A Performance Comparison of Golay and Reed-Muller Coded OFDM Signal for Peak-to-Average Power Ratio Reduction
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Abstract—Multicarrier transmission system such as Orthogonal Frequency Division Multiplexing (OFDM) is a promising technique for high bit rate transmission in wireless communication systems. OFDM is a spectrally efficient modulation technique that can achieve high speed data transmission over multipath fading channels without the need for powerful equalization techniques. A major drawback of OFDM is the high Peak-to-Average Power Ratio (PAPR) of the transmit signal which can significantly impact the performance of the power amplifier. In this paper we have compared the PAPR reduction performance of Golay and Reed-Muller coded OFDM signal. From our simulation it has been found that the PAPR reduction performance of Golay coded OFDM is better than the Reed-Muller coded OFDM signal. Moreover, for the optimum PAPR reduction performance, code configuration for Golay and Reed-Muller codes has been identified.

Keywords—OFDM, PAPR, Perfect Codes, Golay Codes, Reed-Muller Codes

I. INTRODUCTION
Wireless digital communication is rapidly expanding, resulting in a demand for portable wireless systems that are reliable and have high spectral efficiency. Orthogonal Frequency Division Multiplexing (OFDM) has been considered to achieve high data rate transmission in mobile environment. OFDM is a method of transmitting data simultaneously over multiple, equally spaced carrier frequencies using Fourier transform processing for modulation and demodulation [1]. Due to its robustness against the frequency-selective fading, which causes inter symbol interference (ISI) and degrades the performance [2], OFDM has been adopted in some wireless standards such as Digital Audio Broadcasting (DAB), Terrestrial Digital Video Broadcasting (DVB-T), HIPER LAN/2 and IEEE 802.11 standard for WLAN [3] [4]. Moreover, OFDM has been considered for fourth generation (4G) transmission techniques [5].

Due to large number of subcarriers, OFDM systems have a large dynamic signal range with a very high Peak-to-Average Power Ratio (PAPR). As a result of which, the OFDM signal will be clipped when passed through a nonlinear power amplifier at the transmitter end. Clipping degrades the Bit-Error-Rate (BER) performance and causes spectral spreading [6].

One way to solve this problem is to force the amplifier to work in its linear region. But such a solution is not power efficient. Power efficiency is necessary in wireless communication as it provides adequate area coverage, saves power consumption and allow small-size terminals. It is, therefore, important to aim at power efficient operation of the power amplifier with low back-off values and try to prevent the occurrence of signal clipping. This can be achieved by manipulating the OFDM signal before transmission.

To achieve the above objective, several proposals have been suggested and studied in the literature. The clipping [7] is the most simple technique used to reduce PAPR. However, clipping causes in-band and out-of-band distortion which degrades the performance of the system. Multiple signal representation techniques are distortion less techniques for PAPR reduction. One of the most widely used multiple signal representation technique is selective mapping (SLM) [8]. Coding [9] [10] is another distortion less technique which not only reduces the PAPR but also corrects errors.

In this paper we have investigated the performance of Golay and Reed-Muller codes for the reduction of PAPR of an OFDM signal. The rest of the paper is organized as follows. Section II describes the basic OFDM system and defines PAPR. Section III computes the PAPR of a BPSK modulated OFDM signal. Section IV describes about the Golay code. Section V describe Reed-Muller code. Section VI discusses the simulation results. Finally section VII concludes the paper.

II. OFDM AND PAPR
For our analysis emphasis is on examining the PAPR of an OFDM signal. Therefore, the OFDM system model used in this paper is a simplified version of the practical OFDM model as shown in Fig.1. Specifically, we have ignored the guard interval because it does not contribute to the PAPR [11]. Assuming that any pulse shaping in the transmitter is flat over all of the subcarriers, and deal only with the PAPR of the baseband signal. For one OFDM symbol with N subcarriers, the normalized complex baseband signal can be written as:

\[ s(t) = \sum_{k=0}^{N-1} c_k e^{j2\pi k t} \quad 0 \leq t \leq T \]  

where \( c_k \) is the frequency domain information symbol mapped to the \( k_{th} \) subcarrier of the OFDM symbol and \( T \) is the OFDM symbol duration. The peak-to-average power ratio (PAPR) of
the given frequency domain samples, $c = \{c_0, c_1, c_2, \cdots, c_{N-1}\}$ is defined as:

$$\text{PAPR} \triangleq \max_{0 \leq t \leq T} \frac{|s(t)|^2}{E[|s(t)|^2]}$$

(2)

where $E[\cdot]$ denotes a time averaging operator. The distribution of PAPR values is described using the complementary cumulative distribution function (CCDF). The CCDF of the PAPR represents the probability that the PAPR of a data block exceed a given threshold, $\xi$ and is given [16] by

$$\Pr(\text{PAPR} > \xi) = 1 - (1 - e^{-\xi})^N.$$  

(3)

