A High-Frequency Low-Power Low-Pass-Filter-Based All-Current-Mirror Sinusoidal Quadrature Oscillator

A. Leelasantitham, and B. Srisuchinwong

Abstract—A high-frequency low-power sinusoidal quadrature oscillator is presented through the use of two 2nd-order low-pass-current-mirror (CM)-based filters, a 1st-order CM low-pass filter and a CM bilar transfer function. The technique is relatively simple based on (i) inherent time constants of current mirrors, i.e. the internal capacitances and the transconductance of a diode-connected NMOS, (ii) a simple negative resistance $R_n$ formed by a resistor load $R_i$ of a current mirror. Neither external capacitances nor inductances are required. As a particular example, a 1.9-GHz, 0.45-mW, 2-V CMOS low-pass-filter-based all-current-mirror sinusoidal quadrature oscillator is demonstrated. The oscillation frequency ($f_0$) is 1.9 GHz and is current-tunable over a range of 370 MHz to 21.6 %. The power consumption is at approximately 0.45 mW. The amplitude matching and the quadrature phase matching are better than 0.05 dB and 0.15°, respectively. Total harmonic distortions (THD) are less than 0.3 %.

Keywords—Sinusoidal quadrature oscillator, low-pass-filter-based, current-mirror bilar transfer function, all-current-mirror, negative resistance, low power, high frequency, low distortion.

I. INTRODUCTION

Quadrature oscillators (QOs) typically provide two sinusoids with 90° phase difference for a variety of applications such as in receivers for wireless communication systems (GSM, PCS or Bluetooth etc). For example, GSM 1800-MHz or PCS 1.9-GHz receivers require operating frequencies between 1.805 to 1.99 GHz [1]. QOs are important for receivers and examples of reasons are as follows:

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B. Srisuchinwong is with School of Engineering, University of the Thai Chamber of Commerce, 126/1, Vibhavadee-Rangsit Road, Dindaeng, Bangkok 10400, Thailand, (corresponding author to provide phone: (+66 2) 697-6721; fax: (+66 2) 275-4892; e-mail: adisorn_l ee@utcc.ac.th, leedisorn@yahoo.com).

A. Leelasantitham is with School of Communications, Instrumentation and Control Systems, Sirindhorn International Institute of Technology, Bangkok, Thammasat University, 160 Moo 5, Tiwanont Road, Bangkadi, Muang Pathumthani, 12000, Thailand.


Generally, QOs can be either non-linear or linear types. Non-linear QOs such as relaxation and ring QOs are usually realized using periodically switching mechanisms and therefore outputs may not be readily low-distortion sinusoids [6]. In contrast, linear QOs employ frequency-selective networks such as RC or LC circuits and consequently low-distortion sinusoids can be readily generated [7].

As mentioned earlier, the required operating frequencies between 1.805 to 1.99 GHz in the receivers are typically utilized in the GSM 1800 MHz or PCS 1.9 GHz [1]. In the well open literature, no other linear (sinusoidal) QOs using RC techniques have been reported for tuning ranges of high oscillation frequencies from 1.805 to 1.99 GHz. Existing RC techniques for QOs include all-pass filters [8], operational transconductance amplifiers using capacitors (OTA-C) [9], operational transresistance amplifiers (OTRA) [10], current conveyors [11] and negative resistance [7]. Related attempts to use BJT current mirrors (CMs) have been reported but only for RC non-quadrature oscillators [12], [13].

Such RC techniques have suffered not only from relatively low oscillation frequencies between 40 kHz to 8 MHz due to the use of relatively large off-chip capacitors but also from relatively high power consumptions. However, existing RC linear QOs exploiting techniques using internal capacitances of either BJTs [14] or MOS [15] have been demonstrated for high oscillation frequencies at 0.58 GHz [14] and 2.83 GHz [15], whilst their oscillation frequencies are not tuned in the range from 1.805 to 1.99 GHz.

