Recovery of Missing Samples in Multi-channel Oversampling of Multi-banded Signals

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Abstract—We show that in a two-channel sampling series expansion of band-pass signals, any finitely many missing samples can always be recovered via oversampling in a larger band-pass region. We also obtain an analogous result for multi-channel oversampling of harmonic signals.

Keywords—oversampling, multi-channel sampling, recovery of missing samples, band-pass signal, harmonic signal

I. INTRODUCTION

For a bounded and closed band-region \( B \), let \( \text{PW}_B \) be the Paley-Wiener space of finite energy (i.e. square integrable) signals of which frequencies are confined in \( B \). That is,
\[
\text{PW}_B := \{ f(t) \in L^2(\mathbb{R}) : \text{supp} \hat{f}(\xi) \subset B \},
\]
where \( \mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\xi t}dt \) is the Fourier transform of \( f(t) \) with inverse Fourier transform
\[
f(t) = \mathcal{F}^{-1}(\hat{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi t}d\xi.
\]
If a signal \( f(t) \) is single-banded with band-region \( B = [-\pi\omega, \pi\omega] (\omega > 0) \), then \( f(t) \) can be expanded as a Shannon sampling series:
\[
f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\omega}\right) \sin \pi(t-n) \frac{\pi}{n},
\]
in which all samples \( \{f\left(\frac{n}{\omega}\right) : n \in \mathbb{Z}\} \) are independent. However, if we oversample \( f(t) \) with higher rate than the optimal Nyquist rate \( \omega \), then we will obtain multi-channeling, and we may or may not to recover finitely missing samples depending on the nature of the band-region \( B \) and pre-filters used in channeling \( [6,10] \). In this work, we show that in case of band-pass and harmonic signals, any finitely many missing samples can be always recovered through a multi-channel oversampling in a larger band-region of the same type.

II. OVERSAMPLING OF BAND-PASS SIGNALS

Consider a band-pass region \( B = B_- \cup B_+ \), where \( w_0, w > 0 \) and
\[
B_- = [-\pi(\omega_0 + \omega), -\pi\omega_0] \quad \text{and} \quad B_+ = [\pi\omega_0, \pi(\omega_0 + \omega)].
\]
Then the optimal Nyquist rate for signals in \( \text{PW}_B \) is \( \omega \) samples per second. For \( \tau \) with \( 0 < \tau \leq w_0 \), let \( \bar{B} = B_- \cup B_+ \) be another band-pass region, where
\[
\bar{B}_+ = [-\pi(\omega_0 + \omega + \tau), -\pi(\omega_0 + \tau)]
\]
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and
\[
\bar{B}_- = [\pi(\omega_0 - \tau), \pi(\omega_0 + \omega + \tau)].
\]
We take \( \tau \) so that \( \tau := \frac{2\omega_0 + \omega}{2\pi + \omega} \) is a positive integer. Then \( \bar{B}_+ = \bar{B}_- + \pi(2\tau + \omega) \) so that \( \bar{B} \) becomes a so-called selectively tiled band-region \( [4] \) of length \( 2\pi\bar{\omega} \) with \( \bar{\omega} = \omega + 2\tau \). Note that the smallest such \( \tau \) is obtained when we take \( \tau \) to be the integer less than \( 1 + \frac{\pi\omega}{2\omega_0} \).

For any band-pass signal \( f(t) \) in \( \text{PW}_B \), let
\[
c_j(f)(t) := \mathcal{F}^{-1}(A_j(\xi)\hat{f}(\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_j(\xi)\hat{f}(\xi)e^{i\xi t}d\xi
\]
be the channelled output signals of the input signal \( f(t) \). Then \([4,7,8,9]\)
\[
f(t) = \sum_{j=1}^{2} \sum_{n} c_j(f)\left(\frac{2n}{\omega}\right)S_{j,n}(t),
\]
which converges in \( \text{PW}_{\bar{B}} \) and also converges uniformly on \( \mathbb{R} \). By taking Fourier transform on (2), we obtain
\[
\hat{f}(\xi) = \sum_{j=1}^{2} \sum_{n} c_j(f)\left(\frac{2n}{\omega}\right)\phi_{j,n}(\xi),
\]
which converges in \( L^2(\mathbb{R}) \), where
\[
\phi_{j,n}(\xi) = \frac{1}{\bar{\omega}} \sqrt{2} U_j(\xi)e^{-i\bar{\omega}\xi}
\]
and
\[
A(\xi)^{-1} = \begin{bmatrix} U_1(\xi) & U_2(\xi) \\ U_1(\xi + r\pi\bar{\omega}) & U_2(\xi + r\pi\bar{\omega}) \end{bmatrix} \quad \text{on} \quad \bar{B}_-.
\]
If \( f(t) \) is in \( \text{PW}_{\bar{B}} \), i.e., \( \text{supp} \hat{f} \subset B \), then
\[
\hat{f}(\xi) = \sum_{j=1}^{2} \sum_{n} c_j(f)\left(\frac{2n}{\omega}\right)\phi_{j,n}(\xi)\chi_B(\xi)
\]
in \( L^2(\mathbb{B}) \), where \( \chi_B(\xi) \) is the characteristic function of \( B \). By taking inverse Fourier transform on (4), we have
\[
f(t) = \sum_{j=1}^{2} \sum_{n} c_j(f)\left(\frac{2n}{\omega}\right)T_{j,n}(t)
\]
where \( T_{j,n}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_{j,n}(\xi)e^{i\xi t}d\xi \). We may call (5) a two-channel oversampling series expansion of \( f(t) \) in \( \text{PW}_{\bar{B}} \).
III. RECOVERING MISSING SAMPLES

For a band-pass signal \( f(t) \) in \( PW_B \), consider its oversampled expansion (5).

