On some subspaces of Entire sequence space of Fuzzy Numbers

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Abstract—In this paper we introduce some subspaces of fuzzy entire sequence space. Some general properties of these sequence spaces are discussed. Also some inclusion relation involving the spaces are obtained.

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I. INTRODUCTION

The concepts of fuzzy set theory was introduced by Zadeh [1]. Later on sequence of fuzzy numbers have been discussed by Matloka [2] and developed by Mursaleen [3] Nanda [4] and Savas [5], Tripathy and Dutta [6] and many others. The sequence space $\Gamma$ of entire sequences was introduced by Ganapathy Iyer [9]. The space $\Gamma(F)$ of fuzzy entire sequences space was introduced by Kavikumar, Azme Bin Khamis and Kandasamy [8]. Also Orlicz space of Entire sequence of fuzzy numbers was introduced by Subramanian and Metin Basarir [9]. In this article we introduce some subspaces of fuzzy entire sequence space and some of their properties are discussed

II. PRELIMINARIES AND DEFINITIONS

We begin with giving some required definitions and propositions and lemmas. A fuzzy number is a fuzzy set on the real axis i.e. a mapping $u : \mathbb{R} \rightarrow [0, 1]$ which satisfies the following four conditions.

(i) $u$ is normal i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) $u$ is fuzzy convex i.e. $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) $u$ is upper semi-continuous.

(iv) The set $[u]_0 = \{x \in \mathbb{R} : u(x) > 0\}$ is compact (cf. Zadeh [1]) where $\{x \in \mathbb{R} : u(x) > 0\}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$. We denote the set of all fuzzy numbers on $\mathbb{R}$ by $E$ and called it as the space of fuzzy numbers. $\alpha$-level set $[u]_\alpha$ of $u \in E$ is defined by

$$[u]_\alpha = \{t \in \mathbb{R} : u(t) > \alpha\}, \quad \alpha \in [0, 1].$$

The set $[u]_\alpha$ is a closed bounded and non-empty interval for each $\alpha \in [0, 1]$ which is defined by $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$.

$\mathbb{R}$ can be embedded in $E'$ since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\bar{r}$ defined by $\bar{r} = \begin{cases} 1, & (x = r) \\ 0, & (x \neq r). \end{cases}$

Let $u, v, w \in E'$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication and product and division defined on $E'$ by

$$u + v = w \iff [w]_\alpha = [u]_\alpha + [v]_\alpha \quad \text{for all} \quad \alpha \in [0, 1]$$

$$w^-(\alpha) = [u^-(\alpha), v^-(\alpha)] \quad \text{and} \quad w^+(\alpha) = [u^+(\alpha), v^+(\alpha)]$$

and $w = w \iff [w]_\alpha = [u]_\alpha [v]_\alpha \quad \text{for all} \quad \alpha \in [0, 1]$ where it is immediate that

$$w^-(\alpha) = \min\{u^-(\alpha), v^-(\alpha), u^+(\alpha), v^+(\alpha)\}$$

and $w^+(\alpha) = \max\{u^-(\alpha), v^-(\alpha), u^+(\alpha), v^+(\alpha)\}$

Let $W$ be the set of all closed and bounded intervals $A$ of real numbers with endpoints $A$ and $\bar{A}$ i.e., $A = [A, \bar{A}]$. Define the relation $d$ on $W$ by $d(A, B) = \max\{|A - B|\}$. Then it can easily be observed that $d$ is a metric on $W$. Then we can define the metric $D$ on $E'$ by means of a Hausdorff metric $d$ as

$$D(u, v) = \sup_{\alpha \in (0, 1]} d([u]_\alpha, [v]_\alpha)$$

$$= \sup_{\alpha \in (0, 1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}$$

One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \iff \text{if and only if} \quad A \leq B \quad \text{and} \quad A \leq B$$

The partial order relation on $E'$ is defined as follows. $u \leq v$ if and only if $[u]_\alpha \leq [v]_\alpha$ if and only if $u^-(\alpha) \leq v^-(\alpha)$ and $u^+(\alpha) \leq v^+(\alpha)$.

