The Statistical Properties of Filtered Signals

Ephraim Gower, Thato Tsalaile, Monageng Kgwadi and Malcolm Hawksford.

Abstract—In this paper, the statistical properties of filtered or convolved signals are considered by deriving the resulting density functions as well as the exact mean and variance expressions given a prior knowledge about the statistics of the individual signals in the filtering or convolution process. It is shown that the density function after linear convolution is a mixture density, where the number of density components is equal to the number of observations of the shortest signal. For circular convolution, the observed samples are characterized by a single density function, which is a sum of products.

Keywords—Circular Convolution, linear Convolution, mixture density function.

I. INTRODUCTION

A signal is a group of observations, and these are represented in vector form. For example, \( x_i(n) = [x_i(1), x_i(2), \ldots, x_i(K_i)] \) is a vector for the \( i^{th} \) signal, for \( i \in [1, N] \), whose observations are \( x_i(n) \), for \( n \in [1, K_i] \), where \( K_i \) is the length of the \( i^{th} \) signal. Given a vector \( x_i(n) \), the variable \( X_i \) (capital letter with a corresponding subscript) is defined where its possible values are the vector elements or observations, and for \( X_i \) we define the density function \( p(x_i) \) summarizing the statistical properties of its observations.

II. LINEAR CONVOLUTION OF TWO SIGNALS

Let \( z(n) = x_1(n) * x_2(n) \) be the convolution of the two signals \( x_1(n) \) and \( x_2(n) \). The observations of \( z(n) \) are given by:

\[
z(n) = \sum_r x_1(r) x_2(n - r) = x_1(1) x_2(n - 1) + x_1(2) x_2(n - 2) + \cdots + x_1(n - 1) x_2(1)
\]
for \( n \in [1, K_1 + K_2 - 1] \), where \( K_i \) is the total number of observations of the signal \( x_i(n) \), for \( i \in [1, N] \), and \( x_i(r) \) is the \( r^{th} \) observation of \( x_i(n) \), for \( r \in [1, K_i] \). Assuming that \( K_1 \geq K_2 \), an observation of \( z(n) \) is the sum of the product of the observations of the signals \( x_1(n) \) and \( x_2(n) \) as illustrated in (1), where \( K_2 \) is the limiting length of the number of sum terms. Let \( p_j(z) \) be the generating density function of an observation derived from the sum of \( j \) products. For example, \( z(1) \) and \( z(K_1 + K_2 − 1) \) would be drawn from \( p_1(z) \), and \( z(K_2 - 1) \) and \( z(K_1 + 1) \) from \( p_{K_2−1}(z) \). It follows that the variable \( Z \), realised by the observations of the signal \( z(n) \), is a mixture of \( K_2 \) variables where \( p_1(z), p_2(z), \ldots, p_{K_2−1}(z) \) constitute two observations each and \( p_{K_2}(z) \) constitutes \( K_1 - K_2 + 1 \) observations. Therefore the density function of \( Z \) is the mixture

\[
p(z) = \sum_{j=1}^{K_2} w_j p_j(z)
\]

where \( w_j \) is the relative weight of \( p_j(z) \) in the mixture and \( \sum_{j=1}^{K_2} w_j = 1 \). As the number of observations from each density function is known, it is clear that \( w_1 = w_2 = \ldots = w_{K_1−1} = \frac{1}{K_1+K_2−1} \) and \( w_{K_1−1} = \frac{K_1−K_2+1}{K_1+K_2−1} \), thus the density function of the signal \( z(n) \) resulting from the linear convolution of two signals is

\[
p(z) = \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2} p_j(z) + \frac{K_1 - K_2 + 1}{K_1 + K_2 - 1} p_{K_2}(z)
\]

A. The Mean of a Sum of Identical Products

If \( \mu_{X_i} = E[X_i] \) is the mean of \( X_i \), the variable realized by the observations of the signals \( x_i(n) \), for \( i \in [1, N] \), where \( E[\cdot] \) is mathematical expectation, the mean of the product variable \( X_1 X_2 \) is \( \mu_{X_1 X_2} = E[X_1 X_2] \). Thus, the mean of the density function \( p_j(z) \) is

\[
\mu_j = j \cdot \mu_{X_1 X_2}, \quad \text{for } j \in [1, K_2]
\]

since the expectation of a sum is the sum of expectations.

