Two iterative algorithms to compute the bisymmetric solution of the matrix equation

\[ A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \]

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Abstract—In this paper, two matrix iterative methods are presented to solve the matrix equation \( A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \) the minimum residual problem \( \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F = \min_{X_i \in BSR^{n_i \times n_i}} \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F \) and the matrix nearness problem \([X_1, X_2, \ldots, X_l] = \min_{X_i \in S^* \subseteq S} \| [X_1, X_2, \ldots, X_l] - [\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_l] \|_F \), where \( BSR^{n_i \times n_i} \) is the set of bisymmetric matrices, and \( S^* \) is the solution set of above matrix equation or minimum residual problem. These matrix iterative methods have faster convergence rate and higher accuracy than former methods. Paige’s algorithms are used as the frame method for deriving these matrix iterative methods. The numerical example is used to illustrate the efficiency of these new methods.

Keywords—Bisymmetric matrices, Paige’s algorithms, Least square.

I. INTRODUCTION

In this work, we will use the following notations. Let \( R^{m \times n} \) and \( BSR^{m \times n} \) denote the set of \( m \times n \) real matrices and \( n \times n \) real symmetric matrices, respectively. \( S_n(S_n = (e_1, e_2, \ldots, e_n)) \) denotes the \( n \times n \) reverse identity matrix \( (e_i \text{ denotes } i\text{th column of } n \times n \text{identity matrix}) \). The superscript \( T \) represents the transpose of a matrix. In space \( R^{m \times n} \), we define inner product as: \( \langle A, B \rangle = \text{trace}(B^T A) \) for all \( A, B \in R^{m \times n} \) which represents the Frobenius norm \( \| A \|_F = \sqrt{\text{trace}(A^T A)} \). Notation \( A \otimes B \) is Kronecker product. The symbol \( \text{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T \) is a vector formed by the columns of given matrix \( A = (a_1, a_2, \ldots, a_n) \). The bisymmetric matrices play an important role in information theory, linear system theory, linear estimate theory and numerical analysis [3], [13], which can be defined as follows:

Definition 1.1: Let \( S_n \in R^{n \times n} \) be a reverse identity matrix. A matrix \( X \in R^{n \times n} \) is said to be bisymmetric if \( X = X^T = S_nXS_n \).

In this paper, we consider the following three problems.

Problem I. Given \( A_i \in R^{n_i \times n_i}, B_i \in R^{n_i \times q}, i = 1, 2, \ldots, l \) and \( C \in R^{p \times q} \), find matrix group \( [X_1, X_2, \ldots, X_l] \) with \( X_i \in BSR^{n_i \times n_i}, i = 1, 2, \ldots, l \) such that

\[ A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C. \]  

Problem II. Given \( A_i \in R^{n_i \times n_i}, B_i \in R^{n_i \times q}, i = 1, 2, \ldots, l \) and \( C \in R^{p \times q} \), find matrix group \( [X_1, X_2, \ldots, X_l] \) with \( X_i \in BSR^{n_i \times n_i}, i = 1, 2, \ldots, l \) such that

\[ BSR^{n_i \times n_i}, i = 1, 2, \ldots, l \text{ such that} \]

\[ \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F = \min_{X_i \in BSR^{n_i \times n_i}} \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F \]  

Problem III. When problem I or II is consistent. Let \( S_E \) denote its solution group set, of the minimum residual problem for given matrix group \( [X_1, X_2, \ldots, X_l] \) with \( X_i \in R^{n_i \times n_i}, i = 1, 2, \ldots, l \), find \( [X_1, X_2, \ldots, X_l] \in S_E \) with \( X_i \in BSR^{n_i \times n_i}, i = 1, 2, \ldots, l \) such that

\[ [X_1, X_2, \ldots, X_l] = \min_{X_i \in \{X_1, X_2, \ldots, X_l\} \subseteq S_E} \| [X_1, X_2, \ldots, X_l] - [\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_l] \|_F \]  