An absolute value $|u|$ of a fuzzy number $u$ is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\}, & (t \geq 0) \\ 0, & (t > 0) \end{cases}$$
α-level set $[u]_α$ of the absolute value of $u \in E'$ is in the form $[u]_α = [u^-(α), u^+(α)]$ where

$$
|u^−(α)| = \max\{0, u^−(α), -u^+(α)\}
$$

$$
|u^+(α)| = \max\{|u^−(α)|, |u^+(α)|\}
$$

**Definition II.1.** A sequence $u = (u_k) \in w(F)$ is a function $u$ from the set $\mathbb{N}$ into $E'$.

The fuzzy number $u_k$ denotes the value of the function at $k \in \mathbb{N}$ and is called the $k$-th term of the sequence. The set of all fuzzy sequences is denoted by $w(F)$.

**Definition II.2.** A sequence $u = (u_k) \in w(F)$ is called convergent with limit $l \in E'$ if and only if for every $ε > 0$ there exists $n_0 = n_0(ε) \in \mathbb{N}$ such that $D(u_k, u) < ε$ for all $k \geq n_0$.

**Definition II.3.** A sequence $u = (u_k) \in w(F)$ is called bounded if and only if the set of fuzzy numbers consisting of the terms of the sequence $(u_k)$ is a bounded set. That is, a sequence $(u_k) \in w(F)$ is said to be bounded if and only if there exist two fuzzy numbers $m$ and $M$ such that $m \leq u_k \leq M$ for all $k \in \mathbb{N}$.

In this article we define some subspaces $\Gamma(F, λ)$, $χ(F)$ and $Γ(1, d)$ of fuzzy entire sequences space.

**III. MAIN RESULTS**

For each fixed $k$, define the fuzzy metric

$$
D(u_k, v_k) = \sup_{α \in [0, 1]} \max\{|u_k^−(α) − v_k^−(α)|^{1/k}, |u_k^+(α) − v_k^+(α)|^{1/k}\}
$$

Clearly $(E', D)$ is a complete metric space.

We define the space of all entire sequences of fuzzy numbers by

$$
Γ(F) = \{u = (u(k)) \in w(F) : \lim_{k \to \infty} D(u_k, \bar{u}) = 0\}
$$

**Theorem III.1.** $Γ(F)$ is a complete metric space with respect to the metric

$$
d(u, v) = \sup_k D(u_k, v_k)
$$

**Proof:** Let $\{u^i\}$ be a Cauchy sequence of fuzzy numbers in $Γ(F)$. Then for every $ε > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$
d(u^i, u^j) = \sup_k D(u_k^i, u_k^j) < ε \quad \text{for all} \quad i, j \geq n_0
$$

(1)

Since this is true for all $k$, we have

$$
D(v_k^i, v_k^j) < ε \quad \text{for all} \quad i, j \geq n_0
$$

(2)

This leads to the fact that $\{v_k^i\}$ is a Cauchy sequence of fuzzy number in $E'$. Since $(E', D)$ is a complete metric space, $\{v_k^i\} \to v_k$ as $i \to \infty$.

Therefore, $D(u_k^i, u_k) < ε$, since this is true for all $k$, $sup_k D(u_k^i, u_k) < ε$ implies that $u^i \to u$ in $Γ(F)$.

It is easy to see that $u \in Γ(F)$.

Hence $Γ(F)$ is complete. ■

Now we proceed to define some subspaces of $Γ(F)$.

**Definition III.2.** Let $λ = (λ_k)$ denote a fixed sequence of fuzzy numbers such that $λ_k \neq 0$ for all $k$. We define the sequence space $Γ(F, λ)$ as follows:

$$
Γ(F, λ) = \{u \in Γ(F) : λ_k u_k \in Γ(F)\}.
$$

**Theorem III.3.** $Γ(F, λ) = Γ(F)$ if and only if $lim sup\{D(λ_k, \bar{u})\} < ∞$.

**Proof:** Suppose $Γ(F, λ) = Γ(F)$. Let $u \in Γ(F, λ)$. Then $λ_k u_k \in Γ(F)$. Therefore for given $ε > 0$ there exist $n_0$ such that $D(λ_k u_k, \bar{u}) < ε$ for all $k \geq n_0$. Suppose $lim sup\{D(λ_k, \bar{u})\} = ∞$. Then there exist a subsequence $(n_k)$ such that $D(λ_{n_k}, \bar{u}) > M$ for some $M > 0$.

Therefore $sup max\{|λ_k^+ α|^{1/k}, |λ_k^− α|^{1/k}\} > M$. This implies that $|λ_k^+ α|^{1/k} > M$ and $|λ_k^− α|^{1/k} > M$.

