Soft connected spaces and soft paracompact spaces

Fucai Lin

Abstract—Soft topological spaces are considered as mathematical tools for dealing with uncertainties, and a fuzzy topological space is a special case of the soft topological space. The purpose of this paper is to study soft topological spaces. We introduce some new concepts in soft topological spaces such as soft closed mapping, soft open mappings, soft connected spaces and soft paracompact spaces. We also redefine the concept of soft points such that it is reasonable in soft topological spaces. Moreover, some basic properties of these concepts are explored.

Keywords—soft sets, soft open mappings, soft closed mappings, soft connected spaces, soft paracompact spaces.

I. INTRODUCTION

The real world is too complex for our immediate and direct understanding, for example, many disciplines, including medicine, economics, engineering and sociology, are highly dependent on the task of modeling uncertain data. Since the uncertainty is highly complicated and difficult to characterize, classical mathematical approaches are often insufficient to useful models or derive effective. There are some theories: the theory of rough sets [10], the theory of vague sets [2] and the theory of fuzzy sets [13], which can be regarded as mathematical tools for dealing with uncertainties. However, all these theories have their own difficulties. The main reason for these difficulties is, possibly, the inadequacy of the parametrization of the tool of the theory as it was mentioned by Molodtsov in [6]. In [6], Molodtsov introduced the concept of a soft set in order to solve complicated problems, and then Molodtsov presented the fundamental results of the new theory and successfully applied it to several directions such as operations research, game theory, Riemann-integration, theory of probability, smoothness of functions, Perm integration etc.

A soft set is a collection of approximate descriptions of an object. In [6], Molodtsov also proved how soft set theory is free from the parametrization inadequacy syndrome of probability theory, rough set theory, game theory and fuzzy set theory. Soft systems provide a very general framework with the involvement of parameters. Hence research works on soft set theory and its applications in various fields are progressing rapidly. Recently, Shabir and Naz [11] have introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Then some authors have begun to study some basic concepts and properties of soft topological spaces, see [3], [5], [11], [8], [12]. In particular, Akdag, Zorlutuna, Min and Atmaca [12] proved that an ordinary topological space can be considered a soft topological space, and that a fuzzy topological space is a special case of the soft topological space. Of course, a soft topological space is not certain an ordinary topological space.

In the present study, we introduce some new concepts in soft topological spaces such as soft closed mapping, soft open mappings, soft connected spaces and soft paracompact spaces. We also redefine the concept of soft points such that it is reasonable in soft topological spaces.

II. PRELIMINARIES

Definition 2.1: [6] Let U be an initial universe and E be a set of parameters. Suppose that \( P(U) \) denotes the power set of U and \( A \) is a non-empty subset of \( E \). A pair \( (F,A) \) is called a soft set over \( U \), where \( F \) is a mapping given by \( F: A \rightarrow \mathcal{P}(U) \).

In indeed, a soft set over \( U \) is a parameterized family of subsets of the universe \( U \). For a particular \( e \in A \), \( F(e) \) may be considered the set of \( e \)-approximate elements of the soft set \( (F,A) \).

Definition 2.2: [7] For two soft sets \( (F,A) \) and \( (G,B) \) over a common universe \( U \), \( (F;A) \) is a soft subset of \( (G,B) \), denoted by \( (F,A) \subseteq (G,B) \), if \( A \subseteq B \) and \( e \in A \), \( F(e) \subseteq G(e) \).

\( (F,A) \) is called a soft superset of \( (G,B) \), if \( (G,B) \) is a soft subset of \( (F,A) \), \( (F,A) \supseteq (G,B) \).

Definition 2.3: [7] Two soft sets \( (F,A) \) and \( (G,B) \) over a common universe \( U \) are said to be soft equal if \( (F,A) \) is a soft subset of \( (G,B) \) and \( (G,B) \) is a soft subset of \( (F,A) \).

Definition 2.4: [4] The complement of a soft set \( (F,A) \), denoted by \( (F,A)^{c} \), is defined by \( (F,A)^{c} = (F^{c},A) \), \( F^{c}: A \rightarrow \mathcal{P}(U) \) is a mapping given by \( F^{c}(e) = U - F(e) \) for arbitrary \( e \in A \). \( F^{c} \) is called the soft complement function of \( F \). Obviously, \( (F^{c})^{c} \) is the same as \( F \) and \( (F,A)^{c} = (F,A) \).

Definition 2.5: [7] A soft set \( (F,A) \) over \( U \) is said to be a NULL soft set denoted by \( \emptyset \) if for each \( e \in A \), \( F(e) = \emptyset \) (null set).

Definition 2.6: [7] A soft set \( (F,A) \) over \( U \) is said to be an absolute soft set, denoted by \( U_{A} \), if \( e \in A \), \( F(e) = U \).

Obviously, ones have \( U_{A} \subseteq U_{A} \) and \( U_{A} \subseteq U_{A} \).

Definition 2.7: [7] The union of two soft sets \( (F,A) \) and \( (G,B) \) over the common universe \( U \) is the soft set \( (H,C) \), where \( C = A \cup B \) and for arbitrary \( e \in C \),

\[ H(e) = \begin{cases} F(e), & \text{if } e \in A \setminus B, \\ G(e), & \text{if } e \in B \setminus A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases} \]

Definition 2.8: [9] The intersection of two soft sets \( (F,A) \) and \( (G,B) \) over the common universe \( U \) is the soft set \( (H,C) \), where \( C = A \cap B \) and for each \( e \in C \), \( H(e) = F(e) \cap G(e) \).
Note In order to efficiently discuss, we consider only soft sets \((F, E)\) over a universe \(U\) in which all the parameter set \(E\) are same. We denote the family of these soft sets by \(SS(U)\).

**Definition 2.9:** [12] Let \(I\) be an arbitrary index set, and let \(\{ (F, E) \}_{i \in I} \) be a subfamily of \(SS(U)\).

