Large Deviations for Lacunary Systems

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Abstract—Let $X_i$ be a Lacunary System, we established large deviations inequality for Lacunary System. Furthermore, we gained Marcinkiewicz Larger Number Law with dependent random variables sequences.

Keywords—Lacunary system, larger deviations, Locally Generalized Gaussian, Strong law of large numbers.

I. INTRODUCTION


In this paper, we shall establish large deviations inequality for Lacunary System. Further, we shall gain Marcinkiewicz Larger Number Law with dependent random variables sequences.

We give defined of Lacunary system as follows:

Definition 1.1 Given $p > 0$, a sequence of real-valued random variables $\{X_n, n \geq 1\}$ is called a Lacunary System or an $S_p$ system, if there exists a constant $K_p$ such that

$$E\sum_{i=1}^{n} C_i X_i^2 \leq K_p \sum_{i=1}^{n} C_i^2 p^{p/2}$$

for all sequence of real constant $\{C_i\}$ and all $n \geq m$.

Definition 1.2 Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$, set $\mathcal{F}^n = \sigma(X_1, \ldots, X_n)$, denote by the $\sigma$-field generated by the random variables $X_1, X_2, \ldots, X_n$.

1) Let $A \in \mathcal{F}_k, B \in \mathcal{F}_k, k \geq 1, \{X_n, n \geq 1\}$ is called a mixing if

$$|P(A \cap B) - P(A)P(B)| \leq \phi(n)P(A)$$

for some $\phi(n) \downarrow 0$.

2) $\{X_n, n \geq 1\}$ is called mixing if

$$\psi(n) = \sup_{k \in \mathbb{N}} \psi(\mathcal{F}_k, \mathcal{F}_k \cap \mathcal{F}_n) \to 0, n \to \infty,$$

where

$$\psi(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}, P(A)P(B) > 0} \frac{|P(A \cap B) - P(A)P(B)|}{P(A)P(B)}.$$

Definition 1.3 Let $X$ be a real-valued random variable, we call a Locally Generalized Gaussian, if there exists $\alpha > 0$ such that

$$E[\exp(ux)|\mathcal{F}] \leq \exp(u^2\alpha^2/2) \quad a.s.$$

for all $x \in \mathbb{R}$.

II. LARGER DEVIATIONS INEQUALITY

In order to prove larger deviations we need the following lemmas.

Lemma 2.1 Let $X_n$ be a zero-mean $\phi - mixing$ and

$$\sum_{i=1}^{\infty} \phi^{1/2}(i) < \infty, \quad \text{for some } p \geq 2, \quad \sup_i E[|X|^p] < \infty.$$

Then, we deduce (3) from (4).

$$E\sum_{i=1}^{n} a_{ni} X_i^2 \leq c_1 \sum_{i=1}^{n} a_{ni}^2 \leq c_1 \sum_{i=1}^{n} a_{ni}^{p/2}.$$
Proof Since \( \{X_n, n \geq 1\} \) is a Lacunary System, we have
\[
E|S_n|^p \leq K_p \left( \sum_{i=1}^{n} C_i^2 \right)^{p/2}.
\]
By using Markov’s inequality,
\[
P\{ |S_n| \geq nx \} \leq \frac{E|S_n|^p}{(nx)^p}
\]
for every \( p > 1 \), we can obtain (5).

Remark 2 (1) If \( \sum_{i=1}^{n} C_i^2 = O(n) \), we have
\[
E|S_n|^p \leq C(p)n^{-p/2}.
\]
(2) If \( C_i \equiv 1, p > 2 \), by Borel-Cantelli lemma:
\[
\sum_{n=1}^{\infty} P(\{ \sum_{i=1}^{n} X_i \geq nx \}) \leq \sum_{n=1}^{\infty} C(p)n^{-p/2} < \infty,
\]
then
\[
\lim_{n \to \infty} P(\{ \sum_{i=1}^{n} X_i \geq nx \}) = 0. \quad a.s.,
\]

Theorem 2.2 Let \( (X_n, \mathcal{F}_n) \) be a Locally Generalized Gaussian sequence, if \( \sup_n X_n = k < \infty \), then (5) holds for any \( p \geq 2, x \geq 0 \).

Proof Theorem 2.2 holds if only we can prove that Locally Generalized Gaussian sequence is a Lacunary System. Let
\[
A_n = \sum_{i=m}^{n} C_i^2, u = x/k^2A_n, \]
by lemma 1 in [6], then
\[
E(\exp(u \sum_{i=m}^{n} C_i X_i)) = E(\exp(u(S_n - S_{m-1}))) \leq \exp(u^2/k^2A_n/2),
\]
where \( S_n = \sum_{i=1}^{n} C_i X_i \). Since
\[
P(\{ |S_n - S_{m-1}| > x \}) \leq 2\exp(-x^2/2k^2A_n)
\]
for \( p \geq 2 \), by Chebyshev’s inequality, we get
\[
E|\sum_{i=m}^{n} C_i X_i|^p = p \int_{0}^{\infty} x^{p-1} P(\{ |S_n - S_{m-1}| > x \}) dx \leq 2p \int_{0}^{\infty} x^{p-1} \exp(-x^2/2k^2A_n) dx \leq 2^{p/2} k^{p/2} A_n^{p/2} \int_{0}^{\infty} x^{p/2-1} e^{-x} dx = K_p \left( \sum_{i=m}^{n} C_i^2 \right)^{p/2},
\]
where \( K_p = p^{2^{p/2}}k^{p} \int_{0}^{\infty} x^{p/2-1} e^{-x} dx \).

III. THE STRONG LAW OF LARGER NUMBERS

Theorem 3.1 Assume that \( \{X_n, n \geq 1\} \) is a zero-mean \( \psi - mixing \), such that
\[
\sum_{i=1}^{\infty} \psi(i) < \infty, \quad E|X_i|^p, \quad \text{for} \quad p \geq 2.
\]
If there exists \( 1/2 < r \leq 1, \theta = 2r - 1 \) and positive constant \( K \) such that
\[
\sum_{i=1}^{n} a_{ni}^2 \leq Kn^\theta, \quad i = 1, 2, \ldots, n,
\]
then
\[
\sum_{i=1}^{n} a_{ni} X_i \rightarrow 0, \quad a.s.,
\]

Proof Denote \( \sum_{i=1}^{n} a_{ni} X_i \), by Markov’s inequality, we have
\[
P(\{ |S_n| \geq n^r \}) \leq \frac{E(\sum_{i=1}^{n} a_{ni}^2)}{\epsilon n^{pr}}.
\]
From lemma 1.2 and (8), we obtain
\[
\sum_{n=1}^{\infty} P(\{ |S_n| \geq n^r \}) \leq \epsilon \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} a_{ni}^2 \right)^{r/2} \leq \epsilon \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} a_{ni}^2 \right)^{r/2} \leq \epsilon \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} a_{ni} X_i \right)^{r/2} \rightarrow 0, \quad a.s.,
\]

Acknowledgment

This work was supported by the Science Foundation of the Anhui Province (090416222; KJ2010B001), and the Project of PR China for Statistical Research (2008LYO48).

References