Recovery of Missing Samples in Multi-channel Oversampling of Multi-banded Signals

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Abstract—We show that in a two-channel sampling series expansion of band-pass signals, any finitely many missing samples can always be recovered via oversampling in a larger band-pass region. We also obtain an analogous result for multi-channel oversampling of harmonic signals.

Keywords—oversampling, multi-channel sampling, recovery of missing samples, band-pass signal, harmonic signal

I. INTRODUCTION

For a bounded and closed band-region B, let \( PW_B := \{ f(t) \in L^2(\mathbb{R}) : \text{supp} \hat{f}(\xi) \subset B \} \), where \( \mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt \) is the Fourier transform of \( f(t) \) with inverse Fourier transform \( f(t) = \mathcal{F}^{-1}(\hat{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{it\xi}d\xi \).

If a signal \( f(t) \) is single-banded with band-region \( B = [-\pi\omega, \pi\omega] \) \((\omega > 0)\), then \( f(t) \) can be expanded as a Shannon sampling series:

\[
\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{in\pi\omega} \frac{\sin \pi(\xi - n)}{\pi(\xi - n)},
\]

in which all samples \( \{f(n) : n \in \mathbb{Z}\} \) are independent. However, if we oversample \( f(t) \) with higher rate than the optimal Nyquist rate \( \omega \), then the resulting samples are dependent. Using this observation, we may recover finitely many missing samples(2,3,5,8).

When we join oversampling and multi-channeling, we may or may not be able to recover finitely many missing samples depending on the nature of the band-regions \( B \) and pre-filters used in channeling (6,10). In this work, we show that in case of band-pass and harmonic signals, any finitely many missing samples can be always recovered through a multi-channel oversampling in a larger band-region of the same type.

II. OVERSAMPLING OF BAND-PASS SIGNALS

Consider a band-pass region \( B = B_- \cup B_+ \), where \( w_0, w > 0 \) and

\[
B_- = [-\pi(\omega_0 + \omega), -\pi\omega_0] \quad \text{and} \quad B_+ = [\pi\omega_0, \pi(\omega_0 + \omega)].
\]

Then the optimal Nyquist rate for signals in \( PW_B \) is \( w \) samples per second. For \( \tau \) with \( 0 < \tau \leq w_0 \), let \( \hat{B} = \hat{B}_- \cup \hat{B}_+ \) be another band-pass region, where

\[
\hat{B}_- = [-\pi(\omega_0 + \omega + \tau), -\pi\omega_0] \quad \text{and} \quad \hat{B}_+ = [\pi\omega_0, \pi(\omega_0 + \omega + \tau)]
\]

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and

\[
\hat{B}_+ = [\pi(\omega_0 - \tau), \pi(\omega_0 + \omega + \tau)].
\]

We take \( \tau \) so that \( \tau := \frac{2\omega_0 + \omega}{2\pi + \omega} \) is a positive integer. Then \( \hat{B}_+ = \hat{B}_- + \pi(2\tau + \omega) \) so that \( \hat{B} \) becomes a so-called selectively tiled band-region ([4]) of length \( 2\pi\omega \) with \( \omega = \omega_0 + 2\tau \). Note that the smallest such \( \tau \) is obtained when we take \( \tau \) to be the largest integer less than \( 1 + \frac{2\omega_0}{2\pi + \omega} \).

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III. RECOVERING MISSING SAMPLES

For a band-pass signal \( f(t) \) in \( PW_B \), consider its oversampled expansion (5).

**Lemma 1.** We have for any integer \( m \)

\[
c_k(f)\left(\frac{2m}{\omega}\right) = \frac{1}{\pi \omega} \sum_n c_k(f)\left(\frac{2n}{\omega}\right) \int_{B^*} e^{i \frac{2\pi}{\omega} (m-n)\xi} d\xi \tag{6}
\]

for \( k = 1, 2 \).

**Proof:** By (1) and (4), we have

\[
c_k(f)(t) = \frac{1}{\sqrt{2\pi}} \int_B A_k(\xi)f(\xi)e^{it\xi} d\xi = \frac{1}{\pi \omega} \sum_{j=1}^2 \sum_n c_j\left(\frac{2n}{\omega}\right) \int_B A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i\frac{2\pi}{\omega} (m-n)\xi} d\xi.
\]

Hence for any integer \( m \) we have

\[
c_k(f)\left(\frac{2m}{\omega}\right) = \frac{1}{\pi \omega} \sum_{j=1}^2 \sum_n c_j\left(\frac{2n}{\omega}\right) \int_B A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i\frac{2\pi}{\omega} (m-n)\xi} d\xi + \int_B A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i\frac{2\pi}{\omega} (m-n)\xi} d\xi + \int_B A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i\frac{2\pi}{\omega} (m-n)\xi} d\xi,
\]

from which (6) comes since \( A_k(\xi)U_j(\xi) + A_k(\xi + \pi\omega)U_j(\xi + \pi\omega) = \delta_{jk} \) by (3).

**Theorem 1.** For any finite index sets of integers \( I_1 \) and \( I_2 \), any finite missing samples \( \{c_1(f)\left(\frac{2m}{\omega}\right) : m \in I_1\} \cup \{c_2(f)\left(\frac{2n}{\omega}\right) : n \in I_2\} \) can be uniquely recovered.