In many areas of computational mathematics, control and system theory, matrix equations can be encountered. In recent years, there has been an increased interest in solving matrix equations; for example, Dai [2], Huang [4], have studied the linear matrix equation \( AX = C \) with a symmetric and skew-symmetric condition on the solution, Peng [7], [6], Shim [12], Chu [1] have studied the linear matrix equation \( AXB + CYD = E \) with unknown matrices \( X \) and \( Y \) being real or complex. The methods used in these papers included generalized inverse, generalized singular value decomposition (GSVD) and canonical decomposition (CCD) of matrices. Peng [10], [11] has studied the equation \( A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \) with bisymmetric conditions on the solutions. Peng [11] has studied the conjugate gradient method, and show that the solvability of the matrix equation can be judged automatically. By using Paige’s algorithms [5], Peng [9], [8] proposed two matrix iterative methods to get the constrained solutions of \( AXB = C \) and the constrained least squares solutions of \( AXB + CYD = E \), and to solve general coupled matrix equations, respectively. Motivated by the work of Peng [9], [8], we propose two iterative methods to solve the matrix equation \( A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \) with bisymmetric condition on the solution, and matrix nearness problem II. These matrix iterative methods have faster convergence rate and higher accuracy than the iterative methods proposed in above references in some cases. We will use Paige’s algorithms [5], which are based on the bidiagonalization procedure of Golub and Kahan [3] as the framework for deriving these matrix-form iterative methods. The basic idea is that we first transform the problem I into the unconstrained linear problem in vector form which can be solved by Paige’s algorithms by the Kronecker product of matrices, and finally,
we transform the vector-form iterative methods into matrix-form iterative methods.

This paper is organized as follows. In section 2, we shortly recall Paige’s algorithms for solving linear systems and least-squares problem, and so based on Paige’s algorithms, we propose two iterative algorithms to solve problems I, II, III. Finally, in section 3, one numerical example is presented to support the theoretical results of this paper.

II. TWO MATRIX ITERATIVE METHODS

In this section, by extending the idea of Paige’s algorithms, we construct two methods for solving problem I, II. We first shortly recall Paige’s algorithms for solving the minimum norm solution of the following unconstrained linear system:

$$Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$. Paige’s algorithms are based on the Bidiagonalization procedure of Golub and Kahan [3], which are summarized as follows.

Paige’s Algorithm 1

1. $\tau_0 = 1; \xi_0 = -1; \omega_0 = 0; z_0 = 0; w_0 = 0; \beta_1 u_1 = b; \alpha_1 v_1 = \chi T u_1$;
2. For $i = 1, 2, \ldots$
   (a) $\xi_i = -\xi_{i-1}/\omega_i$;
   (b) $z_i = z_{i-1} + \xi_i v_i$;
   (c) $\omega_i = (\tau_{i-1} - \beta_i \omega_{i-1})/\omega_i$;
   (d) $w_i = w_{i-1} + \omega_i v_i$;
   (e) $\beta_{i+1} u_{i+1} = A v_i - \beta_{i+1} u_i$;
   (f) $\tau_i = -\tau_{i-1}/\beta_i$;
   (g) $\alpha_{i+1} v_{i+1} = \chi T u_{i+1} - \beta_{i+1} v_i$;
   (h) $v_i = \beta_{i+1} \xi_i / (\beta_i + \omega_i - \tau_i)$;
   (i) $x_i = z_i - \tau_i \omega_i$;
   (j) Exit if a stopping criterion has been met.

Paige’s Algorithm 2

1. $\theta_1 v_1 = \chi T b; \rho_1 u_1 = A v_1; v_1 = v_1/\rho_1; \xi_1 = \theta_1/\rho_1; x_1 = A x_1$;
2. For $i = 1, 2, \ldots$
   (a) $\theta_{i+1} v_{i+1} = \chi T u_i - \rho_{i+1} v_i$;
   (b) $\rho_{i+1} u_{i+1} = v_{i+1} - \theta_{i+1} u_i$;
   (c) $\omega_i = (v_{i+1} - \theta_i v_i) / \rho_{i+1}$;
   (d) $\xi_i = -(\bar{\xi}_i + 1) / \rho_{i+1}$;
   (e) $x_{i+1} = x_i + \xi_i (1 + 1) v_{i+1}$;
   (f) Exit if a stopping criterion has been met.

The real scalars $\alpha_i$, $\beta_i$, $\rho_i$, and $\theta_i$ are chosen to be nonnegative and such that $\|u_i\|_2 = \|v_i\|_2 = 1$ in Paige’s algorithms, respectively. The stopping criterion may be chosen as $\|v_i\| = \|b - Ax_i\|_2 \leq \epsilon$ or $\|x_i - x_{i-1}\|_2 \leq \epsilon$, where $\epsilon > 0$ is a small tolerance.