Now, define a sequence of fuzzy numbers by

$$
u_k = \begin{cases} 1, & \text{if } k = n_k, \\ 0, & \text{if } k \neq n_k. \end{cases}
$$

Then $λ_k^+ α = 0$ and $λ_k^− α = 1$. Clearly $(u_k) \in Γ(F)$.

But $|λ_k^+ α|^{1/k} > M$ and $|λ_k^− α|^{1/k} > M$, which contradicts the fact that $D(λ_k u_k, \bar{u}) < ε$.

Conversely, Suppose $lim sup\{D(λ_k, \bar{u})\} < ∞$. Then there exist $M > 0$ such that $D(λ_k, \bar{u}) < ε$ for all $k$.

Obviously $Γ(F, λ) \subseteq Γ(F)$. Let $u \in Γ(F)$. Then $D(u_k, \bar{u}) < ε/M$. Now, $D(λ_K u_k, \bar{u}) \leq D(λ_k, \bar{u}) ≤ D(u_k, \bar{u}) < ε$ (see cf. Talo [11]).

Hence $λ_k u_k \in Γ(F)$. From this we get $Γ(F) \subseteq Γ(F, λ)$.

Consequently, $Γ(F) = Γ(F, λ)$. This completes the proof. ■

**Theorem III.4.** If $λ = (λ_k)$ and $μ = (μ_k)$ are any two fixed sequences of fuzzy numbers and if $\{D(γ_k, \bar{u})\} < M$ for all $k$ and for some $M > 0$, where $γ_k = \frac{μ_k}{λ_k}$ then $Γ(F, λ) \subseteq Γ(F, μ)$.

**Proof:** Suppose $\{D(γ_k, \bar{u})\} < M$ for some $M > 0$. Then $sup_{α \in [0, 1]} max\{|γ_k^+ α|^{1/k}, |γ_k^− α|^{1/k}\} < M$.

This implies that $\left|\left(\frac{μ_k}{λ_k}\right)^{+ α}\right|^{1/k} < M$ and $\left|\left(\frac{μ_k}{λ_k}\right)^{− α}\right|^{1/k} < M$. From this we get

$$
|μ_k^+ α| < M |λ_k^+ α|, \quad |μ_k^− α| < M |λ_k^− α| \quad (3a)
$$

and $|μ_k^+ α| < M |λ_k^+ α|, \quad |μ_k^− α| < M |λ_k^− α| \quad (3b)$
Let \( u \in \Gamma(F, \lambda) \). Then \( \lambda u \in \Gamma(F, \lambda) \). Therefore for given \( \varepsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that \( D(\lambda (u_{k}), \overline{u}) < \varepsilon / M \) for all \( k \geq n_0 \).

This implies that
\[
\sup_{\alpha \in [0,1]} \max \{ |(\lambda u)^{-}(\alpha)|^{1/k}, |(\lambda u)^{+}(\alpha)|^{1/k} \} < \varepsilon / M.
\]
This means that \( |(\lambda u)^{+}(\alpha)|^{1/k} < \varepsilon / M \) and \( |(\lambda u)^{-}(\alpha)|^{1/k} < \varepsilon / M \). From this we get
\[
|\lambda^{-}_{k}(\alpha) u^{-}_{k}(\alpha)| < \varepsilon / M, \quad |\lambda^{+}_{k}(\alpha) u^{+}_{k}(\alpha)| < \varepsilon / M \quad (4)
\]

Using (3) and (4) we get
\[
|\mu^{-}_{k}(\alpha) u^{-}_{k}(\alpha)| = |\mu^{+}_{k}(\alpha)| |u^{+}_{k}(\alpha)| < M, \quad |\lambda^{-}_{k}(\alpha) u^{-}_{k}(\alpha)| < M, \quad |\lambda^{+}_{k}(\alpha) u^{+}_{k}(\alpha)| < M.
\]

Similarly we have \( |\mu^{-}_{k}(\alpha) u^{-}_{k}(\alpha)| < \varepsilon \) and \( |\mu^{+}_{k}(\alpha) u^{+}_{k}(\alpha)| < \varepsilon \). Hence
\[
\sup_{\alpha \in [0,1]} \max \{ |(\mu^{-}_{k} u^{-}_{k})^{-}(\alpha)|^{1/k}, |(\mu^{+}_{k} u^{+}_{k})^{+}(\alpha)|^{1/k} \} \]

This implies that \( D(\mu_{k} u_{k}, \overline{u}) < \varepsilon \). Thus \( u \in \Gamma(F, \mu) \).

