A new robust stability criterion for dynamical neural networks with mixed time delays

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Abstract—In this paper, we investigate the problem of the existence, uniqueness and global asymptotic stability of the equilibrium point for a class of neural networks, the neutral system has mixed time delays and parameter uncertainties. Under the assumption that the activation functions are globally Lipschitz continuous, we drive a new criterion for the robust stability of a class of neural networks with time delays by utilizing the Lyapunov stability theorems and the Homomorphic mapping theorem. Numerical examples are given to illustrate the effectiveness and the advantage of the proposed main results.

Keywords—Neural networks, Delayed systems, Lyapunov function, Stability analysis.

I. INTRODUCTION

In recent years, neural networks have been widely used in solving various classes of engineering problems such as control systems, optimization, image processing, associative memory design and signal processing. The key feature of the designed neural network, in such applications, is to be convergent. When a neural network is designed to function as an associative memory, it is desired that the neural network has multiple equilibrium points. Therefore, there has been a great deal of interest to the stability properties of neural networks in the past literature. When a neural network is employed to solve optimization problems, then the neural network must have unique equilibrium point which is globally asymptotically stable. But, in hardware implementation of neural networks, some parameters associated with the dynamical behavior of neural network may be subjected to some changes. Therefore, in order to be able to completely characterize equilibrium and stability properties of the neural network, we must take into account the delay parameters and uncertainties in the mathematical model of the neural network. In the recent literature, many papers have studied the existence, uniqueness and global robust asymptotic stability of the equilibrium point for different classes of delayed neural networks and presented various robust stability conditions [1]-[15]. In [1], the authors have applied many methods to study the existence, uniqueness and global asymptotic stability of the equilibrium point for the class of neural networks with multiple time delays and parameter uncertainties, and got a new robust stability criterion. But the paper have conservation. In the current paper, we will present a new alternative sufficient condition for global robust stability of delayed neural networks with multiple and distributed time delays. At the end of this paper we will give two numerical examples to clarify the problem which we study.

II. PROBLEM STATEMENT

The dynamical behavior of the neural network we consider is assumed to be governed by the following system of ordinary differential equations:

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij})) + \int_{t-a}^{t} d_{ij} f_j(x_j(s))ds + u_i, i = 1, 2, \ldots, n.$$ (1)

Where $n$ is the number of the neurons, $x_i(t)$ denotes the state of the neuron $i$ at time $t$, $f_j(\cdot)$ denotes activation functions, $a_{ij}$, $b_{ij}$ and $d_{ij}$ denote the strengths of connectivity between neurons $j$ and $i$, $\tau_{ij}$ and $\sigma$ represent the time delay in transmitting a signal from the neuron $i$, $u_i$ is the constant input to the neuron $i$, $c_i$ is the charging rate for the neuron $i$.

In order to completely characterize the equilibrium and stability properties of the neural network defined by (1), it will be assumed that the parameters $a_{ij}$ and $b_{ij}$ and $c_i$ and $d_{ij}$ of neural system (1) are uncertain and have bounded norms with being defined in the following intervals:

$$C I := \{C = diag(c_i) : 0 < C < \mathcal{C}, \text{ i.e., } 0 < c_i \leq c_i \leq \mathcal{C}, i = 1, 2, \ldots, n \}$$
$$A I := \{A = (a_{ij})_{n \times n} : 0 < A \leq \mathcal{A}, \text{ i.e., } 0 < a_{ij} \leq a_{ij} \leq \mathcal{A}, i, j = 1, 2, \ldots, n \}$$
$$B I := \{B = (b_{ij})_{n \times n} : 0 < B \leq \mathcal{B}, \text{ i.e., } 0 < b_{ij} \leq b_{ij} \leq \mathcal{B}, i, j = 1, 2, \ldots, n \}$$
$$D I := \{D = (d_{ij})_{n \times n} : 0 < D \leq \mathcal{D}, \text{ i.e., } 0 < d_{ij} \leq d_{ij} \leq \mathcal{D}, i, j = 1, 2, \ldots, n \}.$$ (2)

In order to achieve the task of finding conditions that ensure robust stability of neural network (1), we need to prove that the conditions to be obtained must guarantee that the unique equilibrium point of system (1) is globally asymptotically stable for all $C \in C I, A \in A I, B \in B I, D \in D I$. Therefore, our main goal will be studying the dynamical analysis of neural network (1) under the parameter uncertainties defined by (2).
We will assume that the functions \( f_i \) are Lipschitz conditions satisfying
\[
|f_i(x) - f_i(y)| \leq \ell_i |x - y|, \quad i = 1, 2, \ldots, n, \forall x, y \in R, x \neq y.
\]
Where \( \ell_i > 0 \) denotes a Lipschitz constant. This class of function is denote by \( f \in \mathcal{L} \).