1. The union of these soft sets is the soft set \((H, E)\), where \(H(e) = \bigcup_{i \in I} F_i(e)\) for every \(e \in E\). We write \(\bigcup_{i \in I} (F_i, E) = (H, E)\).
2. The intersection of these soft sets is the soft set \((M, E)\), where \(M(e) = \bigcap_{i \in I} F_i(e)\) for every \(e \in E\). We write \(\bigcap_{i \in I} (F_i, E) = (M, E)\).

**Definition 2.10:** [11] Let \(\tau\) be a collection of soft sets over a universe \(U\) with a fixed set \(E\) of parameters, then \(\tau \subseteq SS(V)\) is called a soft topology on \(U\) with a fixed set \(E\) if

1. \(\emptyset_U, U_E\) belong to \(\tau\);
2. the union of arbitrary number of soft sets in \(\tau\) belongs to \(\tau\);
3. the intersection of arbitrary two soft sets in \(\tau\) belongs to \(\tau\).

**Definition 2.11:** Let \((U, F_1, E)\) and \((U, F_2, E)\) be two soft topological spaces over \(U\). If \(F_1 \subseteq F_2\), then we say that \((U, F_1, E)\) is finer than \((U, F_2, E)\) or \((U, F_2, E)\) is coarser than \((U, F_1, E)\). If \(F_1 \subseteq F_2\) and \(F_2 \not\subseteq F_1\), then we say that \((U, F_2, E)\) is strict finer than \((U, F_1, E)\) or \((U, F_1, E)\) is strict coarser than \((U, F_2, E)\).

**Note:** The soft indiscrete space is the coarsest soft topology, and the soft discrete space is the finest soft topology.

**Definition 2.12:** [12] A soft set \((G, E)\) in soft topological space \((U, \tau, E)\) is called a soft neighborhood of the soft set \((F, E)\) if there is a soft open set \((H, E)\) such that \((F, E) \subseteq (H, E) \subseteq (G, E)\).

**Definition 2.13:** Let \((U, \tau, E)\) be a soft topological space, and let \((G, E)\) be a soft set over \(U\).

1. The soft closure \([11]\) of \((G, E)\) is the soft set \([G, E] = \bigcap\{S(E) : (S, E) \in \tau \text{ and } G \subseteq S(E)\}\);
2. The soft interior \([12]\) of \((G, E)\) is the soft set \((G, E) = \bigcup\{S(E) : (S, E) \in \tau \text{ and } G \subseteq S(E)\}\).

**Definition 2.14:** The soft set \((F, E) \in SS(V)\) is called a soft point in \(U\) if there exist \(x \in U\) and \(e \in E\) such that \(F(e) = \{x\}\) and \(F(e') = \emptyset\) for each \(e' \in E - \{e\}\), and the soft point \((F, E)\) is denoted by \(e_x\).

**Theorem 2.15:** Let \((U, \tau, E)\) be a soft topological space. A soft point \(e_x \in (A, E)\) if and only if each soft neighborhood of \(e_x\) intersects \((A, E)\).

**Proof:** Necessity. Let a soft neighborhood \((B, E)\) of \(e_x\) disjoint from \((A, E)\). Without loss of generality, we may assume that \((B, E)\) is soft open. Then \((B, E)^c\) is soft closed and contains \((A, E)\), and hence \((A, E) \subseteq (B, E)^c\). Since \(e_x \in (B, E)^c\), we have \(e_x \in (A, E)\).

Sufficiency. Let \(e_x \notin (A, E)\). Then \((A, E)\) is a soft open neighborhood of \(e_x\) and disjoint from \((A, E)\), which is a contradiction.

Readers may refer to [7], [9], [11], [12] for notations and terminology not explicitly given here.

**III. SOFT OPEN AND SOFT CLOSED MAPPINGS**

**Definition 3.1:** [5] Let \(SS(U)\) and \(SS(V)\) be families of soft sets. Let \(u : U \to V\) and \(p : A \to B\) be mappings.

Then a mapping \(f_{pu} : SS(U)_A \to SS(V)_B\) is defined as:
1. Let \((F, A)\) be a soft set in \(SS(U)_A\). The image of \((F, A)\) under \(f_{pu}\), written as \(f_{pu}(F, A) = (f_{pu}(F), p(A))\), is a soft set in \(SS(V)_B\) such that
   \[f_{pu}(F)(y) = \begin{cases} \bigcup_{x \in E^{-1}(y) \cap A} u(F(x)), & p^{-1}(y) \cap A \neq \emptyset \smallsetminus \emptyset \\ \emptyset, & \text{otherwise} \end{cases}\]
   for all \(y \in B\).
2. Let \((G, B)\) be a soft set in \(SS(V)_B\). Then the inverse image of \((G, B)\) under \(f_{pu}\), written as \(f^{-1}_{pu}(G, B) = (f^{-1}_{pu}(G), p^{-1}(B))\), is a soft set in \(SS(U)_A\) such that
   \[f^{-1}_{pu}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B, \\ \emptyset, & \text{otherwise} \end{cases}\]
   for all \(x \in A\).

**Definition 3.2:** Let \((U, \tau, A)\) and \((V, \tau^*, B)\) be soft topological spaces. Let \(u : U \to V\) and \(p : A \to B\) be mappings. Let \(f_{pu} : SS(U)_A \to SS(V)_B\) be a function.

1. The function \(f_{pu}\) is soft continuous \([12]\) if \(f^{-1}_{pu}(H, B) \in \tau\) for each \((H, B) \in \tau^*\).
2. The function \(f_{pu}\) is soft open if \(f_{pu}(G, A) \in \tau^*\) for each \((G, A) \in \tau\).
3. The function \(f_{pu}\) is soft closed if \(f_{pu}(G, A) \in \tau^*\) for each \((G, A) \in \tau\).
4. The function \(f_{pu}\) is soft homeomorphism if \(f_{pu}\) is an onto, one to one, soft continuous and soft open mapping.

**Remark 3.3:** (1) The soft continuous mappings may not soft closed and soft open.

(2) The soft closed and soft open mappings may not soft continuous.