For the sake of simplicity we have considered BPSK modulation. PAPR analysis of BPSK modulated OFDM signal is done to understand the reason behind high PAPR. For BPSK modulated OFDM signal, $c_k \in \{-1, +1\}$. Using the technique as described in [12] and assuming $T = 1.0$ equation (1) can be written as:

$$\sqrt{N}s(t) = \sum_{k=0}^{N-1} c_k e^{j2\pi k t}$$

$$N|s(t)|^2 = \left(\sum_{k=0}^{N-1} c_k e^{j2\pi k t}\right)^2$$

$$= \left(\mathbb{R}\left[\sum_{k=0}^{N-1} c_k e^{j2\pi k t}\right]\right)^2 + \left(\mathbb{I}\left[\sum_{k=0}^{N-1} c_k e^{j2\pi k t}\right]\right)^2$$

$$= \left(\sum_{k=0}^{N-1} c_k \cos(2\pi k t)\right)^2 + \left(\sum_{k=0}^{N-1} c_k \sin(2\pi k t)\right)^2$$

$$= \sum_{k=0}^{N-1} c_k^2 \cos^2(2\pi k t) +$$

$$+ 2 \sum_{k=0}^{N-2} \sum_{i=k+1}^{N-1} c_k c_i \cos(2\pi k t) \cos(2\pi i t) +$$

$$+ \sum_{k=0}^{N-1} c_k^2 \sin^2(2\pi k t) +$$

$$+ 2 \sum_{k=0}^{N-2} \sum_{i=k+1}^{N-1} c_k c_i \sin(2\pi k t) \sin(2\pi i t)$$

Substituting for $P_0(t)$ from (6) in (4), the average power of $s(t)$ becomes:

$$E[|s(t)|^2] = E\left[1 + \frac{2P_0(t)}{N}\right]$$

$$= 1 + \frac{1}{N}E[2P_0(t)]$$

$$= 1.$$  

(8)

As the average power of $s(t)$ is unity, the PAPR in (2), when considering its symmetry with respect to the half symbol time becomes:

$$\text{PAPR} \triangleq \max_{0 \leq t \leq 0.5} \left(1 + \frac{2}{N}P_0(t)\right)$$

(9)

where $P_0(t)$ is given by (6). From (6), (7) and (9), it is found that the PAPR is completely characterized by the aperiodic autocorrelations $C_k$. Without any loss of generality we can extend this analysis to other efficient modulation techniques such as QPSK/QAM.

**Example 1.** For the $N = 4$ the aperiodic autocorrelation $C_k$ are given by:

$$C_0 = c_0 c_0 + c_1 c_2 + c_2 c_3$$

$$C_1 = c_0 c_1 + c_1 c_2 + c_2 c_3$$

$$C_2 = c_0 c_2 + c_1 c_3$$

$$C_3 = c_0 c_3.$$  

Table.1 presents the values of $C_k$ and the PAPR values for the BPSK modulated OFDM signal for $N = 4$. From table it is observed that there are six different sets of aperiodic autocorrelation $C_k$ and three different PAPR values, namely 1.77, 2.37 and 4.00. The message symbols can be grouped according to their PAPR values.

Table.1 has motivated us to apply techniques to OFDM signal which could reduce the aperiodic correlation among the
subcarriers so that PAPR is reduced. There are various method to do this such as scrambling [15], here we have focused only on the error control coding techniques such as Golay and Reed-Muller codes. By employing error control coding techniques it gives dual advantage of error control as well as PAPR reduction. Next two sections deals with Golay and Reed-Muller codes.

IV. GOLAY CODES

The binary form of the Golay code is one of the most important types of linear binary block codes. It is of particular significance since it is one of only a few examples of a nontrivial perfect code [13] [17]. A t-error-correcting code can correct a maximum of t errors. A perfect t-error correcting code has the property that every word lies within a distance of t to exactly one code word. Equivalently, the code has \( d_{\text{min}} = 2t+1 \), and covering radius \( r \), where the covering radius \( r \) is the smallest number such that every word lies within a distance of \( r \) to a codeword.

**Theorem 1.** If there is an \((n, k)\) code with an alphabet of \( q \) elements, and \( d_{\text{min}} = 2t+1 \), then
\[
q^n \geq q^k \sum_{i=0}^{t} \binom{n}{i} (q-1)^i.
\]

The inequality in Theorem 1 is known as the Hamming bound. Clearly, a code is perfect precisely when it attains equality in the Hamming bound. Two Golay codes do attain equality, making them perfect codes: the (23, 12) binary code with \( d_{\text{min}} = 7 \), and the (11, 6) ternary code with \( d_{\text{min}} = 5 \). Both codes have the largest minimum distance for any known code with the same values of \( n \) and \( k \).