Based on Paige’s algorithms 1 and 2, we propose two matrix iterative algorithms to solve problem I and II.

We can show that problem I is equivalent to the linear matrix equation

$$Ax = b$$

(4)

where,

$$A = \begin{pmatrix}
\chi T A_1 & \chi T A_2 & \cdots & \chi T A_l \\
(\chi T A_1) T S_{21} & (\chi T A_2) T S_{22} & \cdots & (\chi T A_l) T S_{2l}
\end{pmatrix},

x = \begin{pmatrix}
vec(X_1) \\
vec(X_2) \\
\vdots \\
vec(X_l)
\end{pmatrix},

b = \begin{pmatrix}
vec(C) \\
vec(C) \\
\vdots \\
vec(C)
\end{pmatrix}.

Therefore, the vector form of $\beta_1 u_1 = b$, $\alpha_1 v_1 = A^T u_1$, $\beta_{i+1} u_{i+1} = A v_i - \alpha_{i+1} u_i$, and $\alpha_{i+1} v_{i+1} = A^T u_{i+1} - \beta_{i+1} v_i$, $i = 1, 2, \ldots$ in Paige’s algorithm I can be written in the matrix form

$$\beta_1 = 2 \|C\|_F, \quad U_{1,1} = C / \beta_1, \quad U_{1,2} = C / \beta_1, \quad U_{1,3} = C^T / \beta_1, \quad U_{1,4} = C^T / \beta_1, \quad\alpha_1 = \{\sum_{k=1}^{i} \|A^T U_{k,1} B^T + S_n A^T U_{k,2} B^T S_{n,i} + B, U_{k,3} A^T + S_n, B, U_{k,4} A^T + S_n, B, U_{i,1} A, S_{n,i}\|_F^2 \}^{1/2};$$

$$\alpha_{i+1} V_{i+1} = A^T U_{i+1} + S_n A^T U_{i+1} B^T S_{n,i} + B, U_{i+1} A^T + S_n, B, U_{i,1} A, S_{n,i} - \beta_{i+1} V_{i+1}, i = 1, 2, \ldots, l.$$
\[ \rho_{1} U_{1,1} = \sum_{i=1}^{l} A_i X_{1,i} B_i, \quad \rho_{1} U_{1,2} = \sum_{i=1}^{l} A_i S_i X_{1,i} S_i B_i, \]
\[ \rho_{1} U_{1,3} = \sum_{i=1}^{l} B_i T X_{1,i} A_i, \quad \rho_{1} U_{1,4} = \sum_{i=1}^{l} T X_{1,i} A_i, \]
\[ \theta_{k+1} = \sum_{i=1}^{l} A_i U_{k+1,1} B_i + S_i A_i U_{k+1,2} B_i S_i + B_i U_{k+3} A_i + S_n B_i U_{k+2} A_i S_n - \rho_k V_{k+1,1} \mathbb{P}^{1/2}, \]
\[ \theta_{k+1} V_{k+1,2} = A_i T U_{k+2} B_i + S_i A_i U_{k+2,2} B_i S_i + B_i U_{k+3} A_i + S_n B_i U_{k+2} A_i S_n - \rho_k V_{k+1,2} \mathbb{P}^{1/2}, \]
\[ \theta_{k+1} V_{k+1,3} = \sum_{i=1}^{l} B_i T X_{1,i} A_i + \theta_{k+1} U_{k+3} A_i, \quad \theta_{k+1} V_{k+1,4} = \sum_{i=1}^{l} B_i T X_{1,i} A_i + \theta_{k+1} U_{k+4} A_i - \rho_k V_{k+1,4} \mathbb{P}^{1/2}, \]

Analogous results can be obtained about the minimum residual problem 1. According to above discussion, we introduce two iterative algorithms to compute the unique minimum Frobenius norm solution \([X_1, X_2, \ldots, X_l]\) of the problem 1 as:

**Page 1 B.S.**

1. \( \beta_0 = 1; \quad \xi_0 = -1; \quad \omega_0 = 0; \quad Z_{0,1} = \ldots = Z_{0,l} = 0; \quad W_{0,1} = \ldots = W_{0,l} = 0; \)
2. For \( k = 1, 2, \ldots, \)
   (a) \( \xi_k = -\xi_{k-1} \beta_k / \alpha_k; \)
   (b) \( Z_{k,i} = Z_{k-1,i} + \xi_k V_{k,i} = Z_{k-1,i} + \xi_k / \alpha_k (A_i T U_{k-1} B_i + S_i A_i U_{k-1,2} B_i S_i + B_i U_{k,3} A_i + S_n B_i U_{k-2} A_i S_n - \beta_{k-1} V_{k-1,1}), i = 1, 2, \ldots, l; \)
   (c) \( \omega_k = (\tau_k - \beta_k \omega_{k-1}) / \alpha_k; \)
   (d) \( W_{k,i} = W_{k-1,i} + \omega_k V_{k,i} = W_{k-1,i} + \omega_k / \alpha_k (A_i T U_{k-1} B_i + S_i A_i U_{k-1,2} B_i S_i + B_i U_{k,3} A_i + S_n B_i U_{k-2} A_i S_n - \beta_{k-1} V_{k-1,1}), i = 1, 2, \ldots, l; \)
   (e) \( \beta_{k+1} = \{ \| \sum_{i=1}^{l} A_i V_{k,i} B_i - \alpha_k U_{k,i} \mathbb{P}^{1/2} + \| \sum_{i=1}^{l} A_i S_i V_{k,i} B_i - \alpha_k U_{k,i} \mathbb{P}^{1/2} + \| \sum_{i=1}^{l} B_i T V_{k,i} A_i^T - \beta_k V_{k-1,1} \mathbb{P}^{1/2} \}^{1/2} + \| \sum_{i=1}^{l} B_i T S_i V_{k,i} S_i A_i^T - \beta_k U_{k-1,1} \mathbb{P}^{1/2} \}^{1/2}; \)
   (f) \( \beta_{k+1} U_{k+1,1} = \rho_k V_{k+1,1} B_i - \theta_{k+1} U_{k,1}; \)
   \( \beta_{k+1} U_{k+1,2} = \sum_{i=1}^{l} A_i S_i V_{k+1,1} B_i - \theta_{k+1} U_{k,2}; \)
   \( \beta_{k+1} U_{k+1,3} = \sum_{i=1}^{l} B_i T V_{k+1,1} A_i^T - \theta_{k+1} U_{k,3}; \)
   \( \beta_{k+1} U_{k+1,4} = \sum_{i=1}^{l} B_i T S_i V_{k+1,1} S_i A_i^T - \theta_{k+1} U_{k,4}; \)

**Page 2 B.S.**

1. \( \theta_{1} = \{ \| \sum_{i=1}^{l} A_i T C B_i^T + S_i A_i T C B_i^T S_i + B_i C T A_i + S_n B_i C T A_i S_n \mathbb{P}^{1/2} \}^{1/2} \}
2. For \( k = 1, 2, \ldots, \)
   (a) \( \xi_k = -\xi_{k-1} \beta_k / \alpha_k; \)
   (b) \( Z_{k,i} = Z_{k-1,i} + \xi_k V_{k,i} = Z_{k-1,i} + \xi_k / \alpha_k (A_i T U_{k-1} B_i + S_i A_i U_{k-1,2} B_i S_i + B_i U_{k,3} A_i + S_n B_i U_{k-2} A_i S_n - \beta_{k-1} V_{k-1,1}), i = 1, 2, \ldots, l; \)
   (c) \( \omega_k = (\tau_k - \beta_k \omega_{k-1}) / \alpha_k; \)
   (d) \( W_{k,i} = W_{k-1,i} + \omega_k V_{k,i} = W_{k-1,i} + \omega_k / \alpha_k (A_i T U_{k-1} B_i + S_i A_i U_{k-1,2} B_i S_i + B_i U_{k,3} A_i + S_n B_i U_{k-2} A_i S_n - \beta_{k-1} V_{k-1,1}), i = 1, 2, \ldots, l; \)

\[ \theta_{k+1} = \sum_{i=1}^{l} A_i T U_{k,i} B_i + S_i A_i T U_{k,2} B_i S_i + B_i U_{k,3} A_i + S_n B_i U_{k-2} A_i S_n - \rho_k V_{k,i} \mathbb{P}^{1/2}, \]
2. For \( k = 1, 2, \ldots, \)
\[ \theta_{k+1} V_{k+1,i} = A_i^T U_{k,i} B_i^T + S_n A_i^T U_{k,2} B_i^T S_n + B_i U_{k,3} A_i + S_n B_i U_{k,4} A_i S_n - \rho_k V_{k,i}, \text{i}=1,2,\ldots, l; \]