**Lemma 1.** We have for any integer \( m \)

\[
c_k(f)(\frac{2m}{\omega}) = \frac{1}{\pi \omega} \sum_n c_k(f)(\frac{2n}{\omega}) \int_{B^-} e^{i \frac{\pi}{\omega} (m-n)\xi} d\xi
\]

for \( k = 1, 2 \).

**Proof:** By (1) and (4), we have

\[
c_k(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{B^-} A_k(\xi) \hat{f}(\xi) e^{i\xi t} d\xi
\]

Hence for any integer \( m \) we have

\[
c_k(f)(\frac{2m}{\omega}) = \frac{1}{\pi \omega} \sum_{j,k} c_j(\frac{2n}{\omega}) \int_{B^-} A_k(\xi) U_j(\xi) \chi_B(\xi) e^{i \frac{\pi}{\omega} (m-n)\xi} d\xi
\]

We may write (7-8) in a vector form as :

\[
\begin{aligned}
(I - S_1) c_1 &= g_1 \\
(I - S_2) c_2 &= g_2
\end{aligned}
\]

where

\[
\begin{aligned}
c_1 &= (c_1(f)(\frac{2m_1}{\omega}), \ldots, c_1(f)(\frac{2m_M}{\omega}))^T, \\
c_2 &= (c_2(f)(\frac{2n_1}{\omega}), \ldots, c_2(f)(\frac{2n_N}{\omega}))^T, \\
g_1 &= (g_{11}, \ldots, g_{1M})^T, \\
g_2 &= (g_{21}, \ldots, g_{2N})^T,
\end{aligned}
\]

and

\[
S_1 = \left[ \frac{1}{\omega} r(m_j, m_k) \right]_{j,k=1}^M, \quad S_2 = \left[ \frac{1}{\omega} r(n_j, n_k) \right]_{j,k=1}^N.
\]

Note that \( S_1 \) and \( S_2 \) are self-adjoint. Now for any \( u = (u_1, \ldots, u_M) \in \mathbb{C}^M \),

\[
\langle S_1 u, u \rangle = \frac{1}{\sigma^2} \sum_{j,k=1}^M r(m_j, m_k) u_j u_k
\]

is a harmonic band-region and

\[
B := \bigcup_{i=1}^N [a_i, b_i]
\]

is a harmonic band-region and

\[
\begin{aligned}
b_i - a_i &= \frac{\pi \omega}{2} (1 \leq i \leq N) \\
a_{i+1} - b_i &= 2\pi \omega (1 \leq i < N) \text{ for } \omega, \omega_0 > 0.
\end{aligned}
\]

For \( 0 < \tau \leq \omega_0 \), let \( \tilde{B} := \cup_{i=1}^N \tilde{B}_i \) be another harmonic band-region, where

\[
\tilde{B}_i = [a_i - \pi \tau, b_i + \pi \tau] \text{ for } 1 \leq i \leq N.
\]

We take \( \tau \) so that \( r := \frac{2\pi \omega_0}{\pi \omega} \) is a positive integer. Then \( \tilde{B}_i = \tilde{B}_i + (j - 1)\pi \tau + \omega \) for \( 1 \leq i < j \leq N \) so that \( \tilde{B} \) becomes a so-called selectively tiled band-region of total length \( N\pi \omega \), where \( \omega = \omega + 2\tau \). We now take \( N \) pre-filters \( A_j(\xi) (j = 1, 2, \cdots, N) \) of bounded measurable functions on \( \tilde{B} \). We set \( A(\xi) \) be the \( N \times N \) matrix whose \( (j, k) \)th component is given by

\[
A_{jk}(\xi) = \langle A_j(\xi) \hat{f}(\xi) \rangle (\xi) = \frac{1}{\sqrt{2\pi}} \int_{\tilde{B}} A_j(\xi) \hat{f}(\xi) e^{i\xi t} d\xi
\]

and assume \( |\det A(\xi)| \geq \alpha > 0 \) a.e. on \( \tilde{B} \).
be the channeled output signals. Proceeding as in Section 2, we can obtain an oversampling formula for any harmonic signal \( f(t) \) in \( PW_B \) (but viewed as a signal in \( PW_B \)) as

\[
f(t) = \sum_{j=1}^{N} \sum_{n} c_j(f) \left( \frac{2n}{\omega} \right) T_{j,n}(t). \tag{10}
\]

Then, we have the following multi-channel analog of Theorem 3.2.

**Theorem 2.** For any finite index sets of integers \( I_i (i = 1, 2, \ldots, N) \), any finite missing samples \( \cup_{i=1}^{N} \{ c_i(f) \left( \frac{2m}{\omega} \right) : m \in I_i \} \) from the oversampling (10) can be uniquely recovered.

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**REFERENCES**