**Example 3.4:** Let \(U = V\) be an initial universe set, and \(A\) be the set of parameters. Suppose that \((U, \tau, A)\) is a non-discrete soft topological space, and that \((V, \tau^*, A)\) is the soft discrete topology. Then the identical mapping \(i_U\) from \((U, \tau, A)\) to \((V, \tau^*, A)\) is soft open and soft closed. However, it is easy to see that \(i_U\) is not soft continuous. Moreover, the identical mapping \(i_U\) from \((V, \tau^*, A)\) to \((U, \tau, A)\) is soft continuous. However, it is easy to see that \(i_U\) is not soft open and soft closed.

**Theorem 3.5:** Let \((U, \tau, A)\) and \((V, \tau^*, B)\) be soft topological spaces. Let \(u : U \to V\) and \(p : A \to B\) be onto mappings. Let \(f_{pu} : SS(U)_A \to SS(V)_B\) be a function. Then the following are equivalent:
1. \(f_{pu}\) is soft open;
2. For each soft set \((G, A)\) over \(U\), we have \(f_{pu}((G, A)) \subseteq f_{pu}(G, A)^c\);
3. For each soft set \((F, B)\) over \(V\), we have \(f_{pu}^{-1}((F, B)) \subseteq f_{pu}^{-1}(F, B)^c\);
4. For each soft point \(e_x \in U\) and each soft neighborhood \((U, E)\) at \(e_x\) over \(U\), \(f_{pu}(U, E)\) is a soft neighborhood at soft point \(f_{pu}(e_x)\) over \(V\).

**Proof:** (1) \(\Rightarrow\) (2). Obvious, \(f_{pu}((A, E)) \subseteq f_{pu}(A, E)^c\), and since \(f_{pu}((A, E))^c\) is soft open by (1), we have \(f_{pu}(A, E) \subseteq f_{pu}((A, E))^c\).

(2) \(\Rightarrow\) (4). Let \((U, E)\) be a soft neighborhood at soft point \(e_x\). Then we have \(e_x \in (A, E)^c\) by (2). We have \(f_{pu}(e_x) \in (U, E)^c\). Therefore, \(f_{pu}(U, E)\) is a soft neighborhood at point \(f_{pu}(e_x)\) over \(V\).
(4) ⇒ (3). Let \( e_x \in f_{pV}^{-1}((F, B)) \). Then \( f_{pu}(e_x) \in (F, B) \). Let \((H, A)\) be an arbitrary soft neighborhood at soft point \( e_x \). By (4), \( f_{pu}((H, A)) \) is a soft neighborhood of \( f_{pu}(e_x) \), and hence \( f_{pu}((H, A)) \cap (F, B) \neq \emptyset \) by Theorem 2.15. Then there exists a soft point \( e_y \in (H, A) \) such that \( f_{pu}(e_y) \in (F, B) \), and thus \( e_x \in f_{pu}^{-1}((F, B)) \). Then \((H, A) \cap f_{pu}^{-1}((F, B)) \neq \emptyset \), and therefore, \( e_x \in f_{pu}^{-1}((F, B)) \).

(3) ⇒ (2). Since \((A, G)^* \subseteq (A, G) \subseteq f_{pu}^{-1}((F, G)(A, G)) \) and \((G, A)^* \) is soft open, we have \((G, A)^* \subseteq (f_{pu}(A, G))^* \). Obviously, we have

\[ f_{pu}^{-1}(f_{pu}((A, G)))^* = f_{pu}^{-1}((f_{pu}(A, G))^*)^* \]

and

\[ f_{pu}^{-1}(f_{pu}((A, G)))^* = f_{pu}^{-1}((f_{pu}(A, G))^*)^* \]

since \( f_{pu}^{-1}(f_{pu}((A, G)))^* = f_{pu}^{-1}((f_{pu}(A, G))^*)^* \). Then

\[ f_{pu}^{-1}(f_{pu}((A, G))^*) \subseteq f_{pu}^{-1}((f_{pu}(A, G))^*)^* \]

\[ = f_{pu}^{-1}((f_{pu}(A, G))^*)^* \]

\[ = (f_{pu}(A, G))^* \].

(2) ⇒ (1). Let \((G, A)\) be a soft open set. Then

\[ (G, A) = (G, A)^* \].

By (2), we have \( f_{pu}((G, A))^* \subseteq f_{pu}((G, A))^* \), that is, \( f_{pu}(G, A) \) is soft open. Therefore, \( f_{pu} \) is a soft open mapping.

Theorem 3.6: Let \((U, \tau, A)\) and \((V, \tau^*, B)\) soft topological spaces. Let \( u : U \to V \) and \( p : A \to B \) be mappings. Let \( f_{pu} : SS(U)_A \to SS(V)_B \) be a function. Then the following are equivalent:

1) \( f_{pu} \) is soft closed;
2) For each soft set \((G, A)\) over \( U \), \( f_{pu}(G, A) \subseteq f_{pu}(G, A) \).

Proof: (1) ⇒ (2). Since \( f_{pu}(G, A) \subseteq f_{pu}((G, A)) \) and \( f_{pu}(G, A) \) is soft closed over \( V \), we have \( f_{pu}(G, A) \subseteq f_{pu}(G, A) \).

(2) ⇒ (1). Let \((G, A)\) be a soft closed set over \( V \). By (2), we have \( f_{pu}(G, A) \subseteq f_{pu}(G, A) \).

Since \( G, A = (G, A) \cap f_{pu}((G, A)), f_{pu}((G, A)) \subseteq f_{pu}(G, A) \), and thus \( f_{pu}(G, A) \) is a soft closed set over \( V \).

The proof of the following proposition is an easy exercise.

Proposition 3.7: Let \((U, \tau, A)\) and \((V, \tau^*, B)\) soft topological spaces. Let \( u : U \to V \) and \( p : A \to B \) be mappings. Then \( f_{pu} : SS(U)_A \to SS(V)_B \) be a function. Then \( f_{pu}^{-1}(B, E) \subseteq (A, E) \) if and only if \((B, E) \subseteq (f_{pu}(A, E))^* \).

