Abstract—Vibrations of circular cylindrical shells made of layered composite materials are considered. The shells are weakened by circumferential cracks. The influence of circumferential cracks with constant depth on the vibration of the shell is prescribed with the aid of a matrix of local flexibility coupled with the coefficient of the stress intensity known in the linear elastic fracture mechanics. Numerical results are presented for the case of the shell with one circular crack.

Keywords—Layered shell, axisymmetric vibration, crack.

I. INTRODUCTION

Until the fracture takes place the large class of composites behave as elastic bodies, following the Hooke’s law. Usually the fracture of bodies from such material is brittle, for example, destruction of bodies from fiber-glass. Therefore for investigations of small deformations in bodies from such materials, methods of the classical theory of elasticity of an anisotropic body can be used.

Circular cylindrical shells, made of composite materials, are widely used in many fields of engineering, especially in civil, mechanical, aerospace, marine and chemical industry. Vibration of circular cylindrical shells from composite materials is of interest in a number of different fields.

Since the crack-like defects are practically unavoidable during the manufacturing and operation of structural elements there exists the need for the information about the sensitivity of vibrational parameters of the shell with respect to defects. Vibration and stability of notched beams was investigated by Dimarogonas [1], Chondros and Dimarogonas [2], [3], Rizos et al. [4], Liang et al. [5], Kisa et al. [6], Lellep and Sakkov [7], Krawczuk, Ostachowich [8], [9] making use of the weightless rotational spring model. In [12] Lellep and Roots investigated axisymmetric vibrations of cylindrical shells with circumferential cracks.

According to this concept a beam with a crack can be treated as a structure consisting of two segments. These segments are connected each other with a rotational spring which stiffness is coupled with the stress intensity coefficient of the structure with the crack.

This idea was extended to composite structures and to buckling of composite columns by Nikpour and Dimarogonas [10], [11]. In this paper we will study free axisymmetric vibrations of layered cylindrical shells with cracks.

II. FORMULATION OF THE PROBLEM FOR LAYERED SHELLS

Consider a layered, circular cylindrical shell with length l (see Fig. 1). The shell can be divided into n ring segments. The symbol n denotes the number of total ring segments separated from the rest cylindrical shell by the sections where the thickness variations take place. Every jth ring segment of shell has q layers. Each layer is isotropic with thickness $h_{ij}$, Young’s modulus $E_{ij}$, Poisson’s ratio $\nu_{ij}$, and mass density $\rho_{ij}$ as show in Fig. 1.

Let’s denote

$$\rho_{ij} = \rho_1 d_{ij}$$

where $d_{ij}$ is a constant of proportionality, $d_{ij}$=1 and similarly Young’s modulus for each layer

$$E_{ij}=E_1 e_{ij}$$

where $e_{ij}$ is a constant of proportionality, $e_1=1$.

We will denote the thickness of each layer $h_y$ by

$$h_y=(z_{i+1j}-z_{ij})h_{ij}$$

where $z_{ij}$ is a local coordinate of a layer with the thickness $h_y$ and $z_{ij}$=0.

The mass of jth ring segment will be equal

$$\rho_{ij} h_j \sum_{i=1}^{q} d_y (z_{i+1j} - z_{ij})$$

For the jth ring segment, the free axisymmetric vibration motion can be described by the equations [12].
\[
\frac{\partial N_{ij}}{\partial x} = 0 \\
\frac{\partial^2 M_j}{\partial x^2} - \frac{N_j}{R} - \rho_j h_j \frac{\partial^2 w}{\partial t^2} = 0
\]

(5)

where for calculation of thin shells often use following formula [13]:

\[
N_{ij} = \int_0^{h_i} \sigma_{ij} dz, \quad N_j = \int_0^{h_j} \sigma dz, \quad M_j = \int_0^{h_j} \sigma j dz,
\]

(6)

where

\[
\sigma_{ij} = \left[ E_j / (1 - \nu_j^2) \right] (\varepsilon_i + z \chi) + \nu_j \varepsilon_j,
\]

\[
\sigma_j = \left[ E_j / (1 - \nu_j^2) \right] \varepsilon_j + \nu_j \varepsilon_j + z \chi.
\]

If for all ring segments \( \nu_j = \nu \), by using (1) - (3) and [13]

\[
e_j = \frac{\partial u}{\partial x}, \quad \varepsilon = \frac{w}{R}, \quad \chi = \frac{\partial^2 w}{\partial x^2}.
\]