Remark: The condition stated in Theorem III.5 is not necessary. Let us, now define a sequence \( (\lambda_{k}) = \left( \frac{1}{k} \right) \), where \( k \in E' \) and \( \mu_{k} = \frac{1}{k} \) for all \( k \).

Then \( D(\gamma_{k}, \overline{u}) \) and \( D(\mu_{k}, \overline{u}) \) are bounded sequences.

Thus by Theorem III.4, \( \Gamma(F, \lambda) = \Gamma(F, \mu) = \Gamma(F) \).

Therefore \( \Gamma(F, \lambda) \subseteq \Gamma(F, \mu) \).

But \( D \left( u_{k}, \overline{u} \right) \) is unbounded.

\( \Gamma(F, \lambda) \) is endowed with two topologies, one is the metric topology inherited from \( \Gamma(F) \), its metric being
\[
d(u,v) = \sup_{\alpha \in [0,1]} \max \{ |u^{-}_{k}(\alpha) - v^{-}_{k}(\alpha)|^{1/k}, |u^{+}_{k}(\alpha) - v^{+}_{k}(\alpha)|^{1/k} \}
\]
where \( u,v \in \Gamma(F, \lambda) \). The other is the metric topology \( d_{\lambda} \) given by
\[
d_{\lambda}(u,v) = \sup_{k} D_{\lambda}(u_{k}, v_{k}),
\]
\[
D_{\lambda}(u_{k}, v_{k}) = \sup_{\alpha \in [0,1]} \max \{ |\lambda^{-}_{k}(\alpha) u^{-}_{k}(\alpha) - v^{-}_{k}(\alpha)|^{1/k}, |\lambda^{+}_{k}(\alpha) u^{+}_{k}(\alpha) - v^{+}_{k}(\alpha)|^{1/k} \}
\]
and \( E' \) is complete with respect to \( D_{\lambda} \).

Theorem III.5. If \( \lim \sup \{ D(\lambda_{k}, \overline{u}) \} < \infty \) then \( d \) is finer than \( d_{\lambda} \).

Proof: To prove the result, it is enough to prove that if \( \{ u_{k} \} \) is a sequences of fuzzy numbers converging to \( u \) in \( \Gamma(F, \lambda), d_{\lambda} \) then the sequence converges to \( u \) in \( \Gamma(F, \lambda), d \). Consider the identity mapping \( I \) from \( \Gamma(F, \lambda), d_{\lambda} \) to \( \Gamma(F, \lambda), d \) defined by \( u \mapsto u \). Take \( u = \overline{u} \), where \( \overline{u} \) is the zero element of \( \Gamma(F) \). Since \( (u_{k}) \) converges to \( u = \overline{u} \) in \( \Gamma(F, \lambda), d_{\lambda} \), given \( \varepsilon > 0 \) there exists a positive integer \( n_{0} \) such that \( d(u_{k}, \overline{u}) = \sup D(u_{k}, \overline{u}) < \varepsilon \) for all \( k \geq n_{0} \).

Let \( U = \lim \sup \{ D(\lambda_{k}, \overline{u}) \} \)
\[
d_{\lambda}(u, \overline{u}) = \sup_{k} \left\{ \sup_{\alpha \in [0,1]} \max \{ |\lambda^{-}_{k}(\alpha)|^{1/k} u^{-}_{k}(\alpha)^{1/k}, |\lambda^{+}_{k}(\alpha)|^{1/k} u^{+}_{k}(\alpha)^{1/k} \} \right\}
\]
\[
\leq U \sup_{k} \left\{ \sup_{\alpha \in [0,1]} \max \{ |u^{-}_{k}(\alpha)|^{1/k}, |u^{+}_{k}(\alpha)|^{1/k} \} \right\}
\]
\[
\leq U \sup D(u_{k}, \overline{u}) < U \varepsilon
\]
Hence \( (u_{k}) \) converges to \( \overline{u} \) in \( \Gamma(F, \lambda), d_{\lambda} \).