Golay was in search of perfect code when he noticed that \( \binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3} = 2^{11} = 2^{23-12} \) which indicated the existence of a (23, 12) perfect code that could correct any combination of three or fewer random errors in a block of 23 bits. The (23, 12) Golay code can be generated using a method similar to CRC by using any of the following generator polynomial:

- \( P_1(X) = X^{11} + X^{10} + X^6 + X^5 + X^4 + X^2 + 1 \)
- \( P_2(X) = X^{11} + X^0 + X^7 + X^6 + X^5 + X + 1 \)

this is due to the fact that \( X^{23} + 1 = (X+1)P_1(X)P_2(X) \).

For our analysis, we have constructed Golay codes using Hadamard matrix, which is explained below.

A. Construction of extended binary Golay code \( G_{24} \)

We have used the Hadamard matrix of Paley type of \( p = 11 \) and \( n = p + 1 = 12 \) for the construction of generator matrix for the Golay code. Hadamard Paley type is a normalized Hadamard matrix \( H \) of order \( n = p + 1 \) and \( H \) is of the form
\[
H = \begin{bmatrix} 1 & 1 \\ 1 & Q - I \end{bmatrix}
\]

where \( I \) is a \( p \times p \) identity matrix and \( Q \) is a Jacobsthal matrix. Jacobsthal matrix \( Q = (q_{ij}) \) is a \( p \times p \) matrix whose columns and rows are labeled as 0, 1, 2, \ldots, \( p - 1 \) and \( q_{ij} = \chi(j-i) \).

The Legendre symbol \( \chi(i) \) is defined as:
\[
\chi(i) = \begin{cases} 
0 & \text{if } i \text{ is multiple of } p \\
1 & \text{if the rem}(p \mid i) \text{ is a quadratic residue } \text{ mod } p, \text{ and} \\
-1 & \text{if the remainder is nonresidue}
\end{cases}
\]

For \( p = 11 \), the Jacobsthal matrix \( Q \) is as follows:
\[
Q = \begin{bmatrix} 
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}
\]

Let \( A_p = Q_p - I_p \), where \( I \) is an identity matrix and \( Q \) Jacobsthal matrix, then Hadamard of Paley type of order \( n \) will be of the following form:
\[
H_{p+1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1 & A_p \\
& & & & 1 \\
\end{bmatrix}
\]

Without losing the generality, \( A_{p+1} \) can be obtained similar to \( H_{p+1} \). Now defining a code \( C_{24} \subseteq V = F_2^{24} \) has \( n = 24 \) and dimension \( k = 12 \). Thus, its generator matrix \( G_{24} \) has the form \( G_{24} = (I_{12}, A_{11+1}) \), where \( I_{12} \) is \( 12 \times 12 \) identity matrix and \( A_{12} \) is a Hadamard matrix of order 12 of Paley type. Binary Golay code \( C_{23}[23, 12, 7] \) is obtained by puncturing any column of \( C_{24}[24, 12, 8] \).
V. REED-MULLER CODES

Reed-Muller codes are among the oldest and well known codes [14]. Reed-Muller codes have many interesting properties. They form an infinite family of codes, and larger Reed-Muller codes can be constructed from smaller ones. This particular observation lead us to show that Reed-Muller codes can be defined recursively. Assuming that we are given a vector space $\mathbb{F}_2^m$ and considering the ring $\mathcal{R}_m = \mathbb{F}_2[x_0, x_1, \ldots, x_m]$.

**Definition 1.** A Boolean monomial is an element $p \in \mathcal{R}_m$ of the form:

$$p = x_0^i x_1^r \cdots x_m^{r_{m-1}}$$

where $r_i \in \mathbb{N}$ and $i \in \mathbb{Z}_m$.

A Boolean polynomial is a linear combination of Boolean monomials.

**Definition 2.** Given a Boolean monomial $p \in \mathcal{R}_m$, we say that $p$ is in reduced form if it is square free. For any monomial $q \in \mathcal{R}_m$, the reduced form $q'$ is found by applying the following:

$$x_i x_j = x_j x_i \quad \text{as } \mathcal{R}_m \text{ is a commutative ring}$$

$$x_i^2 = x_i \quad \text{as } 0 \ast 0 = 0 \text{ and } 1 \ast 1 = 1.$$ 

A Boolean polynomial in reduced form is simply a linear combination of reduced-form Boolean monomials (with coefficients in $\mathbb{F}_2$).

