\((R, S)\)-Modules and \((1, k)\)-Jointly Prime \((R, S)\)-Submodules

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Abstract—We introduced the notions of \((1, k)\)-prime ideal and \((1, k)\)-jointly prime \((R, S)\)-submodule as a generalization of prime ideal and jointly prime \((R, S)\)-submodule, respectively. We provide a relationship between \((1, k)\)-prime ideal and \((1, k)\)-jointly prime \((R, S)\)-submodule. Characterizations of \((1, k)\)-jointly prime \((R, S)\)-submodules are also given.

Keywords—\((R, S)\)-module, \((1, k)\)-prime ideal, \((1, k)\)-jointly prime \((R, S)\)-submodule.

I. INTRODUCTION

THROUGHOUT this paper, let \(R\) and \(S\) be rings and \(M\) an abelian group.

Definition 1.1: [1] Let \(R\) and \(S\) be rings and \(M\) an abelian group under addition. We say that \(M\) is an \((R, S)\)-module if there is a function \(\cdot : R \times M \times S \to M\) satisfying the following properties: for all \(r, r_1, r_2 \in R\), \(m, n \in M\) and \(s_1, s_2 \in S\),

(i) \(r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s\)
(ii) \((r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s\)
(iii) \(r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2\)
(iv) \(r_1 \cdot (r_2 \cdot m \cdot s_1) = (r_1 r_2) \cdot m \cdot (s_1 s_2)\).

We usually abbreviate \(r \cdot m \cdot s\) by \(r m s\). We may also say that \(M\) is an \((R, S)\)-module under + and \(\cdot\).

An \((R, S)\)-submodule of an \((R, S)\)-module \(M\) is a subgroup \(N\) of \(M\) such that \(r m s \in N\) for all \(r \in R\), \(m \in M\) and \(s \in S\).

Definition 1.2: [1] Let \(M\) be an \((R, S)\)-module. A proper \((R, S)\)-submodule \(P\) of \(M\) is called jointly prime if for each left ideal \(I\) of \(R\), right ideal \(J\) of \(S\) and \((R, S)\)-submodule \(N\) of \(M\),

\[INJ \subseteq P \implies IMJ \subseteq P \text{ or } N \subseteq P.\]

The structure of an \((R, S)\)-module was created as a generalization of a module structure. The basic results of an \((R, S)\)-module structure have been given by [1] and [2]. Almost all of those results was studied analogous to a module structure such as the primalities of \((R, S)\)-submodules of \((R, S)\)-modules and left multiplication \((R, S)\)-modules; see [1] and [2].

In this paper, we introduce the notions of \((1, 2)\)-prime ideal, \((1, k)\)-prime ideal, \((1, 2)\)-jointly prime \((R, S)\)-submodule and \((1, k)\)-jointly prime \((R, S)\)-submodule obtained equivalent conditions for an \((R, S)\)-submodule to be \((1, k)\)-jointly prime \((R, S)\)-submodule.

II. \((1, 2)\)-JOINTLY PRIME \((R, S)\)-SUBMODULES

In this research, we modify the structure of a jointly prime \((R, S)\)-submodules for more general. Now, we start this section by giving the definition of \((1, 2)\)-jointly prime \((R, S)\)-submodules.

Definition 2.1: A proper \((R, S)\)-submodule \(P\) of \(M\) is called \((1, 2)\)-jointly prime if for each left ideal \(I\) of \(R\), right ideal \(J\) of \(S\) and \((R, S)\)-submodule \(N\) of \(M\),

\[INJ^2 \subseteq P \implies IMJ^2 \subseteq P \text{ or } N \subseteq P.\]

By the dual of \((1, 2)\)-jointly prime, we define \((2, 1)\)-jointly prime as follow.

A proper \((R, S)\)-submodule \(P\) of \(M\) is called \((2, 1)\)-jointly prime if for each left ideal \(I\) of \(R\), right ideal \(J\) of \(S\) and \((R, S)\)-submodule \(N\) of \(M\),

\[I^2N^2 \subseteq P \implies I^2MJ \subseteq P \text{ or } N \subseteq P.\]

It is clear that a jointly prime \((R, S)\)-submodule is \((1, 2)\)-jointly prime and \((2, 1)\)-jointly prime. Next, we give a characterization of \((1, 2)\)-jointly prime and \((2, 1)\)-jointly prime \((rZ, sZ)\)-submodule of \(Z\) where \(r, s \in Z^+\).