B. The Variance of a Sum of Identical Products

Let \( \sigma^2_{X_i} = E[X_i^2] - \mu^2_{X_i} \) be the variance of \( X_i \), for \( i \in [1, N] \). In [15, 16], the exact variance of a product is derived for independent variables \( X_1 \) and \( X_2 \) as:

\[
\sigma^2_{X_1 X_2} = \mu^2_{X_1} \sigma^2_{X_2} + \mu^2_{X_2} \sigma^2_{X_1} + \sigma^2_{X_1} \sigma^2_{X_2},
\]

and if they are not necessarily independent

\[
\sigma^2_{X_1 X_2} = \mu^2_{X_1} \sigma^2_{X_2} + \mu^2_{X_2} \sigma^2_{X_1} + 2 \mu_{X_1} \mu_{X_2} E_{11} + 2 \mu_{X_1} E_{12} + 2 \mu_{X_2} E_{21} + E_{22} - E_{11}^2
\]

where \( E_{ij} = E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})] \). Taking note that the variances of the products in the sum given by (1) are identical (because all observations of \( X_1 \) and \( X_2 \) are drawn independently from \( p(x_1) \) as well as those of \( X_2 \) from \( p(x_2) \), the variance of \( p_j(z) \) is given by:

\[
\sigma^2_j = j \cdot \sigma^2_{X_1 X_2}, \quad \text{for } j \in [1, K_2]
\]

because the variance of a sum is the sum of variances for independent and identical products [17]. If the observations are not drawn independently from their respective distributions, then it is shown in [17] that the net variance is the sum of the permutations of the covariance matrices of all the product variables in the sum. Since in this case the sum terms are identically distributed,

\[
\sigma^2_j = j^2 \cdot \sigma^2_{X_1 X_2}, \quad \text{for } j \in [1, K_2]
\]

C. The Mean of a Linear Convolution of Two Signals

The mean \( \mu_Z \) of the variable \( Z \), representing the signal \( z(n) = x_1(n) X_2(n) \), is given by

\[
\mu_Z = E[Z] = \int_{−\infty}^{\infty} z \cdot p(z) dz
\]

Using (2),

\[
\mu_Z = \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2−1} \int_{\mu_j}^{\mu_j + K_1 - K_2 + 1} z \cdot p_j(z) dz + \int_{\mu_j + K_1 - K_2 + 1}^{\infty} z \cdot p_{K_2}(z) dz
\]

\[
= \frac{2}{K_1 + K_2 − 1} \sum_{j=1}^{K_2−1} \mu_j + \frac{K_1 - K_2 + 1}{K_1 + K_2 - 1} \mu_{X_1 X_2}
\]

By (3),

\[
\mu_Z = \frac{2 \mu_{X_1 X_2}}{K_1 + K_2 - 1} \sum_{j=1}^{K_2−1} j + \frac{K_2(K_1 - K_2 + 1)}{K_1 + K_2 - 1} \mu_{X_1 X_2}
\]

The sum \( \sum_{j=1}^{K_2−1} j = \frac{K_2(K_2−1)}{2} \), therefore the mean of the signal \( z(n) \) can be expressed as

\[
\mu_Z = \frac{K_1 K_2}{K_1 + K_2 - 1} \mu_{X_1 X_2}
\]

D. The Variance of a Linear Convolution of Two Signals

The variance \( \sigma^2_Z \) of the variable \( Z \), realized by the observations of the signal \( z(n) = x_1(n) * x_2(n) \), is given by

\[
\sigma^2_Z = E[Z^2] - \mu^2_Z = \int_{−\infty}^{\infty} z^2 \cdot p(z) dz - \mu^2_Z
\]