**Example 2.** Suppose we have a Boolean polynomial $p = x_0 x_1 x_2^{1500} + x_0^1 x_2^1 + x_1 + 1 \in \mathbb{R}_3$ then, by applying the rules as per definition 1 and 2, we get the reduced form, $p'$ as:

$$p' = x_0 x_1 x_2 + x_0 x_2 + x_1 + 1.$$ 

Consider the mapping $\psi : \mathcal{R}_m \to \mathbb{F}_2^{2m}$, defined as follows:

$$\psi(0) = \underbrace{00 \cdots 0}_{2^m}$$

$$\psi(1) = \underbrace{11 \cdots 1}_{2^m}$$

$$\psi(x_0) = \underbrace{11 \cdots 100 \cdots 0}_{2^m-1}$$

$$\psi(x_1) = \underbrace{11 \cdots 100 \cdots 0}_{2^m-2} \underbrace{11 \cdots 100 \cdots 0}_{2^m-2} \cdots$$

$$\psi(x_t) = \underbrace{11 \cdots 100 \cdots 0}_{2^m-t} \cdots$$

For any monomial $p \in \mathcal{R}_m$, to calculate $\psi(p)$, first we find its reduced form $p' = x_{i_0} x_{i_1} \cdots x_{i_t}$, then $\psi(p) = \psi(x_{i_0}) \ast \psi(x_{i_1}) \ast \cdots \ast \psi(x_{i_t})$.

For the Reed-Muller code $\mathcal{R}_m(r, m)$, the generator matrix is defined as follows:

$$G_{\mathcal{R}_m(r, m)} = \begin{bmatrix}
\psi(1) \\
\psi(x_0) \\
\psi(x_1) \\
\vdots \\
\psi(x_{m-1}) \\
\psi(x_0 x_1) \\
\psi(x_0 x_2 - 2) \\
\vdots \\
\psi(x_{m-2} x_{m-1}) \\
\psi(x_0 x_1 x_2) \\
\vdots \\
\psi(x_{m-r} x_{m-r+1} \cdots x_{m-1})
\end{bmatrix} \quad (13)$$

The matrix $G_{\mathcal{R}_m(r, m)}$ has dimension $k \times n$, where the code dimension is given by $k = \sum_{i=0}^{r} \binom{m}{i}$ and $n = 2^m$. Encoding a message using Reed-Muller code $\mathcal{R}_m(r, m)$ is straightforward. Let $m = (m_1, m_2, \ldots, m_k)$ be a block of length $k$, the encoded message $M_e$ is given by:

$$M_e = \sum_{i=1}^{k} m_i R_i,$$

where $R_i$ is a row of encoding matrix $G_{\mathcal{R}_m(r, m)}$.

VI. SIMULATION RESULTS AND DISCUSSION

In order to evaluate and compare the PAPR reduction performance of Golay and Reed-Muller code, we have considered a text file modulated by 16-QAM and 64-subcarriers are used throughout the simulation. The transmitted signal is oversampled by a factor of $L = 4$. Fig.1 shows the CCDF vs PAPR curve of an OFDM signal, where both (23, 12) and (24, 12) Golay coding schemes have been used to reduce PAPR of OFDM signal. Fig.1, clearly shows that there is a significant reduction in PAPR due to the Golay coding prior to OFDM modulation.

![CCDF vs PAPR curve of an OFDM signal with 16-QAM](image)

**Fig. 2.** CCDF vs. PAPR curve for OFDM signal with Golay coding
Fig. 3. CCDF vs. PAPR curve for OFDM signal with Reed-Muller coding configuration.

Table 1 shows the values of PAPR of an OFDM signal for various coding schemes at $CCDF = 10^{-2}$. In Table 1, the highest PARRG obtained is 2.9171 dB and 1.8433 dB for Golay and Reed-Muller coding schemes respectively for comparable code rate. This shows that the PAPR reduction performance of Golay code is higher than the Reed-Muller code for comparable code rate. Among the two Golay coding configurations, the performance of (23,12) is better than the (24,12) Golay coding while among the various Reed-Muller coding configurations the RM(2,4) gives the best result from both PARRG and code rate perspective.

VII. CONCLUSION

OFDM is a promising technique for high-speed wireless communication systems. A major drawback of conventional OFDM system is the high peak-to-average power ratio. In this paper we have compared the PAPR reduction performance of Golay and Reed-Muller coded OFDM signal. It has been found that the PAPR reduction performance of Golay code is higher than the Reed-Muller code for the comparable code rate. Moreover, among the two Golay code configurations used the performance of (23,12) Golay code is better than that of (24,12) Golay code. For Reed-Muller code, among various configuration considered, the PAPR reduction performance of RM(2,4) is optimum.

ACKNOWLEDGMENT

The authors are indebted to Editor Dr. B. Brojack and the anonymous reviewers, whose comments have greatly improved the quality and presentation of the paper. Also the authors would like to thank DSC members of Department of Information & Communication Technology at Manipal Institute of Technology, Manipal, for their valuable suggestions.

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