Proposition 2.2: Let \(r, s \in Z^+\) and \(p \in Z^{+}_{\infty} \setminus \{1\}\). Then

(i) \(pZ\) is an \((1, 2)\)-jointly prime \((rZ, sZ)\)-module of \(Z\) if and only if \(p = 0\), \(p\) is a prime integer or \(p | r^2s^2\).
(ii) \(pZ\) is a \((2, 1)\)-jointly prime \((rZ, sZ)\)-module of \(Z\) if and only if \(p = 0\), \(p\) is a prime integer or \(p | r^2s^2\).

Proof: (i) \((\Rightarrow)\) Assume that \(pZ\) is a \((1, 2)\)-jointly prime \((rZ, sZ)\)-module of \(Z\). Suppose that \(p \neq 0\) and \(p\) is not a prime integer. Then \(p = mn\) for some integer \(m, n > 1\).

It implies that \((rnZ)(mnZ)(s^2Z) = (rmns^2Z) \subseteq pZ\). Since \(pZ\) is \((1, 2)\)-jointly prime and \(p \nmid n\), \((rmZ)(sZ) \subseteq pZ\).

Note that \((rZ)(mZ)(s^2Z) = (rmZ)(sZ) \subseteq pZ\). Since \(pZ\) is \((1, 2)\)-jointly prime and \(p \nmid m\), \((rZ)(Z)(s^2Z) \subseteq pZ\). Hence \(p | r^2s^2\).

\((\Leftarrow)\) If \(p = 0\) or \(p\) is a prime integer or \(p | r^2s^2\), then it is clear that \(pZ\) is \((1, 2)\)-jointly prime.

Now, we already have an example of \((1, 2)\)-jointly prime but not jointly prime.

Example 2.3: It is clear that \(Z\) is a \((2Z, 3Z)\)-module.

Then \(9Z\) is a \((1, 2)\)-jointly prime \((2Z, 3Z)\)-submodule of \(Z\) but \(9Z\) is not a jointly prime \((2Z, 3Z)\)-submodule of \(Z\).

The following is an example showing that \((1, 2)\)-jointly prime and \((2, 1)\)-jointly prime are exactly different.

Example 2.4: Recall that \(Z\) is a \((2Z, 3Z)\)-module. Then \(4Z\) is \((2, 1)\)-jointly prime \((2Z, 3Z)\)-submodule of \(Z\) but \(4Z\) is not a \((1, 2)\)-jointly prime \((2Z, 3Z)\)-submodule of \(Z\) and \(9Z\)
is a $(1,2)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ but $9\mathbb{Z}$ is not a $(2,1)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$.

Moreover, $p\mathbb{Z}$ is both a $(1,2)$-jointly prime and a $(2,1)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ if and only if $p\mathbb{Z}$ is a jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$.

Note that $(1,2)$-jointly prime and $(2,1)$-jointly prime may be different even if $R$ and $S$ are commutative.

We have a question from Example 2.4 that for general, if $P$ is $(1,2)$-jointly prime and $(2,1)$-jointly prime, then can $P$ be a jointly prime $(R, S)$-submodule? The following is an answer.

**Example 2.5:** It easy to see that $\mathbb{Z}$ is a $(2\mathbb{Z}, 4\mathbb{Z})$-module.

Then $16\mathbb{Z}$ is both a $(1,2)$-jointly prime and $(2,1)$-jointly prime $(2\mathbb{Z}, 4\mathbb{Z})$-submodule of $\mathbb{Z}$ but $16\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z}, 4\mathbb{Z})$-submodule of $\mathbb{Z}$.

**Example 2.6:** Let $R$ be a ring of integer and let

$$R = \begin{cases} 0 & x, y \in \mathbb{Z} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{cases},$$

$$S = \begin{cases} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{cases}, x, y \in \mathbb{Z}$$

and $M$ is the set of all $3 \times 3$ matrices on integer. Then $M$ is an $(R, S)$-module. Since $R^2 = 0 = S^2$, all proper $(R, S)$-submodules of $M$ are both $(1,2)$-jointly prime and $(2,1)$-jointly prime $(R, S)$-submodule of $M$. However, $0$ is not a jointly prime $(R, S)$-submodule of $M$.