The force \( N_j \) and bending moment \( M_j \) (6) can be written as

\[
N_j = \int_0^{h_i} \sigma_{ij} dz + \int_0^{h_j} \sigma_{j} dz + \ldots \int_0^{h_j} \sigma_{j} dz + \ldots
\]

\[
+ \int_0^{h_j} \sigma_{j} dz = b_j E_j h_j \frac{W_j}{R},
\]

\[
M_j = \frac{- E_j h_j^3}{12 (1 - \nu^2)} \frac{\partial^2 w}{\partial x^2},
\]

where

\[
\bar{a}_j = 4 \left( \sum_{i=1}^{q} e_j (z_{i+1,j}^2 - z_{i,j}^2) \right) - 3 \left( \sum_{i=1}^{q} e_j (z_{i+1,j}^2 - z_{i,j}^2) \right)^2 \left( \sum_{i=1}^{q} e_j (z_{i+1,j} - z_{i,j}) \right),
\]

(8)

and

\[
b_j = \sum_{i=1}^{q} e_j (z_{i+1,j} - z_{i,j}).
\]

(9)

By using (4) the equation of motion (5) of \( j \)th ring segment can be described by the equation

\[
\bar{D}_j \frac{\partial^2 w}{\partial t^2} + \frac{E_j h_j}{R^2} b_j w = - \rho_j h_j \varepsilon_j w,
\]

(10)

where \( \bar{D}_j = E_j h_j^3 / 12 (1 - \nu^2) \), \( \nu = \text{const} \) and

\[
c_j = \sum_{i=1}^{q} d_j (z_{i,j} - z_{i,j}).
\]

(11)

Evidently, it is reasonable to look for the general solution of (10) in the form

\[
w(x, t) = X_j(x) T(t)
\]

(12)

It follows from (10) with (12) that

\[
X_j^{IV} - r_j^4 X_j = 0
\]

(13)

where

\[
r_j^4 = \alpha^2 \frac{12 \rho_j (1 - \nu^2)}{E_j h_j^2} \frac{b_j}{\bar{a}_j} \frac{12 (1 - \nu^2)}{R^2 h_j^2} \frac{c_j}{\bar{a}_j},
\]

(14)

Frequency of free vibrations of a layered shell will be equal

\[
\omega = \sqrt{\frac{E_j}{\rho_j} \frac{1}{R} \frac{k^4 R^2}{12 (1 - \nu^2)} \frac{\bar{a}_j}{b_j} + \frac{\bar{a}_j}{c_j}},
\]

(7)

where \( r_j = k/\sqrt{h_j} \).

The general solution of the linear fourth order equation (13) can be presented as

\[
X_j(x) = A \sin (r_j x) + B \cos (r_j x) + \ldots
\]

\[
+ C \sinh (r_j x) + D \cosh (r_j x)
\]

(15)

Assume that the ends of the shell are simply supported. We arrive at the boundary conditions at the points \( x=0 \) and \( x=l \)

\[
w(0) = 0, \quad M(0) = 0,
\]

\[
w(l) = 0, \quad M(l) = 0.
\]

The continuity and jump conditions at \( x=a \) are (see [12])

\[
w(a+0) - w(a-0) = 0,
\]

\[
w'(a+0) - w'(a-0) = \frac{72 \pi}{E h_j^2} f(s_i) M_s(a-0),
\]

\[
M_s(a-0) = M_s'(a-0) = 0,
\]

\[
M_s'(a-0) = M_s'(a+0).
\]
Let us consider now the case when \( n=2 \). By using equation (7) we can rewrite the equation for definition of characteristic number \( k \) (see [12]) as

\[
((1-g_s)\cos kl_4 \cosh kl_4 + (g_2 - g_s) \sinh kl_4 + \\
+ g_s \cos kl_4 \sinh kl_4)((1-g_s)\cosh kl_4 - \\
- (g_2 - g_s) \sinh kl_4 + g_s \cosh kl_4 \sinh kl_4) - \\
- ((1+ g_s) \cosh kl_4 + (g_2 + g_s) \sinh kl_4 + \\
+ g_s \cos kl_4 \sinh kl_4)((1 + g_s) \cosh kl_4 - \\
- ((g_2 + g_s) \sinh kl_4 + g_s \cosh kl_4 \sinh kl_4) - \\
- (g_2 + g_s) \sinh kl_4 + g_s \cosh kl_4 \sinh kl_4) = 0,
\]

where

\[
g_1 = (h_0 \bar{a}_j / h_{i} \bar{a}_j)^2,
\]
\[
g_2 = (h_i / h_o)^{1/2},
\]
\[
g_4 = g_2,
\]
\[
g_3 = 6 \bar{a}(s_i) \sqrt{h_{o} g_{i} k}
\]

and

\[
l_1 = a / \sqrt{h_{o}}, \quad l_4 = (l - a) / \sqrt{h_{i}}.
\]

Here \( \bar{a}_0 \) and \( \bar{a}_1 \) are the values of \( \bar{a}_j \) for \( j=0 \) and \( j=1 \), respectively.