(b) \[ \rho_{k+1} = \min \| \sum_{i=1}^l A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1} \|_F^2 + \| \sum_{i=1}^l A_i S_n, V_{k+1,i} S_n B_i - \theta_{k+1} U_{k,2} \|_F^2 + \| \sum_{i=1}^l B_i^T V_{k+1,i} S_n - \theta_{k+1} U_{k,3} \|_F^2 + \| \sum_{i=1}^l B_i^T S_n, V_{k+1,i} S_n, A_i^T - \theta_{k+1} U_{k,4} \|_F^2 \]^{1/2};

\[ \rho_{k+1} U_{k+1,1} = \sum_{i=1}^l A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1}; \]

\[ \rho_{k+1} U_{k+1,2} = \sum_{i=1}^l A_i S_n, V_{k+1,i} S_n, B_i - \theta_{k+1} U_{k,2}; \]

\[ \rho_{k+1} U_{k+1,3} = \sum_{i=1}^l B_i^T V_{k+1,i} S_n - \theta_{k+1} U_{k,3}; \]

\[ \rho_{k+1} U_{k+1,4} = \sum_{i=1}^l B_i^T S_n, V_{k+1,i} S_n, A_i^T - \theta_{k+1} U_{k,4}; \]

(c) \[ W_{k+1,i} = (V_{k+1,i} - \theta_{k+1} W_{k,i})/\rho_{k+1} \text{, i}=1,2,\ldots, l; \]

(d) \[ \xi_{k+1} = -\xi_k \theta_{k+1}/\rho_{k+1}; \]

(e) \[ X_{k+1,i} = X_{k,i} + \xi_{k+1} W_{k+1,i}, \text{ i}=1,2,\ldots, l; \]

(f) Exit if a stopping criterion has been met.

In this section, we compare Paige 1 B.S and Paige 2 B.S numerically with the method proposed in [11], denoted by Peng-M. All the tests were performed by Matlab 7.1. We choose the initial iterative matrix groups in the Peng’s method as zero matrix group in suitable size. All the following examples are used to illustrate the performance of three methods to compute the minimum Frobenius norm bisymmetric solution group \([X_1, X_2, \ldots, X_l]\) of the matrix equation 1 an the minimum residual 2.

### Example 3.1

Suppose that the matrices \(A_1, B_1, A_2, B_2, \) and \(C\) are given

\[
\begin{bmatrix}
1 & 3 & 1 & 3 & 1 \\
3 & -7 & 3 & -7 & 3 \\
3 & -2 & 3 & -2 & 3 \\
11 & 6 & 11 & 6 & 11 \\
-5 & 5 & -5 & 5 & -5 \\
9 & 4 & 9 & 4 & 9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 4 & -1 & 4 & -1 \\
5 & -1 & 5 & -1 & 5 \\
3 & 9 & 3 & 9 & 3 \\
7 & -8 & 7 & -8 & 7 \\
3 & 4 & 3 & 4 & 1 \\
3 & -1 & 3 & -3 & -1 \\
3 & -5 & 3 & -5 & 2 \\
3 & 5 & -3 & 5 & 2 \\
-5 & 4 & -1 & 5 & 4 \\
-2 & 3 & 5 & -2 & 3 \\
3 & 5 & -1 & 3 & 5 \\
2 & 6 & 3 & 2 & 6 \\
1 & 11 & 7 & 1 & 11 \\
4 & -1 & 4 & -5 & 4 \\
\end{bmatrix}
\]

The above given matrices \(A_1, B_1, A_2, B_2, \) and \(C\) are such that the matrix equation \(A_1 X_1 B_1 + A_2 X_2 B_2 = C\) has bisymmetric solution pairs \([X_1, X_2]\). Figure 1 describes the convergence rate of the function \(R(k) = \|C - A_1 X_1 B_1 - A_2 X_2 B_2\|_F\) of the above two methods and conjugate gradient method.

### III. Numerical examples

In this section, we compare Paige 1 B.S and Paige 2 B.S numerically with the method proposed in [11], denoted by Peng-M. All the tests were performed by Matlab 7.1. We choose the initial iterative matrix groups in the Peng’s method as zero matrix group in suitable size. All the following examples are used to illustrate the performance of three methods to compute the minimum Frobenius norm bisymmetric solution group \([X_1, X_2, \ldots, X_l]\) of the matrix equation 1 an the minimum residual 2.
Fig. 1. The results obtained for Example 3.1

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