Theorem III.6. \( (\Gamma(F, \lambda), d_{\lambda}) \) is a complete metric space if and only if
\[
\lim \inf \{ D(\lambda_{k}, \overline{u}) \} > 0.
\]
Proof: Let \( \{ u^{i} \} \) be a Cauchy sequence of fuzzy numbers in \( \Gamma(F, \lambda) \). Therefore for given \( \varepsilon > 0 \) there exist \( n_{0} \in \mathbb{N} \) such that \( d_{\lambda}(u^{i}, u^{j}) = \sup \{ |(\lambda_{k}^{-})(\alpha)|^{1/k} u^{-}_{k}(\alpha)^{1/k} - |(\lambda_{k}^{-})(\alpha)|^{1/k} u^{-}_{k}(\alpha)^{1/k} \} \)
\[
< \varepsilon / L \quad \text{for all } i,j \geq n_{0}
\]
Since this is true for all \( k, D_{\lambda}(u^{i}_{k}, u^{j}_{k}) \) \( < \varepsilon \) for all \( i,j \geq n_{0} \).

This implies that
\[
\sup_{\alpha \in [0,1]} \max \{ |\lambda^{-}_{k}(\alpha)|^{1/k} u^{-}_{k}(\alpha)^{1/k} - |\lambda^{+}_{k}(\alpha)|^{1/k} u^{+}_{k}(\alpha)^{1/k} \} < \varepsilon \quad (5)
\]
\[
\sup_{\alpha \in [0,1]} \max \{ |\lambda^{-}_{k}(\alpha)|^{1/k} u^{-}_{k}(\alpha)^{1/k} - |\lambda^{+}_{k}(\alpha)|^{1/k} u^{+}_{k}(\alpha)^{1/k} \} < \varepsilon \quad (6)
\]

Using (5) and (6) we get
\[
|u^{(i)}_{k}(\alpha) - u^{(j)}_{k}(\alpha)|^{1/k} < \varepsilon / L \quad \text{and}
\]
\[
|u^{(i)+}_{k}(\alpha) - u^{(j)+}_{k}(\alpha)|^{1/k} < \varepsilon / L \quad \text{for all } i,j \geq n_{0}
\]
Hence \( \{ u^{(i)}_{k} \} \) is a Cauchy sequence in \( E' \) and since \( (E', D) \) is complete, \( \{ u^{(i)}_{k} \} \rightarrow u_{k} \) as \( i \rightarrow \infty \).

Hence \( D(u^{(i)}_{k}, u_{k}) < \varepsilon / L \) for all \( i,j \geq n_{0} \).

Letting \( j \rightarrow \infty \) in (7) we get
\[
|u^{(i)}_{k}(\alpha) - u^{(j)}_{k}(\alpha)|^{1/k} < \varepsilon / L \quad \text{and}
\]
\[
|u^{(i)+}_{k}(\alpha) - u^{(j)+}_{k}(\alpha)|^{1/k} < \varepsilon / L
\]
Now
\[
|\lambda^{-}_{k}(\alpha)|^{1/k} |u^{(i)-}_{k}(\alpha) - u^{(j)-}_{k}(\alpha)|^{1/k} < \varepsilon \quad \text{and}
\]
\[
|\lambda^{+}_{k}(\alpha)|^{1/k} |u^{(i)+}_{k}(\alpha) - u^{(j)+}_{k}(\alpha)|^{1/k} < \varepsilon
\]
Hence
\[
\sup_{k} \sup_{\alpha \in [0,1]} \max \{ |\lambda^{-}_{k}(\alpha)|^{1/k} |u^{(i)-}_{k}(\alpha) - u^{(j)-}_{k}(\alpha)|^{1/k}, |\lambda^{+}_{k}(\alpha)|^{1/k} |u^{(i)+}_{k}(\alpha) - u^{(j)+}_{k}(\alpha)|^{1/k} \} < \varepsilon
\]
Thus $u_k^{(i)} \to u$ in $(\Gamma(F, \lambda), d_\lambda)$. Since each $(u^i)$ is in $\Gamma(F)$ we have

$$D(u_k^{(i)}, \overline{0}) < \frac{\varepsilon}{L}. \tag{10}$$

Using (9) and (10),

$$D(\lambda_k, u_k, \overline{0}) \leq D(\lambda_k, \overline{0})D(u_k, \overline{0}) \quad (\text{See [11]})$$

$$\leq D(\lambda_k, \overline{0})[D(u_k, \overline{0}) + D(u_k^{(i)}, \overline{0})]$$