Theorem 3.8: Let \((U, \tau, A)\) and \((V, \tau^*, B)\) soft topological spaces. Let \( u : U \to V \) and \( p : A \to B \) be mappings. Let \( f_{pu} : SS(U)_A \to SS(V)_B \) be a function. Then \( f_{pu} \) is soft closed if and only if, for each soft point \( e_y \) in \( V_B \) and each soft open set \((F, E)\) with \( f_{pu}^{-1}(e_y) \subseteq (F, E) \) in \( U_E \), there exists a soft open set \((W, B)\) in \( V_B \) such that \( e_y \in (W, B) \) and \( f_{pu}^{-1}(W, B) \subseteq (F, E) \).

Proof: Necessity. Let \( f_{pu} \) be a soft closed mapping. For each soft point \( e_y \) in \( V_B \) and each soft open set \((F, E)\) with \( f_{pu}^{-1}(e_y) \subseteq (F, E) \) in \( U_E \), we put \((W, B) = (f_{pu}(F, E))^* \). Then \((W, B)\) is soft open. By Proposition 3.7, we have \( e_y \in (W, B) \) and \( f_{pu}^{-1}(W, B) \subseteq (F, E) \).

Sufficiency. Let \((G, E)\) be a soft closed set in \( U_E \). Take arbitrary soft point \( e_y \in f_{pu}(G, E) \). Then \( f_{pu}^{-1}(e_y) \subseteq (G, E) \). By the assumption, there exists a soft open set \((W, B)\) such that \( e_y \in (W, B) \) and \( f_{pu}^{-1}(W, B) \subseteq (G, E) \), and hence \((W, B)\) is a soft neighborhood of \( e_y \). By Proposition 3.7, we have \((W, B) \cap f_{pu}(G, E) = \emptyset \), and thus \( f_{pu}(G, E) \) is soft closed.

IV. SOFT CONNECTED SPACES

Definition 4.1: Let \((U, \tau, E)\) be a soft topological space, and \((F_1, E), (F_2, E)\) be two soft sets over \( U \). The soft sets \((F_1, E), (F_2, E)\) are said to be soft separated if \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = \emptyset \) and \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = \emptyset \).

Remark 4.2: Two disjoint soft open sets over \( U \) may not be a soft separated.

Example 4.3: Let \( u = \{h_1, h_2, h_3\}, E = \{e_1, e_2\} \) and \( \tau = \{\emptyset, U_E, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\} \) are soft sets over \( U \), defined as follows:

\[ f_{pu}(e_1) = \{h_2, h_3\}, f_{pu}(e_2) = \{h_3\}; \]

\[ f_{pu}(e_1) = \{h_2, h_3\}, f_{pu}(e_2) = \{h_3, h_2\}; \]

\[ f_{pu}(e_1) = \{h_1, h_2\}, f_{pu}(e_2) = \{h_1\}; \]

\[ f_{pu}(e_1) = \{h_1, h_2\}, f_{pu}(e_2) = \{h_1, h_3\}; \]

\[ f_{pu}(e_1) = \emptyset, f_{pu}(e_2) = \{h_2\}. \]

Then \( \tau \) defines a soft topology on \( U \) and hence \((U, \tau, E)\) is a soft topological space over \( U \). It is easy to see that \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = \emptyset \). However, \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = \emptyset \), and hence \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = \emptyset \).

Definition 4.4: Let \((U, \tau, E)\) be a soft topological space. If there exist two non-empty soft separated sets \((F_1, E), (F_2, E)\) such that \( f_{pu}(F_1, E) \subseteq f_{pu}(F_2, E) \), then \((F_1, E)\) and \((F_2, E)\) are said to be a soft division for soft topological space \((U \tau, E)\). \((U, \tau, E)\) is said to be soft disconnected if \((U, \tau, E)\) has a soft division. Otherwise, \((U, \tau, E)\) is said to be soft connected.

It is easy to see that each soft indiscemible space is soft connected, and that each soft discrete non-trivial space is not soft connected.

Theorem 4.5: Let \((U, \tau, E)\) be a soft topological space. Then the following conditions are equivalent:

1) \((U, \tau, E)\) has a soft division;
2) There exist two disjoint soft closed sets \((F_1, E)\) and \((F_2, E)\) such that \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = U_E \);
3) There exist two disjoint soft open sets \((F_1, E)\) and \((F_2, E)\) such that \( f_{pu}(F_1, E) \cap f_{pu}(F_2, E) = U_E \);
4) \((U, \tau, E)\) has a proper soft open and soft closed set in \(U\).

Proof: (1)\(\Rightarrow\)(2). Let \((U, \tau, E)\) have a soft division \((F_1, E)\) and \((F_2, E)\). Then
\[
(F_1, E) \cap (F_2, E) = \emptyset
\]
and
\[
(F_1, E) = (F_1, E) \cap ((F_1, E) \cup (F_2, E)) = ((F_1, E) \cap (F_1, E)) \cup ((F_1, E) \cap (F_2, E)) = (F_1, E).
\]
Therefore, \((F_1, E)\) is a soft closed set in \(U\). Similar, we can see that \((F_2, E)\) is also a soft closed set in \(U\).

(2)\(\Rightarrow\)(3). Let \((U, \tau, E)\) have a soft division \((F_1, E)\) and \((F_2, E)\) such that \((F_1, E)\) and \((F_2, E)\) are soft closed. Then \((F_1, E)\) and \((F_2, E)\) are soft open sets in \(U\). Then it is easy to see that \((F_1, E) \cap (F_2, E) = \emptyset\) and \((F_1, E) \cup (F_2, E) = U\).

(3)\(\Rightarrow\)(4). Let \((U, \tau, E)\) have a soft division \((F_1, E)\) and \((F_2, E)\) such that \((F_1, E)\) and \((F_2, E)\) are soft open in \(U\). Then \((F_1, E)\) and \((F_2, E)\) are also soft closed in \(U\).

