroup and f a semiprime fuzzy ideal of S. Then f is completely semiprime.

Proof. Let f be a semiprime fuzzy ideal of a commutative ordered semigroup S. If $f(a) \neq f(a^2)$ for some $a \in S$, then $f(a^2) > f(a) = \inf f((aSa])$. Thus there exist $b \in (aSa]$ such that $f(b) = f(a)$, i.e., $\exists t \in S$ such that $b \leq ata$, and so we have

$$f(a^2) > f(a) = f(b) \geq f(ata) = f(a^2 t).$$

But this contradicts to the fact that f is a fuzzy ideal of S. Hence, $f(x) = f(x^2)$ for all $x \in S$. We have thus shown, by Theorem 3.3, that f is a completely semiprime fuzzy ideal of S.

An ordered semigroup $(S, \cdot, \leq)$ is called *regular* if, for each element a of S, there exists an element x in S. such that $a \leq axa$. Equivalent definition:

(1) $A \subseteq (ASA], \forall A \in S$.

(2) $a \in (aSa], \forall a \in S$ [10].

Theorem 4.6: Let S be a commutative ordered semigroup. Then S is regular if and only if every nonconstant fuzzy ideal of S is semiprime.

Proof. Assume first that f be a fuzzy ideal of a commutative ordered semigroup S and g a fuuzy ideal of S such that $g^2 \subseteq f$. Since S is regular, then for any $a \in S$, there exist $x \in S$ such that $a \leq axa$. Thus we have

$$\begin{aligned} (g \circ g)(a) &= \bigvee_{(y,z) \in A_a} [\min\{g(y), g(z)\}] \\ &\geq \min\{g(ax), g(a)\} = g(a), \end{aligned}$$

which implies that $g^2 \supseteq g$. On the other hand, since g is a fuzzy ideal of S, then, by Lemma 2.7(4), we have $g^2 = g \circ g \subseteq g \circ S \subseteq g$. Hence, $g = g^2 \subseteq f$. It thus follows that f is a semiprime fuzzy ideal of S.

Conversely, assume that every fuzzy ideal of the commutative ordered semigroup S is semiprime and let $a \in S$. Then, by Lemma2.3, $f_{I(a^2)}$ is a semiprime fuzzy ideal of S. It thus follows, by Corollary 4.5, that $f_{I(a^2)}(t^2) = f_{I(a^2)}(t)$ for any $t \in S$. Thus, $f_{I(a^2)}(a^2) = 1 = f_{I(a^2)}(a),\ \forall a \in S$. So $a \in I(a^2)$. Thus, since S is commutative, $a \leq xa^2$ for some $x \in S$. There, $a \leq axa$ and S is regular.

Theorem 4.7: If f is a prime fuzzy ideal of an ordered semigroup S, then $\max\{f(a), f(b)\} = \inf f((aSb])$ for all $a, b \in S$.

Proof. Suppose that $\max\{f(a), f(b)\} \neq \inf f((aSb])$ for some $a, b \in S$. Then $\max\{f(a), f(b)\} < \inf f((aSb])$. Indeed: Since$f$ is a fuzzy ideal of S, then

$$\begin{aligned} \inf f((aSb]) &= \inf\{f(c) \mid c \in (aSb]\} \\ &= \inf\{f(c) \mid c \leq atb,\ t \in S\} \\ &\geq \inf\{f(atb) \mid t \in S\} \geq \max\{f(a), f(b)\}. \end{aligned}$$

Put $\inf f((aSb]) = m$. Define two fuzzy subsets g and h of S as follows:

$$g(x) = \begin{cases} m, & \text{if } x \in (SaS], \\ 0, & \text{if } x \notin (SaS], \end{cases}$$

and

$$h(x) = \begin{cases} m, & \text{if } x \in (SbS], \\ 0, & \text{if } x \notin (SbS]. \end{cases}$$