$$\leq L \left( \frac{\varepsilon}{L} + \frac{\varepsilon}{L} \right)$$

Hence $u \in \Gamma(F, \lambda)$. Thus $\Gamma(F, \lambda)$ is complete. Conversely suppose $\lim \inf \{D(\lambda_k, \overline{0})\} = 0$. Then there exist a sequence $\{D(\lambda_k, \overline{0})\}$ which is steadily decreasing and tends to zero. Consider a sequence $\{P_n\}$ of polynomials where $P_n(x) = 1 + x^{n_1} + x^{n_2} + \ldots + x^{n_k}$. Clearly this sequence is a Cauchy sequence in $(\Gamma(F, \lambda), d_\lambda)$. But it fails to converge to a point in $(\Gamma(F, \lambda), d_\lambda)$. This completes the proof.

Definition III.7. For each fixed $k$, we define a fuzzy metric

$$D_\lambda(u_k, v_k) = \sup_{\alpha \in [0, 1]} \max \left\{ k|u_k^-(\alpha) - k|v_k^-(\alpha)|^{1/k}, k|u_k^+(\alpha) - k|v_k^+(\alpha)|^{1/k} \right\}$$

where $u = (u_k)$ and $v = (v_k)$ are sequences of fuzzy numbers and we can easily see that $(E', D_\lambda)$ is complete.

Definition III.8. The subspace $\chi(F)$ of $\Gamma(F)$ is defined by

$$\chi(F) = \left\{ u = (u_k) \in \Gamma(F) : \lim_{k \to \infty} D_\lambda(u_k, \overline{0}) = 0 \right\}.$$ 

In other words given $\varepsilon > 0$ there exists a positive integer $n_0 \in \mathbb{N}$ such that $D_\lambda(u_k, \overline{0}) < \varepsilon$ for all $k \geq n_0$.

Theorem III.9. $\chi(F)$ is a complete metric space (See [10]).

Definition III.10. For each fixed $k$ we define a fuzzy metric by

$$\overline{D}(u_k, v_k) = \sup_{\alpha \in [0, 1]} \max \left\{ k|u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, k|u_k^+(\alpha) - v_k^+(\alpha)|^{1/k} \right\}$$

where $u = (u_k)$ and $v = (v_k)$ are sequences of fuzzy numbers and it is clear that $(E', \overline{D})$ is complete.

Definition III.11. We define the subsequence $\Gamma(F, 1, d)$ of $\Gamma(F)$ by

$$\Gamma(F, 1, d) = \left\{ u = (u_k) \in \Gamma(F) : \lim_{k \to \infty} \overline{D}(u_k, \overline{0}) = 0 \right\}$$

In other words given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $\overline{D}(u_k, \overline{0}) < \varepsilon$.

Theorem III.12. $\Gamma(F, 1, d)$ is a complete metric space with respect to the metric $\overline{D}(u, v) \sup D(u_k, v_k)$.

Theorem III.13. $\chi(F)$ is a proper closed subspace of $\Gamma(F, 1, d)$.

Proof: Consider the sequence $(u_k)$ defined by $(u_k) = \left( \frac{1}{k} \right)$ where $k \in E'$. Then $(u_k) \in \Gamma(F, 1, d)$ but $(u_k) \notin \chi(F)$. Therefore $\chi(F)$ is a proper subspace of $\Gamma(F, 1, d)$. Let $u \in \Gamma(F, 1, d)$ be a limit point of $\chi(F)$. Then there exist a sequence $(u^i)$ in $\chi(F)$ such that $u^i \to u$. Therefore for given $\varepsilon > 0$ there exist $n_0$ such that

$$\overline{D}(u^i, u) = \sup_{k} D(u_k^{(i)}, u_k) < \varepsilon \quad \text{for all} \quad k \geq n_0.$$

This implies that $k|u_k^{(i)}(\alpha) - u_k^-(\alpha)|^{1/k} < \varepsilon$ and $k|u_k^{(i)}(\alpha) - u_k^+(\alpha)|^{1/k} < \varepsilon$ for all $k \geq n_0$.

Now our aim is to prove $u \in \chi(F)$.

$$\sup_{k} \left[ k|u_k^{(i)}(\alpha) - u_k^-(\alpha)|^{1/k} \right] \leq \sup_{k} \left[ k|u_k^{(i)}(\alpha) - u_k^+(\alpha)|^{1/k} \right]$$

Therefore $u \in \chi(F)$ and is closed.

REFERENCES