(4)\(\Rightarrow\)(1). Let \((U, \tau, E)\) has a proper soft open and closed set \((F, E)\) in \(U\). Put \((H, E) = (F, E)\). Then \((H, E)\) and \((F, E)\) are non-empty soft closed set in \(U\). \((H, E) \cap (F, E) = \emptyset\) and \((H, E) \cup (F, E) = U\). Therefore, \((H, E)\) and \((F, E)\) is a soft division of \(U\).

Theorem 4.6: Let \((U, \tau, E)\) be a soft topological space. Then the following conditions are equivalent:

1) \((U, \tau, E)\) is soft connected;
2) There exist two disjoint soft closed sets \((F_1, E)\) and \((F_2, E)\) such that
\[
(F_1, E) \cup (F_2, E) = U;
\]
3) There exist two disjoint soft open sets \((F_1, E)\) and \((F_2, E)\) such that
\[
(F_1, E) \cup (F_2, E) = U;
\]
4) \((U, \tau, E)\) at most has two soft closed and soft open sets in \(U\), that is, \(\emptyset\) and \(U\).

Note: By Theorem 4.6, the soft topological space in Example 4.15 is a soft disconnected space since the soft set \((F_2, E)\) is soft open and soft closed in \(U\).

Lemma 4.7: Let \((U, \tau, E)\) be a soft topological space over \(U\) and \(V\) be a non-empty subset of \(U\). If \((F_1, E)\) and \((F_2, E)\) are soft sets in \(V\), then \((F_1, E)\) and \((F_2, E)\) are a soft separation of \(V\) if and only if \((F_1, E)\) and \((F_2, E)\) are a soft separation of \(U\).

Proof: We have
\[
(F_1, E) \cap (F_2, E) = (F_1, E) \cap (F_2, E) \cap (V, E) = (F_1, E) \cap (V, E) \cap (F_2, E) = (F_1, E) \cap (F_2, E) = \emptyset.
\]
Therefore, the lemma holds.

Lemma 4.8: Suppose that \((U, \tau, E)\) is a soft topological space over \(U\), and that \(V\) is a non-empty subset of \(U\) such that \((V, \tau, E)\) is soft connected. If \((F_1, E)\) and \((F_2, E)\) are a soft separation of \(U\) such that \((F_1, E) \subseteq (F_1, E) \cup (F_2, E)\), then \(V \subseteq (F_1, E) \cup (F_2, E)\).

Proof: Since \(V \subseteq (F_1, E) \cup (F_2, E)\), we have \(V = (V \cap (F_1, E)) \cup (V \cap (F_2, E))\). By Lemma 4.7, \((V \cap (F_1, E)) \cup (V \cap (F_2, E))\) is a soft separation of \(V\). Since \((V, \tau, E)\) is soft connected, we can see that \(V \cap (F_1, E) = \emptyset\) or \(V \cap (F_2, E) = \emptyset\). Therefore, \(V \subseteq (F_1, E) \cup (F_2, E)\).

Lemma 4.9: Let \(\{(a, \tau_a, E) : a \in J\}\) be a family of non-empty soft connected spaces of soft topological space \((U, \tau, E)\). If \(\cap_{a \in J} U_a = \emptyset\), then \(\cup_{a \in J} U_a, \tau_{a \in J} E_u, E\) is a soft connected subspace of \((U, \tau, E)\).

Proof: Let \(V = \cup_{a \in J} U_a\). Choose a soft point \(e \in V\). Let \((C, E)\) and \((D, E)\) be a soft division of \(\cup_{a \in J} U_a, \tau_{a \in J} E_u, E\). Without loss of generality, we can assume that \(e \in (C, E)\) or \(e \in (D, E)\). For each \(a \in J\), since \(U_a \subseteq U\) is soft connected, it follows from Lemma 4.8 that \((U_a, \tau_a, E) \subseteq (C, E)\) or \((U_a, \tau_a, E) \subseteq (D, E)\). Therefore, we have \(V \subseteq (C, E)\) since \(e \in (C, E)\), and then \(D, E = \emptyset\), which is a contradiction. Thus \(\cup_{a \in J} U_a, \tau_{a \in J} E_u, E\) is a soft connected subspace of \((U, \tau, E)\).

Theorem 4.10: Let \(\{(a, \tau_a, E) : a \in J\}\) be a family non-empty soft connected spaces of soft topological space \((U, \tau, E)\). If \(\cap_{a \in J} U_a \neq \emptyset\) for arbitrary \(a, \beta \in J\), then \(\cup_{a \in J} U_a, \tau_{a \in J} E_u, E\) is a soft connected subspace of \((U, \tau, E)\).

Proof: Fix an \(a_0 \in J\). For arbitrary \(\beta \in J\), put \(A_\beta = U_{a_0} \cup U_\beta\). By Lemma 4.9, each \((A_\beta, \tau_{A_\beta}, E)\) is soft connected. Then \(\cup_{a \in J} U_a, \tau_{a \in J} E_u, E\) is a family non-empty soft connected spaces of soft topological space \((U, \tau, E)\). If \(\cap_{a \in J} U_a \neq \emptyset\), we have \(\cup_{a \in J} U_a = \cup_{a \in J} U_a, \tau_{a \in J} E_u, E\). It follows from Lemma 4.9 that \(\cup_{a \in J} U_a, \tau_{a \in J} E_u, E\) is a soft connected subspace of \((U, \tau, E)\).

Theorem 4.11: Let \((U, \tau, E)\) be a soft topological space with over \(U\) and \((V, \tau, E)\) be a soft connected subspace of \((U, \tau, E)\). If \(V \subseteq A \subseteq B \subseteq (D, E)\), then \((A, \tau_A, E)\) is a soft connected subspace of \((U, \tau, E)\). In particular, \((Y, E)\) is a soft connected subspace of \((U, \tau, E)\).