Alternative LC techniques using CMOS [16], [17], [18] offer high oscillation frequencies between 1.8 to 1.97 GHz whilst their power consumptions are relatively high between 15 to 50 mW. Recently, non-linear QOs have exploited techniques using internal capacitances of either MOS [19], [20] or BJTs [21], [22] for high oscillation frequency between 1.8 to 2.5 GHz. However, the ratios of the oscillation frequency ($f_0$) to the unity-gain frequency ($f_T$) [7] of a transistor are in the region of 0.1 to 0.2 whilst the power
consumptions are relatively high between 7.01 to 100 mW.

In this paper, a high-frequency low-power all-current-mirror sinusoidal quadrature oscillator using two 2nd-order low-pass CM-based filters, a 1st-order CM low-pass filter and a CM bilinear transfer function. The technique is relatively simple based on (i) inherent time constants of current mirrors, i.e. the internal capacitances and the transconductance of a diode-connected NMOS, (ii) a simple negative resistance \( R_n \) formed by a resistor load \( R_l \) of a current mirror. Neither external capacitances nor inductances are required.

As a particular example, a 1.9-GHz, 0.45-mW, 2-V CMOS low-pass-filter-based all-current-mirror sinusoidal quadrature oscillator is demonstrated. The oscillation frequency \( (f_0) \) is 1.9 GHz and is current-tunable over a range of 370 MHz or 21.6 %. The power consumption is at approximately 0.45 mW. The amplitude matching and the quadrature phase matching are better than 0.05 dB and 0.15°, respectively. Total harmonic distortions (THD) are less than 0.3 %. At 2 MHz offset from the 1.9 GHz, the carrier to noise ratio (CNR) is 90.01 dBc/Hz whilst the figure of merit called a normalized carrier-to-noise ratio (CNRnorm) is 153.03 dBc/Hz. The ratio of harmonic distortions (THD) are less than 0.3 %. At 2 MHz, the carrier to noise ratio (CNR) is 90.01 dBc/Hz whilst the figure of merit called a normalized carrier-to-noise ratio (CNRnorm) is 153.03 dBc/Hz.

II. PROPOSED TECHNIQUES

A. Circuit Descriptions

Figs. 1 and 2 show the small-signal block diagrams and the circuit configuration, respectively, of the 1.9-GHz, 0.45-mW, 2-V CMOS low-pass-filter-based all-current-mirror sinusoidal quadrature oscillator. As shown in Fig. 1, the circuit for the low-pass-filter-based technique consists of four simple cascaded current-mirror (CM) filters connected together in a close loop as follows:

(a) a 2nd-order low-pass CM-based filter \( F_1 \) consists of

(a.1) a 1st-order CM low-pass filter (LPF) \( F'_1 \) formed by a current mirror \( (Q_1, Q_2) \),

(a.2) a 1st-order CM low-pass filter (LPF) \( F''_1 \) formed by a current mirror \( (Q_3, Q_4) \),

(b) a 2nd-order low-pass CM-based filter \( F_2 \) consists of

(b.1) a 1st-order CM low-pass filter (LPF) \( F'_2 \) formed by a current mirror \( (Q_5, Q_6) \),

(b.2) a 1st-order CM low-pass filter (LPF) \( F''_2 \) formed by a current mirror \( (Q_7, Q_8) \),

(c) a 1st-order CM low-pass filter (LPF) \( F_3 \) formed by a current mirror \( (Q_9, Q_{10}) \),

(d) a CM bilinear transfer function (BLT) \( F_4 \) described in terms of a negative resistance \( R_n = -R_l \) where \( R_l \) is a resistor load of a current mirror \( (Q_{11}, Q_{12}) \).

In terms of DC analysis, PMOS transistors \( Q_{11} \) to \( Q_{18} \) and a resistor \( R_l \) form sets of DC current mirrors \( (Q_{11} \text{ to } Q_{18}, R_l) \) for the current-steering circuits and therefore provide DC currents \( I, 2I \) or \( 4I \) for \( F_1, F_2 \) or \( F_3 \) where \( G_m \) is an appropriate gain factor. A resistor \( R_l \) provides a DC current \( G_mI \) to the output of \( F_3 \) where \( G_m \) is an appropriate gain factor.