For each $(R, S)$-submodule $P$ of $M$ and $k \in \mathbb{Z}^+$, let

$$(P : M)_{R,S} = \{ r \in R : rM^S \subseteq P \}. $$

**Proposition 2.7:** Let $P$ be an $(R, S)$-submodule of an $(R, S)$-module $M$ and $k \in \mathbb{Z}^+$.

(i) $(P : M)_{R,S}$ is a subgroup of $R$ under addition.

(ii) $(P : M)_{R,S} \subseteq (P : M)_{R,S+1}$.

(iii) If $S^2 = S$, then $(P : M)_{R,S}$ is an ideal of $R$.

**Proof:** The proof is straightforward.

Next, we introduce a particular nonempty subset of $R$ which play a role in this research.

Let $R$ be a ring and $T$ a proper ideal of $R$. Then $T$ is said to be a $(1,2)$-prime ideal of $R$ if for each ideal $A$ and $B$ of $R$, if $AB^2 \subseteq T$, then $A \subseteq T$ or $B^2 \subseteq T$. A prime ideal of $R$ is a $(1,2)$-ideal of $R$ but the converse is not true. We show by observing the following example.

**Example 2.8:** Let $p$ be an integer. If $p = 0$ or $p$ is a prime integer or $p = q^2$ where $q$ is a prime integer, then $p\mathbb{Z}$ is a $(1,2)$-prime ideal of $\mathbb{Z}$.

It clear that $4\mathbb{Z}$ is a $(1,2)$-prime ideal of $\mathbb{Z}$ but is not a prime ideal of $\mathbb{Z}$.

**Proposition 2.9:** Let $P$ be an $(R, S)$-submodule of an $(R, S)$-module $M$ such that $(P : M)_{R,S^2}$ is a proper ideal of $R$. If $P$ is a $(1,2)$-jointly prime $(R, S)$-submodule of $M$, then $(P : M)_{R,S^2}$ is a $(1,2)$-ideal of $R$.

**Proof:** Assume that $P$ is a $(1,2)$-jointly prime $(R, S)$-submodule of $M$. Let $A$ and $B$ be ideals of $R$ such that $AB^2 \subseteq (P : M)_{R,S^2}$. Hence $(AB^2)MS^2 \subseteq S(AB^2)MS^2 \subseteq P$. Thus $A(B^2MS^2)S^2 \subseteq P$. Since $P$ is a $(2,1)$-jointly prime, $AMS^2 \subseteq P$ or $B^2MS^2 \subseteq P$. Therefore $A \subseteq (P : M)_{R,S^2}$ or $B^2 \subseteq (P : M)_{R,S^2}$. This means that $(P : M)_{R,S^2}$ is a $(1,2)$-prime ideal of $R$.

The converse of Proposition 2.9 is invalid. For example, $6\mathbb{Z}$ is a $(2,2)$-submodule of $\mathbb{Z}$. We see that $(6\mathbb{Z} : (2\mathbb{Z}, 2\mathbb{Z})^2 = 3\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$, of course, $3\mathbb{Z}$ is a $(1,2)$-prime ideal of $\mathbb{Z}$ but $6\mathbb{Z}$ is not a $(1,2)$-jointly prime $(2,2)$-submodule of $\mathbb{Z}$.

**III. $(1, k)$-JOINTLY PRIME $(R, S)$-SUBMODULES**

In this section, we extend the notion of $(1,2)$-jointly prime to $(1, k)$-jointly prime where $k \in \mathbb{Z}^+$. Similarly, we also extend the notion of $(2,1)$-jointly prime to $(k, 1)$-jointly prime where $k \in \mathbb{Z}^+$.