Proof: Let \((C, E)\) and \((D, E)\) be a soft division of \((A, \tau_A, E)\). By Lemma 4.8, we have \(A \subseteq (C, E)\) or \(A \subseteq (D, E)\). Without loss of generality, we may assume that \(A \subseteq (C, E)\). By Lemma 4.7, we have \((C, E) \cap (D, E) = \emptyset\), and hence \(A \subseteq (C, E)\) which is a contradiction.

Theorem 4.12: The image of soft connected spaces under a soft continuous map are soft connected.

Proof: Let \((U, \tau, A)\) and \((V, \tau, B)\) be two soft topological spaces, where \((U, \tau, A)\) is soft connected, and let \(f\) be a soft pu-continuous mapping from \(U\) to \(V\). Obvious, the restricted mapping is soft continuous, and without loss of generality, we may assume that \(u(U) = u(V)\) and \(p(A) = B\). Suppose that \((V, \tau, B)\) is soft disconnected. By Theorem 4.6, there exists a proper soft open and soft closed set \((F, E)\) in \(V\). Since \(f\) is soft continuous, \(f^{-1}(F, E)\) is a proper soft open and soft closed set in \(U\) by [12, Theorem 6.3], which is a contradiction.

Proposition 4.13: [11] Let \((U, \tau, E)\) be a soft topological space. Then the collection \(\tau_\alpha = \{F(\alpha) : (F, E) \in \tau\}\) for
each $a \in E$, defines a topology on $U$.

**Remark 4.14:** There exists a soft connected soft topological space $(U, \tau, E)$ such that $(U, \tau_0)$ is a disconnected topological space for some $a \in E$.

**Example 4.15:** Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \emptyset, \{U_E, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$. The soft sets over $U$, defined as follows:

- $F_1(e_1) = \{h_1, h_2\}$, $F_1(e_2) = U$;
- $F_2(e_1) = \{h_1, h_2\}$, $F_2(e_2) = U$;
- $F_3(e_1) = \{h_2, h_3\}$, $F_3(e_2) = U$;
- $F_4(e_1) = \{h_2, h_3\}$, $F_4(e_2) = U$;
- $F_5(e_1) = \emptyset$, $F_5(e_2) = U$. $(U, \tau, E)$ defines a soft topology on $U$ and hence $(U, \tau, E)$ is a soft topological space over $U$. Obvious, $(U, \tau, E)$ is soft connected. However, $(U, \tau_1)$ is discrete space, and hence $(U, \tau_1)$ is disconnected. Therefore $(U, \tau, E)$ be a soft topological space. A subcollection $\gamma$ of $\tau$ is said to be a base for $\tau$ if every member of $\gamma$ can be expressed as a union of members of $\gamma$.

**Definition 4.16:** Let $(U, \tau, E)$ be a soft topological space. A subcollection $\gamma$ of $\tau$ is said to be a base for $\tau$ if every member of $\gamma$ can be expressed as a union of members of $\gamma$.

**Definition 4.17:** Let $\{(U^a, \tau_0, E_0)\}_{\alpha \in I}$ be a family of soft topological spaces. Let us take as a basic for a soft topology on the product space $(\prod_{\alpha \in I} U^a, \tau_{\alpha}, \prod_{\alpha \in I} E_0)$ the collection of all soft sets $\{(\prod_{\alpha \in I} F_{\alpha}, \prod_{\alpha \in I} E_0) \mid \tau_{\alpha} \subseteq \bigcup \{\tau_{\alpha} \mid \alpha \in I \setminus J\} \}$.

**Theorem 4.18:** A finite product of soft connected spaces is soft connected.

**Proof:** We prove the theorem first for the product of two soft connected spaces $(U, \tau_1, A)$ and $(V, \tau_2, B)$. Choose a fixed point $a \in U \times V$. Obvius, $(U \times B, \tau_1 \times \tau_2, A \times B)$ is soft connected for each $u \in U$, $(u \times \tau_2, \tau_1 \times \tau_2 |_{u \times V}, A \times B)$ is also soft connected, and put $\tau_u(U \times B) \cup (u \times V)$. Then each $(T_u, \tau_1 \times \tau_2 |_{u \times V}, A \times B)$ is soft connected. Since $a \times b \in T_u$ for each $u \in U$, it follows from Theorem 4.10 that $(\bigcup_{u \in U} (T_u, \tau_1 \times \tau_2 |_{u \times V}, A \times B))$ is soft connected.

**Theorem 4.19:** The product of a family of soft connected spaces is soft connected.

**Proof:** Let $\{(U_0, \tau_0, E_0) : \alpha \in J\}$ be a family of soft connected spaces, and put $U = \prod_{\alpha \in J} U_\alpha$. Take a fixed point $a = (a_\alpha)$ of $U$. Take arbitrary finite subset $K$ of $J$, let $U_K$ denote the subset of $U$ consisting of all points $u = (u_\alpha)$ such that $a_\alpha = a_\alpha$ for $\alpha \not\in K$. By Theorem 4.12, it is easy to see that $U_K, \prod_{\alpha \in K} \tau_0 |_{U_K}, \prod_{\alpha \in K} E_0)$ is soft connected. Let $V = \bigcup (U_K : K$ is finite subset of $J)$. It is easy to see that $U_{\prod_{\alpha \in K} \tau_0 |_{U_K}, \prod_{\alpha \in K} E_0}$ is soft connected.

**Definition 4.20:** Given a soft topological space $(U, \tau, E)$, define an equivalence relation on $U_E$ by setting $e_x \sim e_y$ if there exists a soft connected subspace of $(U, \tau, E)$ containing both soft points $e_x$ and $e_y$. The equivalence classes are called the soft components (or the soft connected components) of $U_E$.

Reflexivity and symmetry of the relation are obvious. Transitivity follows by noting that if $A_G$ is a soft connected subspace containing soft points $e_x$ and $e_y$ and if $B_G$ is a soft connected subspace containing soft points $e_y$ and $e_z$, then $A_G \cup B_G$ is a subspace containing soft points $e_x$ and $e_z$ that is soft connected because $A_G$ and $B_G$ have the soft point $e_y$ in common.