In terms of small-signal (SS) analysis, the four CM filters \( F_1, F_2, F_3 \) and \( F_4 \) can be described in terms of current gains \( F_1(s), F_2(s), F_3(s) \) and \( F_4(s) \), respectively, where the physical frequencies \( s = j\omega \). Firstly, the current gain \( F_1(s) = i_{o1} / i_{in} \) where \( i_{o1} \) and \( i_{o1} \) are input and output SS currents of \( F_1 \) at nodes \( N \) and \( S \), respectively. Secondly, the current gain \( F_2(s) = i_{o2} / i_{o1} \) where \( i_{o2} \) and \( i_{o2} \) are input and output SS currents of \( F_2 \) at nodes \( N' \) and \( S' \), respectively. Finally, as will be seen later in Section IIID, the current gain \( F_3(s) = i_{o3} / i_{o2} \) where \( i_{o3} \) and \( i_{o3} \) are input and output SS currents of \( F_3 \) at nodes \( T \) and \( U \), respectively. The oscillation frequency \( (f_0) \) is the unity-gain frequency \( (f_1) \) of a transistor is 0.25. Comparisons to other approaches are also included.

B. Current-Mirror Filters \( F_1, F_2 \) and \( F_3 \)

With reference to Fig. 2, let the effect of channel-length modulation of a transistor be negligible. A transconductance \( g_{m} \) of a MOS transistor \( Q_i \) for \( i = 1 \) to 10 is equal to \( g_{m} = 2I_0/(V_{GS} - V_{T}) \) where \( V_{GS} \) is the gate source voltage of \( Q_i \), \( V_T \) is the threshold voltage and \( I_0 \) is the bias current of \( Q_i \). Table I summarizes the small-signal analysis of the three CM filters \( F_1, F_2 \) and \( F_3 \) in terms of the small-signal currents \( i_i \), \( i_s \) and \( i_i \), the output currents \( i_o1, i_o2, i_o3 \), the resulting current gains \( F_1(s), F_2(s), F_3(s) \) and the internal time constants \( \tau_1 = C_1 / g_{m1}, \tau_2 = C_2 / g_{m2}, \tau_3 = C_3 / g_{m3} \) and \( \tau_4 = C_4 / g_{m4} \) where \( C_1, C_2, C_3, C_4, C_5 \) and \( C_6 \) are the total internal capacitances at nodes \( N, P, N', P' \) and \( T \), respectively, of individual current
mirrors. The results shown in Table I are provided in the Appendix A1.

<table>
<thead>
<tr>
<th>Filters</th>
<th>Related Currents</th>
<th>Output Currents</th>
<th>Current Gains</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>$i_n = \frac{i_{iO}}{(1+s\tau_1)}$</td>
<td>$i_{iO}$ = $\frac{G_{dm}}{(1+s\tau_2)}$</td>
<td>$F_1(s) = \frac{i_{iO}}{i_{iO} + \frac{G_{dm}}{1+s\tau_2}}$</td>
</tr>
<tr>
<td>F2</td>
<td>$i_n = \frac{i_{iO}}{(1+s\tau_2)}$</td>
<td>$i_{iO}$ = $\frac{G_{dm}}{(1+s\tau_2')}$</td>
<td>$F_2(s) = \frac{i_{iO}}{i_{iO} + \frac{G_{dm}}{1+s\tau_2'}}$</td>
</tr>
<tr>
<td>F3</td>
<td>$i_n = \frac{i_{iO}}{(1+s\tau_1)}$</td>
<td>$i_{iO}$ = $\frac{G_{m1}}{(1+s\tau_1)}$</td>
<td>$F_3(s) = \frac{i_{iO}}{i_{iO} + \frac{G_{m1}}{1+s\tau_1}}$</td>
</tr>
</tbody>
</table>

C. Current-Mirror Negative Resistance $R_N$

By setting $F(s) = F_1(s)$, it follows from Table I that

$$i_{iO} = \frac{1}{F(s)}$$

(1)