III. NUMERICAL RESULTS

For an illustration of the method offered in that article the simply supported shell has been considered (See Fig. 2).

The shell under consideration has a uniform shell wall with Young’s modulus \( E_m \) for \( x \in (0, a) \) whereas it consists of two layers with Young’s moduli \( E_m \) and \( E_s \) respectively, and thickness \( h_i \) for \( x \in (a, l) \), as shown in Fig. 2. Geometrical parameters for the one-stepped shell are: \( l=0.6; \ h_0=0.006; \ h_1=\gamma h_0; \ \gamma=0.7 \). It is assumed herein that the material of the shell segment with \( h_0 \) is a homogeneous elastic material - aluminum-lithium alloy with \( E_m=76 \) GPa. The shell segment with \( h_1 \) has two layers. The inner layer is made of the same material as the other segment and the material of the top layer is a fiber-glass. In the segment with \( h_1 \), \( \nu \) is the volume fraction of fibres. We will consider four kinds of fiber-glasses with \( E_s=20 \) GPa, 35 GPa, 50 GPa, 152 GPa (\( s= E_s / E_m \)), respectively. In this case the equation for definition of characteristic number \( k \) is

\[
((1-g_s)\cos kl_4 \cosh kl_4 + (g_2 - g_s) \sinh kl_4 + \\
+ g_s \cos kl_4 \sinh kl_4)((1-g_s)\cosh kl_4 - \\
- (g_2 - g_s) \sinh kl_4 + g_s \cosh kl_4 \sinh kl_4) - \\
- ((1+ g_s) \cosh kl_4 + (g_2 + g_s) \sinh kl_4 + \\
+ g_s \cos kl_4 \sinh kl_4)((1 + g_s) \cosh kl_4 - \\
- ((g_2 + g_s) \sinh kl_4 + g_s \cosh kl_4 \sinh kl_4) - \\
- (g_2 + g_s) \sinh kl_4 + g_s \cosh kl_4 \sinh kl_4) = 0,
\]

where

\[
g_1 = (h_i / h_o)(f \bar{v} + (1 - \nu)^2),
\]
\[
g_2 = (h_i / h_o)^{1/2},
\]
\[
g_4 = g_2,
\]
\[
g_3 = 6 \bar{a}(s_i) \sqrt{h_{o} g_{i} k}
\]

and

\[
l_1 = a / \sqrt{h_{o}}, \quad l_4 = (l - a) / \sqrt{h_{i}}.
\]

The results of calculations regarding to the shell with simply supported ends are presented in Figs. 3, 4. The influence of the crack \( c/h \) on the characteristic number \( k \) for the fixed values \( \nu=0.2; \ \beta=0.2; \ \gamma=0.7 \) and different values of \( s \) is depicted in Fig. 3. In Fig. 4 different curves corresponding to different values of \( \nu \) are presented in the case \( s=2; \ \beta=0.2; \ \gamma=0.7 \). Here \( \bar{\beta}=a_1/l \ , \ \bar{\gamma}=h_1/h_0 \), as in previous sections of the study.

Calculations carried out showed that the characteristic number \( k \) of the shell decreases when the crack depth increases as might be expected.

Calculations were made by means of the package Mathcad.
Fig. 3 Frequency parameters $k$ for simply supported shells with one-step thickness variation and crack, the case $\nu=0.2; \beta=0.2; \gamma=0.7$

Fig. 4 Frequency parameters $k$ for simply supported shells with one-step thickness variation and crack, the case $f=2; \beta=0.2; \gamma=0.7$

IV. CONCLUDING REMARKS

The natural frequency of vibrations is determined for various non-homogeneous materials.

Calculations carried out showed that the crack location and its dimensions have strong influence on the natural frequency of vibrations. Results of calculations showed that when the crack depth increases then the frequency of natural vibrations decreases. This theoretical results needs in the good computer engineering.

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