**Theorem 4.21:** The soft components of soft topological space $(U, \tau, E)$ are soft connected disjoint soft subspace of $U_E$ whose union is $U_E$, such that each non-empty soft connected subspace of $U_G$ intersects only one of them.

**Proof:** Being equivalence classes, the soft components of $U_E$ are disjoint and their union is $U_E$. Let $A_G$ be an arbitrary soft connected subspace. Then that intersects only one of them. For if $A_G$ intersects the soft components $C_G$ and $D_G$ of $E_G$, say in soft points $e_x$ and $e_y$, respectively, then $e_x \sim e_y$ by definition; this cannot happen unless $C_G = D_G$.

Next we shall show the soft component $C_G$ is soft connected. Choose a soft point $e_x$, of $C_G$. For each soft point $e_x$ of $C_G$, we know that $e_x \sim e_x$, hence there exists a soft connected subspace $L_{E_x}$ containing $e_x$ and $e_x$. Obvious, each $L_{E_x} \subseteq C_G$. Therefore, $C_G = \bigcup_{e_x \in C_G} L_{E_x}$. Since the soft subspace $L_{E_x}$ are soft connected and have the soft point $e_x$, it is soft connected by a definition.

**V. SOFT PARACOMPACT SPACES**

**Definition 5.1:** Let $(U, \tau, E)$ be a soft topology space, and $A$ be a collection of soft sets of $U_E$.

1) The collection $A$ is said to be locally finite in $U_E$ if each soft point of $U_E$ has a soft neighborhood that intersects only finitely many elements of $A$.

2) A collection $B$ of soft sets of $U_E$ is said to be a refinement of $A$ if for each element $B \in B$, there exists an element $A \in A$ containing $B$. If the elements of $B$ are soft open sets, we call $B$ a soft open refinement of $A$; if they are soft closed, we call $B$ a soft closed refinement.

**Proposition 5.2:** Let $A$ be a locally finite collection of soft subsets of $U_E$. Then

1) Any subcollection of $A$ is locally finite;
2) The collection $B = \{(F_E) : (F, E) \in A\}$ is locally finite;
3) $\bigcup_{(F,E) \in A} (F, E) = \bigcup_{(F,E) \in A} (F, E)$.

**Proof:** Statement (1) is trivial.

2) Note that any soft open set $(G, E)$ that intersects the soft set $(F, E)$ necessarily intersects $(F, E)$. Thus if $(G, E)$ is a soft neighborhood of soft point $e_x$ that intersects only finitely many elements $(F, E)$ of $A$, then $(F, E)$ can intersect at most the same number of soft sets of the collection.

3) Let $\bigcup_{(F,E) \in A} (F, E) \subseteq (Y, E)$. Obvious, $\bigcup_{(F,E) \in A} (F, E) \subseteq (Y, E)$. We prove the reverse inclusion, under the assumption of local finiteness. Let $e_x \in (Y, E)$; let $(G, E)$ be a soft neighborhood of $e_x$ that intersects only finitely many elements of $A$, say $(F_1, E), \ldots, (F_n, E)$. Then $e_x$ belongs to at least one of the soft sets $(F_1, E), \ldots, (F_n, E)$. For otherwise, the soft set $(G, E) \cap \bigcup_{(F_i, E) \in A} (F_i, E)$ would be a soft neighborhood of $e_x$ that intersects no element.
of \( A \), and therefore it does not intersect \((Y, E)\), which is a contradiction with \( e_x \in \langle Y, E \rangle \).

**Definition 5.3:** A soft topological space \((U, \tau, E)\) is soft paracompact if each soft open covering \( A \) of \( \mathcal{U}_C \) has a locally finite soft open refinement \( B \) that covers \( \mathcal{U}_C \).

**Definition 5.4:** [12] A soft topological space \((U, \tau, E)\) is soft compact if each soft open covering \( A \) of \( \mathcal{U}_C \) has a finite subcover.

**Definition 5.5:** A soft topological space \((U, \tau, E)\) is soft Lindelöf if each soft open covering \( A \) of \( \mathcal{U}_C \) has a countable subcover.

**Proposition 5.6:** Each soft compact space is soft Lindelöf, and each soft Lindelöf space is soft paracompact.

**Proposition 5.7:** Let \((U, \tau, E)\) be a soft paracompact space. If \( E = \{x\} \), then \((U, \tau, E)\) is soft paracompact if and only if the collection \( \eta = \{ (F, \epsilon) : (F, E) \in \tau \} \) is a compact topology on \( U \).

It is well known that a Lindelöf space may not compact and a paracompact space may not Lindelöf. Therefore, it follows from Proposition 5.7 that a soft Lindelöf space may not soft compact and a soft paracompact space may not soft Lindelöf.

**Theorem 5.8:** Each soft paracompact soft \( T_2 \) space is soft normal.

**Proof:** Let \((U, \tau, E)\) be a soft paracompact soft \( T_2 \) space. First prove soft regularity. Let \( e_x \) be a soft point of \( \mathcal{U}_C \) and let \((A, E)\) be a soft closed set of \( \mathcal{U}_C \) disjoint from \( e_x \). The \( T_2 \) condition enables us to take, for each soft point \( e_y \) in \((A, E)\), an open soft set \((B^{e_y}, E)\) about \( e_y \) whose soft closure is disjoint from \( e_x \). Let \( A = \{ (B^{e_y}, E) : e_y \in (A, E) \} \cup \{ (A, E) \} \). Then \( A \) is a soft open covering of \( \mathcal{U}_C \). Since \((U, \tau, E)\) is soft paracompact, there exists a locally finite soft open refinement \( B \) that covers \( \mathcal{U}_C \). Form the subcollection \( C \) of \( B \) consisting of each element of \( B \) that intersects \((A, E)\). Then \( C \) covers \((A, E)\). Moreover, if \( C \subset C \), then the soft closure of \( C \) is disjoint from \( e_x \). Since \( C \) intersects \((A, E)\), it lies in some soft open set \((B^{e_x}, E)\), whose soft closure is disjoint from \( e_x \). Let \( Y = \bigcup_{(B^{e_x}, E) \in C} \). Obiously, \((V, E)\) is soft open in \( \mathcal{U}_C \) containing \((A, E)\). Since \( C \) is locally finite, \((V, E) = \bigcup_{(B^{e_x}, E) \in C} \). By Proposition 5.2. Then \( (V, E) \) is disjoint from \( e_x \). Thus soft regularity is proved.