The input current $i_n$ to filter $F_1$ is equal to

$$i_n = \frac{\nu_{in}}{Z_{in}}$$

(2)

$$Z_{in} = \frac{1}{g_{m1}} \frac{1}{1+s\tau_1}$$

(3)

where $Z_{in}$ is the input impedance of filter $F_1$ at node $N$ seen by $i_n$ (see Appendix A1 for details), $\nu_{in}$ is the small-signal voltage across $Z_{in}$ at node $N$ with respect to node $O$ and node $O$ is the small-signal ground. On the one hand, let $i_{iO}$ be $i_{iO} + i_n$ where $i_{iO}$ enters node $N$ passing through an impedance $Z_1$ and then leaves node $O$. The impedance $Z_1$ can be found from (1), (2), (3) and $i_{iO}$ as shown in (4).

$$\nu_{iO} = Z_{in} \frac{i_{iO}}{1+s\tau_1} = Z_{i1}$$

(4)

On the other hand, let $i_{iO}$ be a small-signal current that enters node $N$ passing through $R_L$ and then leaves node $O$. As $R_L$ is a positive resistance, it follows that

$$\nu_{iO} = R_L$$

(5)

where $\nu_{iO}$ is the small-signal voltage across $R_L$ at node $N$ with respect to node $O$. As $i_{iO} = -i_{iO}$, therefore, $R_L = -\nu_{iO} / i_{iO}$. Both (4) and (5) follow the passive sign convention and therefore $\nu_{iO}$ is positive when $i_{iO}$ is passing through $Z_i$ and $\nu_{iO}$ is negative when $i_{iO}$ is passing through $R_L$. The Kirchhoff’s current law at node $N$ yields $i_{iO} - i_{iO} = 0$ and therefore $i_{iO}$ in (4) can be substituted with $i_{iO}$. The Kirchhoff’s voltage law around the loop that consists of $Z_i$ and $R_L$ between nodes $N$ and $O$ yields $-\nu_{iO} = -\nu_{iO}$ and therefore $\nu_{iO}$ of (4) can be substituted with $-\nu_{iO}$. It follows from (4) that $\nu_{iO} = \frac{\nu_{iO}}{1} = -\nu_{iO} / i_{iO}$. As a result, (4) = (5) and therefore $R_L = Z_{in} / [1 + F(s)]$. Consequently, $i_{iO} / i_{iO} = 1 / [1 + F(s)]$.

$$F_3(s) = \frac{-R_n}{(R_n + Z_{in})}$$

(6)

Equation (7) describes a negative resistance $R_n = -\frac{\nu_{iO}}{i_{iO}} = -\frac{\nu_{iO}}{i_{iO}}$ between nodes $N$ and $O$, as shown in Fig. 1. It can be seen from (7) that $R_n$ is a simple negative resistance based on an existing resistor $R_L$ of the current mirror ($Q_{11}, Q_{12}, R_L$).

D. Current-Mirror Bilinear Transfer Function $F_4$ using Negative Resistance $R_n$

Equation (6) describes the current gain $F_3(s) = i_{in} / i_{iO}$ of filter $F_3$ in terms of the negative resistance $R_n$. Substituting $Z_{in}$ in (6) with (3) yields

$$F_3(s) = \frac{A_n(1+s\tau_3)}{1+s\tau_3}$$

(8)

$$F_3(s) = \frac{A_n(1+s\tau_3)}{1+s\tau_3}$$

(9)

where $A_n = \frac{g_{m1}R_{L}}{1+g_{m1}R_n}$. It can be seen from (8), (9) and (10) that $F_3(s)$ is a CM bilinear transfer function.

E. Proposed Low-Pass-Filter-Based All-Current-Mirror Sinusoidal Quadrature Oscillation

It follows from Figs. 1 and 2 that a loop gain $L(s) = F_1(s) \cdot F_2(s) \cdot F_3(s) \cdot F_4(s)$ where $F_1(s), F_2(s), F_3(s)$ and $F_4(s)$ are described in Table I and $F_4(s)$ is described in (8) and (9). Therefore $L(s) = \frac{G_0}{A_0} \frac{1}{1+s\tau_3} \frac{1}{1+s\tau_3} \frac{1}{1+s\tau_3'} \frac{1}{1+s\tau_3'} \frac{1}{1+s\tau_3}$. As $A_n$ is equal to $\tau_3$ (i.e. $g_m = g_m$ and $C_b = C_b'$), therefore

$$L(s) = \frac{G_0}{A_0} \frac{1}{(1+s\tau_3) \frac{1}{1+s\tau_3} \frac{1}{1+s\tau_3} \frac{1}{1+s\tau_3}}$$