**Definition 3.1:** Let $k \in \mathbb{Z}^+$ and $M$ be an $(R, S)$-module. A proper $(R, S)$-submodule $P$ of $M$ is called a $(1, k)$-jointly prime if for each left ideal $I$ of $R$, right ideal $J$ of $S$ and $(R, S)$-submodule $N$ of $M$.

$$INJ^k \subseteq P \implies IMJ^k \subseteq P \lor N \subseteq P.$$ 

Dually, a proper $(R, S)$-submodule $P$ of $M$ is called $(k, 1)$-jointly prime if for each left ideal $I$ of $R$, right ideal $J$ of $S$ and $(R, S)$-submodule $N$ of $M$.

$$I^kNJ \subseteq P \implies I^kMJ \subseteq P \lor N \subseteq P.
$$

Note here that jointly prime and $(1,1)$-jointly prime are identical.

**Proposition 3.2:** Let $r, s, k \in \mathbb{Z}^+$ and $p \in \mathbb{Z}^+_0 \setminus \{ 1 \}$. Then

(i) $p\mathbb{Z}$ is a $(k, 1)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-module of $\mathbb{Z}$ if and only if $p = 0, p$ is a prime integer or $p = r^k$.

(ii) $p\mathbb{Z}$ is a $(1, k)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-module of $\mathbb{Z}$ if and only if $p = 0, p$ is a prime integer or $p = r^k$.

**Proof:** (⇒) Assume that $p\mathbb{Z}$ is a $(1, k)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-module of $\mathbb{Z}$. Suppose that $p \neq 0$ and $p$ is not a prime integer. Then $p = mn$ for some integer $m, n > 1$. It implies that $(rm\mathbb{Z})(s\mathbb{Z}) = (rm\mathbb{Z})^k \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is a $(1, k)$-jointly prime and $p \nmid m, (rm\mathbb{Z})^k \subseteq p\mathbb{Z}$. Note that $(r\mathbb{Z})(m\mathbb{Z})^k \subseteq p\mathbb{Z}$ and $p$ is a $(1, k)$-jointly prime and $p \nmid m, (r\mathbb{Z})^k \subseteq p\mathbb{Z}$. Hence $p \nmid r^k$.

(⇐) If $p = 0$ or $p$ is a prime integer or $p = r^k$, then it is clear that $p\mathbb{Z}$ is a $(1, k)$-jointly prime.

**Proposition 3.3:** Let $k \in \mathbb{Z}^+$ and $M$ be an $(R, S)$-module. Then

(i) If $P$ is a $(1, 1)$-jointly prime, then $P$ is $(1, k + 1)$-jointly prime.

(ii) If $P$ is $(k, 1)$-jointly prime, then $P$ is $(k + 1, 1)$-jointly prime.

**Proof:** Assume that $P$ is a $(1, k)$-jointly prime $(R, S)$-submodule of $M$. Let $I$ be a left ideal of $R$, $N$ an $(R, S)$-submodule of $M$ and $J$ be a right ideal of $S$ such that $INJ^{k+1} \subseteq P$. Note that $I(INJ)^{k+1} \subseteq P^2NJ^{k+1} \subseteq P$. Since $P$ is a $(1, k)$-jointly prime, $IMJ^{k+1} \subseteq P$ or $NJ^{k+1} \subseteq P$. Note that $J^{k+1} \subseteq J^k \subseteq J$. If $IMJ^k \subseteq P$, then $IMJ^{k+1} \subseteq P$. Assume that $INJ \subseteq P$. Then $INJ^{k+1} \subseteq P$. Since $P$ is a $(1, k)$-jointly prime, $IMJ^k \subseteq P$ or $N \subseteq P$.

The following example shows that the converse of Proposition 3.3 is false in general.
Example 3.4: Recall that \( Z \) is a \((2Z, 3Z)\)-module. Then 27\( Z \) and 54\( Z \) are \((1, 3)\)-jointly prime \((2Z, 3Z)\)-submodule of \( Z \) but 27\( Z \) and 54\( Z \) are not \((1, 2)\)-jointly prime \((2Z, 3Z)\)-submodule of \( Z \).

We obtain the following diagram from Proposition 3.3. Note that the order pair \((m, n)\) means \( P \) is a \((m, n)\)-jointly prime \((R, S)\)-submodule of \( M \).

\[
\begin{array}{c}
(1,1) \rightarrow (1,2) \rightarrow \ldots \rightarrow (1,n) \rightarrow \ldots \\
\downarrow \\
(2,1) \\
\vdots \\
\downarrow \\
(m,1) \\
\vdots \\
\end{array}
\]

In this point, we present a generalization of a \((1,2)\)-prime ideal of \( R \) which is called \((1, k)\)-prime ideal of \( R \) where \( k \in Z^+ \). Let \( R \) be a ring and \( T \) a proper ideal of \( R \) and \( k \in Z^+ \).