The integral \( \int_{−\infty}^{\infty} z^2 \cdot p(z) dz = \sum_{j=1}^{K_2−1} w_j \int_{−\infty}^{\infty} z^2 p_j(z) dz \), and since

\[
\sigma^2_j = \int_{−\infty}^{\infty} (z - \mu_j)^2 p_j(z) dz = \int_{−\infty}^{\infty} z^2 p_j(z) dz - \mu^2_j,
\]

it implies that \( \int_{−\infty}^{\infty} z^2 p_j(z) dz = \sigma^2_j + \mu^2_j \), thus

\[
\sigma^2_Z = \sum_{j=1}^{K_2} w_j (\sigma^2_j + \mu^2_j) - \mu^2_Z.
\]
For independent observations of $X_i$ from $p(x_i)$ (for the signal $x_i(n)$), using (2), (3) and (4)

$$\sigma_Z^2 = \frac{2}{K_1 + K_2 - 1} \left( \sum_{j=1}^{K_2-1} j + \mu_{X_1X_2} \sum_{j=1}^{K_2-1} j^2 \right) + K_1 - K_2 + 1 \frac{K_2^2 \mu_{X_1X_2}}{K_1 + K_2 - 1} - \mu_Z^2$$

Noting that $\sum_{j=1}^{K_2} j = \frac{K_2(K_2+1)}{2}$ and the sum of squares $\sum_{j=1}^{K_2-1} j^2 = \frac{K_2(K_2+1)(2K_2-1)}{6}$, the variance of the signal $z(n)$ can be expressed as

$$\sigma_Z^2 = \frac{K_2}{K_1 + K_2 - 1} - \mu_Z^2$$

where $\mu = \left( K_1 \sigma_{X_1X_2} + 3K_1K_2 - K_2^2 + 1 \right) / \mu_{X_1X_2}$

If the observations of are not drawn independently, then by (2), (3) and (5)

$$\sigma_Z^2 = (\sigma_{xy}^2 + \mu_{xy}^2) - \mu_Z^2$$

where $\mu = \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2-1} j^2 + \frac{K_2^2(K_1 - K_2 + 1)}{K_1 + K_2 - 1}$

After simplification,

$$\sigma_Z^2 = \frac{K_2(3K_1K_2 - K_2^2 + 1)}{3(K_1 + K_2 - 1)} (\sigma_{xy}^2 + \mu_{xy}^2) - \mu_Z^2$$

### III. LINEAR CONVOLUTION OF TWO OR MORE SIGNALS

Let $z^{(N)}(n) = x_1(n) * x_2(n) * \cdots * x_N(n)$ where the superscript $(N)$ in $z^{(N)}(n)$ is used to indicate the convolution of $N$ signals. Based on this notation, it follows that $z^{(N)}(n) = z^{(N-1)}(n) * x_N(n)$ which means that the observations of $z^{(N)}(n)$ are given by

$$z^{(N)}(n) = \sum_{r} z^{(N-1)}(r)x_N(n-r), \text{ for } n \in [1, K^{(N)}]$$

with $K^{(N)}$ as the length of the resulting convolution signal $z^{(N)}(n)$, for $v \in [2, N]$. It can be shown that

$$K^{(v)} = \sum_{i=1}^{v} K_i - v + 1,$$

where $K_i$ is the length of the signal $x_i(n)$, for $i \in [i, N]$, and it is assumed that $K_1 \geq K_2 \geq \cdots \geq K_N$. In order to use equations (3), (4) and (5) it is necessary to know the exact number of product terms in the sum (9) for each and every observation of $z^{(N)}(n)$. If we let $\text{num}[z^{(v)}(n)]$ be the total number of product terms for the observation $z^{(v)}(n)$, then in general