To prove soft normality, one only repeats the same argument, replacing \( e_x \) by a soft closed set throughout and replacing the soft \( T_2 \) condition by soft regularity.

**Theorem 5.9:** Each soft closed subspace of a soft paracompact space is soft paracompact.

**Proof:** Let \((U, \tau, E)\) be a soft paracompact space, and \( Y \subset U \) such that \( \mathcal{U}_C \) is soft closed in \( U \). Let \( A \) be a soft covering of \( Y_C \) by soft open in \( Y_C \). For every \((A, E) \in A\), take a soft open set \((A', E)\) of \( U \) such that \((A', E) \supseteq (A, E)\). Cover \( Y_C \) by the soft open \((A', E)\), along with the soft open set \( Y_C \). Suppose that \( B \) is a locally finite soft open refinement of this soft covering that covers \( Y_C \). Then the collection

\[ C = \{ (B, E) \supseteq Y_C : (B, E) \in B \} \]

is the required locally finite soft open refinement of \( A \).

**Remark 5.10:** By Proposition 5.7, it is easy to see the following two facts:

1. A soft paracompact subspace of a soft Hausdorff space \((U, \tau, E)\) need not be soft closed in \( \mathcal{U}_C \).
2. A soft subspace of a soft paracompact need not be soft paracompact.

**Lemma 5.11:** Let \((U, \tau, E)\) be a soft topological space. If each soft open covering of \((U, \tau, E)\) has a locally finite soft closed refinement, then every soft open covering of \((U, \tau, E)\) has a locally finite soft open refinement.

**Proof:** Let \( A \) be a soft open covering of \((U, \tau, E)\), and let \( B = \{ (F, E) : s \in S \} \) be a locally finite soft closed refinement of \( A \). For each soft point \( e_x \in \mathcal{U}_C \), choose a soft open neighborhood \((V_x, E)\) of \( e_x \) such that \((V_x, E) \) intersects finitely many elements of \( B \). Let \( C = \{ (F, E) : e_x \in F \} \), and let \( D \) be a locally finite soft closed refinement of \( C \). For each \( s \in S \), put

\[ (W_s, E) = \bigcup \{ (D, E) : (D, E) \in D, (D, E) \supseteq (F, E) \} \setminus \emptyset \].

Obviously, each \((W_s, E)\) is soft open and contains \((F, E)\). Moreover, for each \( s \in S \) and each \( (D, E) \in D \), we have

\[ (W_s, E) \supseteq (D, E) \neq \emptyset \] if and only if \( (F, E) \supseteq (D, E) \).

For each \( s \in S \), choose a \((A_s, E) \in A \) such that \((F, E) \subseteq (A_s, E) \), and let \((G_s, E) = (A_s, E) \cap (W_s, E) \). Then \((G_s, E) : s \in S \) is a soft open covering and refines \( A \). It is easy to see that each element of \( D \) intersects only finitely many \((G_s, E)\). Therefore, \((G_s, E) : s \in S \) is locally finite.

**Lemma 5.12:** Each \( \sigma \)-locally finite soft open covering has a locally finite refinement.

**Proof:** Let \( U = \bigcup_{\alpha \in \mathcal{U}_b} \mathcal{U}_b \) be a \( \sigma \)-locally finite soft open covering for some soft topological space, where each \( \mathcal{U}_b \) is locally finite. Put \( V_1 = U_1, V_n = (F, E) \cap \bigcup_{k=n}^\infty (F, E)_{\in \mathcal{U}_b} \), where \( U_k = (F, E) \cap (F, E)_{\in \mathcal{U}_b} \). Then it is easy to see that \( V = \bigcup_{\alpha \in \mathcal{U}_b} \mathcal{U}_b \) is a locally finite soft open covering and refines \( U \).

**Lemma 5.13:** Let \((U, \tau, E)\) be soft regular, if each soft open covering of \((U, \tau, E)\) has a locally finite refinement, then it has a locally finite soft closed refinement.

**Proof:** Let \( U = \{ (F, E) : \alpha \in A \} \) be an arbitrary soft open covering. Then, for each soft point \( e_x \in \mathcal{U}_C \), there exists some \((F, E) \in U \) such that \( e_x \in (F, E) \). By soft regularity, there is an soft neighborhood \((V_x, E)\) such that \( e_x \in (V_x, E) \subseteq (V_x, E) \cap (F, E) \). Put \( V = \bigcup_{e_x \in \mathcal{U}_C} (V_x, E) \in \mathcal{U}_C \). Then \( V \) is a soft open covering and refines \( U \). By the assumption, there is a locally finite soft covering \( W = \{ (W_\beta, E) : \beta \in B \} \) such that \( W \) refines \( V \). Then \( (W_\beta, E) : \beta \in B \) is a locally finite soft closed covering and refines \( U \).

By Lemmas 5.11, 5.12 and 5.13, we have the following theorem.

**Theorem 5.14:** Let \((U, \tau, E)\) be soft regular. Then the following conditions on \( U \) are equivalent:

1. \((U, \tau, E)\) is soft paracompact;
2. Every soft open covering has a \( \sigma \)-locally finite soft open refinement;
3. Every soft open covering has a locally finite refinement;
4. Every soft open covering has a locally finite soft closed refinement.
VI. Conclusion

In the present work, we have continued to study the properties of soft topological spaces. We mainly introduce soft open mappings, soft closed mappings, soft connected spaces and soft paracompact spaces. Moreover, we have also established several interesting results and presented its fundamental properties with the help of some examples. We hope that the findings in this paper will help researcher enhance and promote the further study on soft topology to carry out a general framework for their applications in practical life.

REFERENCES