Then \( T \) is said to be a \((1, k)\)-prime ideal of \( R \) if for each ideal \( A \) and \( B \) of \( R \), if \( AB^k \subseteq T \), then \( A \subseteq T \) or \( B^k \subseteq T \).

Proposition 3.5: Let \( k \in Z^+ \) and \( P \) be an \((R, S)\)-submodule of an \((R, S)\)-module \( M \) such that \((P : M)_{R,S} \) is a proper ideal of \( R \). If \( P \) is a \((1, k)\)-jointly prime \((R, S)\)-submodule of \( M \), then \((\bar{P} : M)_{R,S} \) is a \((1, k)\)-prime ideal of \( R \).

Note that \( (X)_l \) and \( (X)_r \) is the left ideal generated by \( X \) and the right ideal generated by \( X \), respectively, for any subset \( X \) of a ring \( R \). \( (Y) \) is the \((R, S)\)-submodule generated by \( Y \) for any \((R, S)\)-module \( Y \) of an \((R, S)\)-module \( M \) Next result needs the following lemma.

Lemma 3.6: Let \( M \) be an \((R, S)\)-module and \( k \in Z^+ \). The following statements hold:

(i) For all left ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \((R, S)\)-submodule \( N \) and \( P \) of \( M \),

\[
INJ^k \subseteq P \implies I^2N(J)^k \subseteq P.
\]

(ii) For all right ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \((R, S)\)-submodule \( N \) and \( P \) of \( M \),

\[
INJ^k \subseteq P \implies (I)_rN(J)^k \subseteq P.
\]

(iii) For all right ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \((R, S)\)-submodule \( P \) of \( M \),

\[
(I)_rM(J^k)^k \subseteq P \implies (I)_r(INJ^k)^k \subseteq P.
\]

(iv) For all right ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \((R, S)\)-submodule \( P \) of \( M \),

\[
INJ^k \subseteq P \implies I^2N(J)^k \subseteq P.
\]

(v) For all right ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \((R, S)\)-submodule \( P \) of \( M \),

\[
I^2M(J)^k \subseteq P \implies I(IMJ^k)(J)^k \subseteq P.
\]

(vi) For all left ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \((R, S)\)-submodule \( N \) and \( P \) of \( M \),

\[
INJ^k \subseteq P \implies (I)_rN(J)^k \subseteq P.
\]

(vii) For all left ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \((R, S)\)-submodule \( P \) of \( M \),

\[
(I)_rM(J^k)^k \subseteq P \implies (I)_r(INJ^k)^k \subseteq P.
\]

Proof: (i) Let \( I \) be a left ideal of \( R \), \( J \) a left ideal of \( S \), \( N \) and \( P \) be \((R, S)\)-submodules of \( M \). Assume that \( INJ^k \subseteq P \). Then

\[
I^2N(J)^k = I^2N(J + JS)^k \\
\subseteq I^2N(J^k + J^kS) \\
\subseteq INJ^k + I(INJ)^kS \\
\subseteq P.
\]

(ii) Let \( I \) be a right ideal of \( R \), \( J \) a left ideal of \( S \), \( N \) and \( P \) be \((R, S)\)-submodules of \( M \). Assume that \( INJ^k \subseteq P \). Then

\[
(I)_rN(J)^k = (I + RI)N(J)^k \\
\subseteq INJ^k + R(INJ)^kJ^k \\
\subseteq INJ^k + R(INJ)^kJ^k \\
\subseteq P.
\]

(iii) Let \( I \) be a right ideal of \( R \), \( J \) a left ideal of \( S \), \( N \) and \( P \) be \((R, S)\)-submodules of \( M \). Assume that \((I)_rM(J^k)^k \subseteq P \). Then

\[
(I)_r(IZM(J^k)^k + R(IZM)^kS)^k \\
\subseteq (I)_r(IZM(J)^k + R(IIZM)^kS)^k \\
\subseteq Z(I)_rM(J)^k + R(IIZM)^kJ^k \\
\subseteq P.
\]