(11)

For a sinusoidal oscillation to be sustained at the angular oscillation frequency $\omega_0$, the magnitude $|L(s)|$ and the phase angle $\angle L(s)$ of the loop gain $L(s)$ are equal to unity and zero, respectively. Upon substituting $s$ in (11) with $j\omega_0$ and setting $|L(s)| = 1$, therefore the required value of $G_0$ to sustain steady-state sinusoidal oscillations is equal to

$$G_0 = \sqrt{\frac{1}{(1+\omega_0^2\tau_3^2)(1+\omega_0^2\tau_3')(1+\omega_0^2\tau_3')(1+\omega_0^2\tau_3')}}$$

(12)

Upon setting $\angle L(s) = 0^\circ$ or $-360^\circ$, it follows that $\angle F_1(s) + \angle F_2(s) + \angle F_3(s) + \angle F_4(s) + 180^\circ = 0^\circ$ where a symbol ‘$\angle x$’ indicates a phase angle of $x$. Setting $\angle L(s) = \angle F_1(s) + \angle F_2(s)$ and setting $\angle L(s) = \angle F_2(s) + \angle F_3(s)$ yield a quadrature oscillation if

$$\omega_0 = \frac{1}{\tau_b \sqrt{\frac{A_0}{A_n} \frac{\tau_3}{\tau_3}}}$$

(13)

On the other hand, it follows from (13) that $\angle L(s) = \angle F_1(s) + \angle F_2(s) = -90^\circ$ yields the oscillation frequency $\omega_0$

$$\omega_0 = \frac{1}{\tau_b \sqrt{\frac{A_0}{A_n} \frac{\tau_3}{\tau_3}}}$$

(14)

Analytic treatments for the results shown in (14) are provided in the Appendix A2. On the other hand, it follows from (13)
that \( \varnothing_b = \angle F_2(s) + \angle F_3(s) = -90^\circ \) yields the oscillation frequency \( \omega_0 \)

\[
\omega_0 = \frac{1}{\tau_b} \sqrt{\tau_b \left( \frac{1 + \tau_a'}{\tau_a} + \tau_a' \right)}
\]  

(15)

Analytic treatments for the results shown in (15) are provided in the Appendix A3. As a result, (14) = (15) and therefore \( A_0 \) is equal to

\[
A_0 = \frac{\tau_a}{\tau_a} + \left( \frac{\tau_a'}{\tau_a}\right) + \left( \frac{\tau_a'}{\tau_a}\right)
\]  

(16)

As mentioned earlier in Section IIIB, \( \tau_a = C_a/g_{m1} \) and \( \tau_b = C_b/g_{m3} \), Substituting \( \tau_a \) and \( \tau_b \) in (14) with \( \tau_a = C_a/g_{m1} \) and \( \tau_b = C_b/g_{m3} \) yields \( \omega_0 = (g_{m3}/C_a) / [A_0(C_a/g_{m1})(g_{m3}/C_b)]^{1/2} \). As \( \tau_a \) can be equal to \( \tau_b \) (i.e. \( g_{m1} = g_{m3} \) and \( C_a = C_b \)), therefore

\[
\omega_0 = \frac{1}{2I} \sqrt{A_0 C_a / C_b}
\]  

(17)

It can be seen from (17) that \( \omega_0 \) is tunable through the bias current \( I \). Such an oscillator employs a low-pass-filter-based all-current-mirror technique based on (i) inherent time constants of current mirrors as described in (14) or (17), i.e. the internal capacitances and the transconductance of a diode-connected NMOS, (ii) a simple negative resistance \( R_N \) formed by a resistor load \( R_L \) of a current mirror as described in (7).