$$q^{(N)}_n = \sum_{i=1}^{n} \text{num}[z^{(N-1)}(i)], \text{ for } n \in [1, K_N - 1]$$

$$q^{(N)}_n = \sum_{i=1}^{K_N} \text{num}[z^{(N-1)}(i)], \text{ for } n \in [K_N, K^{(N-1)}]$$

$$q^{(N)}_n = \sum_{i=1}^{n} \text{num}[z^{(N-1)}(K^{(N-1)} + K_N - i)], \text{ for } n \in [K^{(N-1)} + 1, K^{(N)}]$$

where $q^{(N)}_n = \text{num}[z^{(N)}(n)]$ given $N \geq 3$, because it has already been shown that for $N = 2$, $q^{(2)}_n = \text{num}[z^{(2)}(n)]$

$$q^{(2)}_n = \begin{cases} n, & \text{for } n \in [1, K_2 - 1] \\ K_2, & \text{for } n \in [K_2, K_1] \\ K_1 + K_2 - n, & \text{for } n \in [K_2 + 1, K_1] \end{cases}$$

It follows that,

$$q^{(N)}_n = \begin{cases} \sum_{i=1}^{n} q^{(N-1)}_i, & \text{for } n \in [1, K_N - 1], \\ \sum_{i=1}^{K_N} q^{(N-1)}_i, & \text{for } n \in [K_N, K^{(N-1)}], \\ \sum_{i=1}^{n} q^{(N-1)}_i + K^{(N-1)}, & \text{for } n \in [K^{(N-1)} + 1, K^{(N)}] \end{cases}$$

for $N \geq 3$. From (12), it is observed that the total number of product terms in each observation of $z^{(N)}(n)$ on expanding in equation (9) from $z^{(N-1)}(n) \rightarrow z^{(N-2)}(n) \ast z_{N-1}(n) \rightarrow \cdots \rightarrow x_1(n) \ast x_2(n) \ast \cdots \ast x_{N-1}(n)$ is evaluated in a recursive manner with the help of (11). Based on this result, then mean of $p_j(z^{(N)})$ is

$$\mu_{Z_j^{(N)}} = q^{(N)}_{j_1} \mu_{X_1X_2 \ldots X_n},$$

and for observations of $X_i$ drawn independently from $p(x_i), \forall i \in [1, N]$, the variance is given by

$$\sigma_{Z_j^{(N)}}^2 = q^{(N)}_{j_1} \sigma_{X_1X_2 \ldots X_n},$$

otherwise

$$\sigma_{Z_j^{(N)}}^2 = q^{(N)}_{j_1} \sigma_{X_1X_2 \ldots X_n}$$

for $j \in [1, K_N]$. The evaluation of the exact variance of the product $\prod_{i=1}^{N} X_i$, given $N \geq 2$, for independent and not necessarily independent variables is fully covered in [6]. The density function of the signal $z^{(N)}(n)$ is the mixture model

$$p(z^{(N)})(n) = \sum_{j=1}^{K_N} w_j p_j(z^{(N)})$$

Using (12), it follows that $w_1 = w_2 = \ldots = w_{K_N - 1} = K_N^{(N-1)}$, since the first expression is a reversal of the last expression. That is, the density functions $p_1(z^{(N)}), p_2(z^{(N)}), \ldots, p_{K_N - 1}(z^{(N)})$ constitute two observations each towards the final mixture density given by (16). The last relative weight is given by $w_{K_N} = \frac{K^{(N-1)} + K_N - 1}{K^{(N-1)}}$. 