Then, by Lemma 2.7, g and h are two fuzzy ideals of S. Furthermore, $g \circ h \subseteq f$. Indeed: If $(g \circ h)(x) = \{m, 0\},\ \forall x \in S$, and $(g \circ h)(x) = m$ if and only if there exist $y \in (SaS],\ z \in (SbS]$ such that $x \leq yz$. Put $y \leq sat$, and $z \leq pbq$ for some $s, t, p, q \in S$. Then $x \leq yz \leq satpbq$. Thus, since f is a fuzzy ideal of S, we have

$$\begin{aligned} f(x) &\geq f(yz) \geq f(satpbq) \geq f(atpb) \\ &\geq \inf f((aSb]) = m = (g \circ h)(x). \end{aligned}$$

Since f is a prime fuzzy ideal of S, and we have $g \subseteq f$ or $h \subseteq f$. Assume $g \subseteq f$. Again define a fuzzy subset δ of S as follows:

$$\delta(x) = \begin{cases} m, & \text{if } x \in (S^1 a S^1], \\ 0, & \text{if } x \notin (S^1 a S^1]. \end{cases}$$

Then, by Lemma 2.7, δ is a fuzzy ideal of S. Thus we have, by the same way to the proof of Theorem 4.3, $\delta^4 \subseteq g \subseteq f$.

Since f is prime, and so, by Lemma 2.7(3), we have $\delta^2 \subseteq f$ and $\delta \subseteq f$. But $m = \delta(a) \leq f(a)$ leads to a contradiction to $\max\{f(a), f(b)\} < m$. Similarly in case of $h \subseteq f$, we have a contradiction.

Now we give a characterization of prime fuzzy ideals of an ordered semigroup S.

Theorem 4.8: Let f be a fuzzy subset of an ordered semigroup S. Then f is a prime fuzzy ideal of S if and only if f satisfies the following conditions:

(1) $|Im(f)| = 2$, which means that range of f consists of exactly two points of [0,1].

(2) $\max\{f(a), f(b)\} = \inf f((aSb]),\ \forall a, b \in S$.

Proof. Let f be a prime fuzzy ideal of an ordered semigroup S. Then we obtain (1) and (2) from Lemma 2.10 and Theorem 4.7.

Conversely, assume that (1) and (2) hold. Put $Im(f) = \{s, t\}$ $(s < t)$. Let g and h be two fuzzy ideals of S such that $g \circ h \subseteq f$. If $g \nsubseteq f$ and $h \nsubseteq f$, then there exists $x, y \in S$ such that $g(x) > f(x)$ and $h(y) > f(y)$. Thus $g(x) = h(y) = t$ and $f(x) = f(y) = s$. From $\max\{f(x), f(y)\} = \inf f((xSy])$, there exists $b \in (xSy]$ such that $f(b) = s$, i.e., $\exists t \in S$ such that $b \leq xty$ and $f(xty) \leq f(b) = s$. Since $g \circ h \subseteq f$, we have

$$\begin{aligned} s &\geq f(xty) \geq (g \circ h)(xty) \\ &= \bigvee_{(u,v) \in A_{xty}} [\min\{g(u), h(v)\}] \\ &\geq \min\{g(xt), h(y)\} \\ &\geq \min\{g(x), h(y)\} = t, \end{aligned}$$

which contradicts that $s < t$. This completes the proof.

V. Conclusion

In study the structure of an ordered semigroup S, we notice that the fuzzy ideals (left, right ideals) of S with special properties always play an important role. The ordered fuzzy points of an ordered semigroup S are key tools to describe the algebraic subsystems of S. In this paper we have introduced the concepts of completely semiprime and semiprime fuzzy ideals of an ordered semigroup S and given some characterizations for an ordered semigroup S to be a semilattice of simple ordered semigroups by completely semiprime fuzzy ideals of ordered semigroups.Furthermore, we introduce the notion of semiprime fuzzy ideals of ordered semigroup S and establish the relations between completely semiprime fuzzy ideals and semiprime fuzzy ideals of S. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of ordered semigroups. Hopefully, some new results in these topics can be obtained in the forthcoming paper.

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