(iv) Let \( I \) be a right ideal of \( R \), \( J \) a left ideal of \( S \), \( N \) and \( P \) be \((R, S)\)-submodules of \( M \). Assume that \( I^2M(J)^k \subseteq P \). Then

\[
I^2N(J)^k = I^2N(J + JS)^k \\
\subseteq INJ^k + (I + IJS)^k \\
\subseteq INJ^k + 1N^kJ^k \\
\subseteq P.
\]

(v) Let \( I \) be a right ideal of \( R \), \( J \) a right ideal of \( S \), \( N \) and \( P \) be \((R, S)\)-submodules of \( M \). Assume that \( I^2M(J)^k \subseteq P \). Then

\[
I(IMJ^k)(J)^k = I(IZM(J)^k + R(IZM)^kS)^k \\
\subseteq Z(I^2M(J)^k) + R(IMJ)^kJ^k \\
\subseteq Z(I^2M(J)^k) + P.
\]
(vi) Let \( I \) be a left ideal of \( R \), \( J \) a right ideal of \( S \), \( N \) and \( P \) be \( (R, S) \)-submodules of \( M \). Assume that \( INJ^k \subseteq P \). Then

\[
(I)_r N(J^2)^k = (I + IR)N^k J^k \\
\subseteq INJ^k J^k + (IR)N^k J^k \\
\subseteq INJ^k + I(RNJ^k) J^k \\
\subseteq INJ^k \\
\subseteq P.
\]

(vii) Let \( I \) be a left ideal of \( R \), \( J \) a right ideal of \( S \), \( N \) and \( P \) be \( (R, S) \)-submodules of \( M \). Assume that \( (I)_r M(J^2)^k \subseteq P \). Then

\[
(I)_r IMJ^k J^k = (I)_r (IMJ^k J^k) + (IR)IMJ^k S J^k \\
\subseteq Z((I)_r IMJ^k J^k) + (I)_r RIMJ^k SJ^k \\
\subseteq Z((I)_r M(J^2)^k + (I)_r M(J^2)^k \\
\subseteq P + P, \\
\subseteq P.
\]

Next, we obtain equivalent conditions for an \( (R, S) \)-submodule to be \( (1, k) \)-jointly prime \( (R, S) \)-submodules.

**Theorem 3.7:** Let \( M \) be an \( (R, S) \)-module and \( P \) a proper \( (R, S) \)-submodule of \( M \) and \( k \in \mathbb{Z}^+ \). The following statements are equivalent:

(i) \( P \) is an \( (1, k) \)-jointly prime \( (R, S) \)-submodule of \( M \).

(ii) For all left ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \( (R, S) \)-submodule \( N \) of \( M \),

\[
INJ^k \subseteq P \implies IMJ^k \subseteq P \text{ or } N \subseteq P.
\]

(iii) For all right ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \( (R, S) \)-submodule \( N \) of \( M \),

\[
INJ^k \subseteq P \implies IMJ^k \subseteq P \text{ or } N \subseteq P.
\]

(iv) For all right ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \( (R, S) \)-submodule \( N \) of \( M \),

\[
INJ^k \subseteq P \implies IMJ^k \subseteq P \text{ or } N \subseteq P.
\]

(v) For all left ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \( m \in M \),

\[
I(m) J^k \subseteq P \implies IMJ^k \subseteq P \text{ or } m \in P.
\]

(vi) For all left ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \( m \in M \),

\[
I(m) J^k \subseteq P \implies IMJ^k \subseteq P \text{ or } m \in P.
\]

(vii) For all right ideal \( I \) of \( R \), left ideal \( J \) of \( S \) and \( m \in M \),

\[
I(m) J^k \subseteq P \implies IMJ^k \subseteq P \text{ or } m \in P.
\]

(viii) For all right ideal \( I \) of \( R \), right ideal \( J \) of \( S \) and \( m \in M \),

\[
I(m) J^k \subseteq P \implies IMJ^k \subseteq P \text{ or } m \in P.
\]

**Proof:** This follows from Lemma 3.6.

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