The mean of $z^{(N)}(n)$ is
\[ \mu_{Z^{(N)}} = \int_{-\infty}^{\infty} z^{(N)} p(z^{(N)}) dz^{(N)} = \sum_{j=1}^{K_N} w_{jj} \mu_{Z^{(N)}_j} \]

Substituting for the relative weights and using (13),

\[ \mu_{Z^{(N)}} = \frac{\mu_{X_1X_2...X_N}}{K_N} \zeta = \frac{\mu_{X_1X_2...X_N}}{K_N} g \]  
(17)

where \[ \zeta = 2 \sum_{j=1}^{K_N-1} q_j^{(N)} + (K_N^{(N)} - K_N + 1) q_{K_N}^{(N)} \]

and \[ g = 2q_{(N+1)}^{(K_N-1)} + (K_N^{(N)} - K_N + 1) q_{K_N}^{(N)} \]

since \( \sum_{j=1}^{K_N-1} q_j^{(N)} = q_{K_N}^{(N)} \) based on the recursive definitions of (12).

Using

\[ \sigma_{Z^{(N)}} = \int_{-\infty}^{\infty} (z^{(N)})^2 p(z^{(N)}) dz^{(N)} - \mu_{Z^{(N)}}^2 = \sum_{j=1}^{K_N} \psi_j \left( \sigma_{Z^{(N)}_j}^2 + \mu_{Z^{(N)}_j}^2 \right) - \mu_{Z^{(N)}}^2 \]

the variance of \( z^{(N)}(n) \) given the observations of \( x_i(n) \) are drawn independently from \( p(x_i) \), \( \forall i \in N \), is given by

\[ \sigma_{Z^{(N)}}^2 = \frac{1}{K_N} \left( \sigma_{X_1X_2...X_N}^2 \varphi + \mu_{X_1X_2...X_N}^2 \xi \right) - \mu_{Z^{(N)}_j}^2 \]  
(18)

where \( \varphi = 2q_{K_N}^{N+1} + (K_N^{(N)} - K_N + 1) q_{K_N}^{(N)} \)

and \( \xi = 2 \sum_{j=1}^{K_N-1} (q_j^{(N)})^2 + (K_N^{(N)} - K_N + 1)(q_{K_N}^{(N)})^2 \)

otherwise,

\[ \sigma_{Z^{(N)}}^2 = \left( \sum_{j=1}^{K_N-1} (q_j^{(N)})^2 + (K_N^{(N)} - K_N + 1)(q_{K_N}^{(N)})^2 \right) \psi - \mu_{Z^{(N)}_j}^2 \]  
(19)

where \( \psi = 2 \sum_{j=1}^{K_N-1} (q_j^{(N)})^2 + (K_N^{(N)} - K_N + 1)(q_{K_N}^{(N)})^2 \)

and \( \mu_{Z^{(N)}} \) is given by (17).

IV. CIRCULAR CONVOLUTION

With circular convolution, the analysis is somewhat simpler because the number of product terms is constant for all observations of the signal \( z^{(N)}(n) \). Due to zero padding, it follows that \( z^{(N)}(n) \), for \( n \in [1, K_N^{(N)}] \), has \( K_N \) product terms in the sum (where \( K_N \) is the length of the shortest variable). That is, all observations are drawn from the same density function \( p(z^{(N)}) = p_{K_N}(z^{(N)}) \). Therefore, the mean of the signal \( z^{(N)}(n) \) is given by

\[ \mu_{Z^{(N)}} = K_N \cdot \mu_{X_1X_2...X_N} \]  
(20)

If the observations are drawn independently,

\[ \sigma_{Z^{(N)}}^2 = K_N^2 \cdot \sigma_{X_1X_2...X_N}^2 \]  
(21)

otherwise,

\[ \sigma_{Z^{(N)}}^2 = K_N^2 \cdot \sigma_{X_1X_2...X_N}^2 \]  
(22)

V. CONCLUSION

The generalized density functions, and mathematical expressions for the mean and variance of the convolution (linear and circular) of signals have been derived based on the assumption that the statistical properties of the individual signals being processed are a known prior. The variance expressions rely on the statistical dependence between observations of a given signal, whereas the mean expression is not. It has been shown that the linear convolution signal has a mixture density, with the number of density components equal to the length of the shortest signal. The circular convolution signal has a relatively simpler description as observations are characterized by a single density function.

REFERENCES