### III. SIMULATION RESULTS

The performance of the circuit shown in Fig. 2 has been simulated through SPICE. As mentioned earlier, transistors are modeled by Alcatel Mietec 0.5 μm CMOS C05MD Technology (AMC) of EUROPRACTICE. The minimum width \( W \) and length \( L \) of a transistor are 0.8 μm and 0.5 μm, respectively. The unity-gain frequency \( (f_T) \) of an NMOS \( Q_i \) in this particular example is approximately 7.56 GHz. The supply voltage \( V_{dd} = 2 \) V and \( R_1 = 18 \) kΩ. For purposes of simulation, the values of \( G_0 \) and \( R_L \) are practically chosen to be 1.1 and 14 kΩ, respectively.

Fig. 3 depicts the resulting cosine and sine oscillograms of the quadrature currents \( i_{o1} \) and \( i_{o3} \) at \( 1.9 \) GHz and \( I = 20 \) μA where the oscillation frequency \( f_0 = \omega_0/(2\pi) \) is measured to be 1.9 GHz. Fig. 4 illustrates plots of the oscillation frequencies (GHz) and the amplitudes (dB) of \( i_{o1} \) versus bias current \( I \), where the dotted lines indicate the expected analysis and the solid lines indicate the SPICE analysis. As shown in Fig. 4,
the oscillation frequencies are tunable over a range from 1.53 to 1.9 GHz by the bias current $I$ from 13 to 20 $\mu$A, respectively, and therefore the tuning range is approximately 370 MHz or 21.6%.

Fig. 5 depicts the amplitude matching (dB) in terms of the ratio $I_{O3} / I_{O1}$ as well as the quadrature phase matching (degrees) in terms of $(\theta_{O3} - \theta_{O1})$ of the quadrature currents versus frequency. The amplitude matching is as near as 0.029 dB whilst the quadrature phase matching for $-90^\circ$ is better than 0.15°. Fig. 6 shows the power spectrum levels (dBm) of the fundamental frequency at 1.9 GHz and the next harmonics of the oscillogram $I_{O1}$ previously depicted in Fig. 3 using a commercially available fast Fourier transform (FFT) program. As shown in Fig. 6, the distortions are due mainly to the presence of the second harmonics, which is approximately 51.5 dB down from the fundamental frequency, and they remain essentially at the same magnitude over the entire operational bias-current range (13 $\mu$A to 20 $\mu$A). Consequently, the total harmonic distortions (THD) are less than 0.3%.

As shown in Fig. 6, the phase noise is equal to $-90.01$ [dBc/Hz] at 2 MHz offset from the 1.9 GHz carrier. In other words, $\text{CNR} = 90.01$ dBc/Hz at $\Delta f = 2$ MHz and $f_0 = 1.9$ GHz. It can be seen from Fig. 2 that the total current consumption of the oscillator is equal to $8I + 3G_0I$. For $I = 20$ mA and $G_0 = 1.1$, the power dissipation (PDC) is only 0.452 mW. Consequently, the figure of merit $[23]$ called $\text{CNR}_{\text{norm}} = 153.03$ dBc/Hz.

IV. CONCLUSION

The high-frequency low-power all-current-mirror sinusoidal quadrature oscillator has been presented through the use of two 2nd-order low-pass current mirror (CM)-based filters (F1 and F2), a 1st-order CM low-pass filter (F3) and a CM bilinear transfer function (F4). The bilinear transfer function (F4) is described in terms of a negative resistance ($R_N = -R_L$) where $R_L$ is a resistor load of a current mirror. The technique is relatively simple based on (i) inherent time constants of current mirrors, i.e. the internal capacitances and the transconductance of a diode-connected NMOS, (ii) a simple negative resistance $R_N$ formed by a resistor load $R_L$ of a current mirror. Neither external capacitances nor inductances are required.