**Proof:** Set \( I_1 = \{m_1, m_2, \ldots, m_M\} \) if \( I_1 \neq \phi \) and \( I_2 = \{n_1, n_2, \ldots, n_N\} \) if \( I_2 \neq \phi \). Then we have from (6)

\[
c_1(f)\left(\frac{2m_1}{\omega}\right) = \frac{1}{\pi \omega} \sum_{k=1}^M r(m_k, m_k)c_1(f)\left(\frac{2m_k}{\omega}\right) + g_{1j} \tag{7}
\]

for \( 1 \leq j \leq M \) and

\[
c_2(f)\left(\frac{2n_1}{\omega}\right) = \frac{1}{\pi \omega} \sum_{k=1}^N r(n_k, n_k)c_2(f)\left(\frac{2n_k}{\omega}\right) + g_{2j} \tag{8}
\]

for \( 1 \leq j \leq N \) where \( g_{1j} \)'s and \( g_{2j} \)'s are known quantities and

\[
r(m, n) := \int_{B^*} e^{i \frac{2\pi}{\omega} (m-n)\xi} d\xi \text{ for } m, n \in \mathbb{Z}.
\]

We may write (7-8) in a vector form as:

\[
\begin{align*}
(I - S_1)c_1 &= g_1, \\
(I - S_2)c_2 &= g_2
\end{align*}
\]

where

\[
\begin{align*}
c_1 &= \left(c_1(f)\left(\frac{2m_1}{\omega}\right), \ldots, c_1(f)\left(\frac{2m_M}{\omega}\right)\right)^T, \\
c_2 &= \left(c_2(f)\left(\frac{2n_1}{\omega}\right), \ldots, c_2(f)\left(\frac{2n_N}{\omega}\right)\right)^T, \\
g_1 &= (g_{11}, \ldots, g_{1M})^T, \\
g_2 &= (g_{21}, \ldots, g_{2N})^T,
\end{align*}
\]

and

\[
S_1 = \frac{1}{\pi \omega} \sum_{j=1}^M r(m_j, m_j), \quad S_2 = \frac{1}{\pi \omega} \sum_{j=1}^N r(n_j, n_j).
\]

Note that \( S_1 \) and \( S_2 \) are self-adjoint. Now for any \( u = (u_1, \ldots, u_M) \in \mathbb{C} \setminus \{0\} \),

\[
\langle S_1u, u \rangle = \frac{1}{\pi \omega} \sum_{j=1}^M \sum_{k=1}^M r(m_j, m_k)u_k \overline{u}_j = \int_{B^*} \sum_{j=1}^M |\overline{u}_j|^2 |\overline{\xi}|^2 \chi_B(\xi) d\xi \leq \int_{B^*} \sum_{j=1}^M |\overline{u}_j|^2 |\overline{\xi}|^2 d\xi = \sum_{j=1}^M |u_j|^2
\]

since \( \{\frac{1}{\pi \omega} e^{i \frac{2\pi}{\omega} \xi}\}_{\xi \in \mathbb{R}} \) is an orthonormal basis of \( L^2(B^*) \). Hence, \( \lambda \) cannot be an eigenvalue of \( S_1 \). Similarly, \( \lambda \) cannot be an eigenvalue of \( S_2 \). Therefore, both equations in (9) have unique solutions \( c_1 \) and \( c_2 \).

Above process can be readily extended to multi-channel oversampling of harmonic signals (see [1] and Chapter 13 in [4]). Let \( f(t) \) be a harmonic signal in \( PW_B \), where

\[
B := \bigcup_{j=1}^N [a_j, b_j]
\]

is a harmonic band-region and

\[
b_i - a_i = \pi \omega (1 \leq i \leq N) \\
a_{i+1} - b_i = 2\pi \omega_0 (1 \leq i < N) \text{ for } \omega, \omega_0 > 0.
\]

For \( 0 < \tau \leq \omega_0 \), let \( \tilde{B} := \bigcup_{j=1}^N \tilde{B}_j \) be another harmonic band-region, where

\[
\tilde{B}_j = [a_i - \pi \tau, b_i + \pi \tau] \text{ for } 1 \leq i \leq N.
\]

We take \( \tau \) so that \( r := \frac{2\pi \omega_0}{2\pi \omega + \omega} \) is a positive integer. Then \( \tilde{B}_j = \tilde{B}_j + (j - 1)\tau \pi(2\tau + \omega) \) for \( 1 \leq i < j \leq N \) so that \( \tilde{B} \) becomes a so-called selectively tiled band-region of total length \( \tau \pi \omega \), where \( \omega = \omega + 2\tau \). We now take \( N \) pre-filters \( A_j(\xi) (j = 1, 2, \ldots, N) \) of bounded measurable functions on \( \tilde{B} \). We set \( A(\xi) \) be the \( N \times N \) matrix whose \( (j,k) \)th component is given by

\[
A_{jk}(\xi) = A_j(\xi + (k - 1)\tau \pi \omega) \text{ and assume } |\det A(\xi)| \geq \alpha > 0 \text{ a.e. on } \tilde{B}_j.
\]

Let

\[
c_j(f)(t) := \mathcal{F}^{-1}(A_j(\xi)\hat{f}(\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{B^*} A_j(\xi)\hat{f}(\xi)e^{it\xi} d\xi
\]
be the channeled output signals. Proceeding as in Section 2, we can obtain an oversampling formula for any harmonic signal $f(t)$ in $\mathcal{P}W_B$ (but viewed as a signal in $\tilde{PW}_B$) as

$$f(t) = \sum_{j=1}^{N} \sum_{n} c_j(f) \left( \frac{2n}{\omega} \right) T_{j,n}(t). \quad (10)$$

Then, we have the following multi-channel analog of Theorem 3.2.

**Theorem 2.** For any finite index sets of integers $I_i (i = 1, 2, \ldots, N)$, any finite missing samples $\cup_{i=1}^{N} \{ c_i(f)(\frac{2m}{\omega}) : m \in I_i \}$ from the oversampling (10) can be uniquely recovered.

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**REFERENCES**