As a particular example of the second technique, a 1.9-GHz, 0.45-mW, 2-V CMOS low-pass-filter-based all-current-mirror sinusoidal quadrature oscillator has been demonstrated in this Chapter. The oscillation frequency ($f_0$) is 1.9 GHz and is current-tunable over a range of 370 MHz or 21.6%. The power consumption is at approximately 0.45 mW. The amplitude matching and the quadrature phase matching are better than 0.05 dB and 0.15°, respectively. Total harmonic distortions (THD) are less than 0.3%. At 2 MHz offset from the 1.9 GHz, the carrier to noise ratio (CNR) is 90.01 dBc/Hz whilst the figure of merit called a normalized carrier-to-noise ratio (CNR$_{\text{norm}}$) is 153.03 dBc/Hz. The ratio of the oscillation frequency ($f_0$) to the unity-gain frequency ($f_T$) of a transistor is 0.25. Comparisons to other approaches have also been included.
APPENDIX A3

Analytical treatments for the results shown in equation (15)

\[ \alpha_3 = -90^\circ = \angle F_2(s) + \angle F_1(s) \]

\[ \theta = -90^\circ = \frac{G_s}{(1+s)(G_1+G_2 \sigma^2)} \quad \text{and} \quad s = \sigma_0 \]

\[ -90^\circ = -\tan^{-1}(\alpha_0 \sigma_0 \beta_0) - \tan^{-1} \left( \frac{\sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0 \sigma_0} \right) + \tan^{-1} \left( \frac{\sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0 \sigma_0} \right) \]

\[ 90^\circ = \tan^{-1}(\alpha_0 \sigma_0 \beta_0) + \tan^{-1} \left( \frac{\sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0 \sigma_0} \right) + \tan^{-1} \left( \frac{\sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0 \sigma_0} \right) \quad (A3.1) \]

\[ \tan^{-1}(\alpha_0 \sigma_0 \beta_0) = C \]

\[ \tan^{-1}(\alpha_0 \sigma_0 \beta_0) = D \]

\[ \tan^{-1}(\alpha_0 \sigma_0 \beta_0) = E \]

Take \( \tan \theta \) in equation (A3.1):

\[ \tan 90^\circ = \tan \left[ (C + D) + E \right] \]

\[ \tan 90^\circ = \frac{\tan C + \tan D + \tan E}{1 - \tan C \tan D \tan E} \]

\[ \theta = \frac{1}{1 - \tan C \tan D \tan E} \quad (A3.2) \]

Substituting C, D and E in equation (A3.3):

\[ 1 = \frac{1}{1 - \tan C \tan D \tan E} \]

\[ \frac{1}{1 - \alpha_0 \sigma_0 \beta_0} \left[ \frac{1}{1 - \alpha_0 \sigma_0 \beta_0} \right] = 0 \]

\[ \alpha_0 \sigma_0 \beta_0 = 1 \]

\[ \alpha_0 \sigma_0 \beta_0 = 1 \]

\[ \theta = 1 \]

Dividing equation (A3.4) with \( \alpha_0 \sigma_0 \beta_0 \) and Rearranging

\[ \alpha_0 \sigma_0 \beta_0 = \frac{1}{1 - \frac{\alpha_0 \sigma_0 \beta_0}{(1 + \frac{\alpha_0 \sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0})}} \]

\[ \alpha_0 \sigma_0 \beta_0 = \frac{1}{1 - \frac{\alpha_0 \sigma_0 \beta_0}{(1 + \frac{\alpha_0 \sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0})}} \]

\[ \alpha_0 \sigma_0 \beta_0 = \frac{1}{1 - \frac{\alpha_0 \sigma_0 \beta_0}{(1 + \frac{\alpha_0 \sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0})}} \]

\[ \alpha_0 \sigma_0 \beta_0 = \frac{1}{1 - \frac{\alpha_0 \sigma_0 \beta_0}{(1 + \frac{\alpha_0 \sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0})}} \]

\[ \alpha_0 \sigma_0 \beta_0 = \frac{1}{1 - \frac{\alpha_0 \sigma_0 \beta_0}{(1 + \frac{\alpha_0 \sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0})}} \]

\[ \alpha_0 \sigma_0 \beta_0 = \frac{1}{1 - \frac{\alpha_0 \sigma_0 \beta_0}{(1 + \frac{\alpha_0 \sigma_0 \beta_0}{\alpha_0 \sigma_0 \beta_0})}} \]