In order to obtain our robust stability result, we will need to make use of some commonly used vector and matrix norms. Let \( v = (v_1, v_2, \ldots, v_n)^T \) be a vector of dimension \( n \) and \( Q = (q_{ij})_{n \times n} \) be a real \( n \times n \) matrix. For \( v = (v_1, v_2, \ldots, v_n)^T \) and \( Q = (q_{ij})_{n \times n} \), \( |v| \) will be denote \( v = (|v_1|, |v_2|, \ldots, |v_n|)^T \) and \( |S| \) will be denote \( Q = (|q_{ij}|)_{n \times n} \). Then, we consider the following norms:
\[
\begin{align*}
\|v\|_1 &= \sum_{i=1}^{n} |v_i|, \\
\|v\|_2 &= \left\{ \sum_{i=1}^{n} v_i^2 \right\}^{\frac{1}{2}}, \\
\|v\|_{\infty} &= \max_{1 \leq i \leq n} |v_i|, \\
\|Q\| &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |q_{ij}|, \\
\|Q\|_2 &= \left\{ \sum_{i=1}^{n} |q_{ij}|^2 \right\}^{\frac{1}{2}}, \\
\|Q\|_{\infty} &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |q_{ij}|.
\end{align*}
\]

Lemma 1 [2]. If the map \( H(x) \in C^0 \) satisfies the following conditions:
\[
\begin{align*}
(i) & H(x) \neq H(y) \text{ for all } x \neq y, \\
(ii) & H(x) \to \infty \text{ as } H(x) \to \infty,
\end{align*}
\]
then, \( H(x) \) is homeomorphic of \( R^n \).

Lemma 2 [3]. Let \( A \) be any real matrix such that \( A \in A_1 \) with \( A_1 \) being defined as
\[
A_1 := \{ A = (a_{ij})_{n \times n} : 0 < A \leq \overline{A}, \text{ i.e.} 0 < a_{ij} \leq \overline{a}_{ij}, i, j = 1, 2, \ldots, n \}
\]
Then, the following inequality holds:
\[
\|A\|_2 \leq \|A\|_{\infty} \leq \|A\|\leq \overline{A},
\]
where \( \overline{A} = (\overline{a}_{ij})_{n \times n} \) with \( \overline{a}_{ij} = \max \{|a_{ij}|, |a_{ji}|\} \).

Lemma 3 [4]. Let \( A \) be any real matrix such that \( A \in A_1 \) with \( A_1 \) being defined as
\[
A_1 := \{ A = (a_{ij})_{n \times n} : 0 < A \leq \overline{A}, \text{ i.e.} 0 < a_{ij} \leq \overline{a}_{ij}, i, j = 1, 2, \ldots, n \}
\]
Then, the following inequality holds:
\[
\|A\|_2 \leq \sqrt{\|A^*\|_2 + \|A_+\|_2 + 2\|A^T\|_2\|A^*\|_2}
\]
where \( A^* = \frac{1}{2}(\overline{A} + A), A_+ = \frac{1}{2}(\overline{A} - A) \).

Lemma 4 [4]. Let \( A \) be any real matrix such that \( A \in A_1 \) with \( A_1 \) being defined as
\[
A_1 := \{ A = (a_{ij})_{n \times n} : 0 < A \leq \overline{A}, \text{ i.e.} 0 < a_{ij} \leq \overline{a}_{ij}, i, j = 1, 2, \ldots, n \}
\]
Then, the following inequality holds:
\[
\|A\|_2 \leq \|A^*\|_2 + \|A_+\|_2
\]
where \( A^* = \frac{1}{2}(\overline{A} + A), A_+ = \frac{1}{2}(\overline{A} - A) \).

Lemma 1 is very important for the prove of the existence and uniqueness of equilibrium point of the neural network defined by (1). And others define different upper bound norms for the interval matrices, which will play key roles in the following proofs.

III. MAIN RESULT

Now, we can studying the robust stability of the equilibrium point defined by (1).

1. Existence and uniqueness of equilibrium point.

Theorem 1. For the neural network defined by (1), assume that the network parameters satisfy (2) and \( f \in \mathcal{L} \). Then, the neural network model (1) has a unique equilibrium point for every input vector \( u = [u_1, u_2, \ldots, u_n]^T \), if the following condition holds:
\[
\varepsilon = c_m - \ell_M \|Q\|_2 - \ell_M \sqrt{\|B\|_1 \|B\|_{\infty}} - \lambda_{max}(D^*) > 0
\]
where \( c_m = \min(e_0), \ell_M = \max(\ell_i), \|Q\|_2 = \min{\|A^*\|_2 + \|A_+\|_2 + \|Bf(x - f(y)) + D\sigma f(x) + f(y)\|_2}}, \)
where \( A^* = \frac{1}{2}(\overline{A} + A), A_+ = \frac{1}{2}(\overline{A} - A), D = \frac{1}{2}(\overline{A} + A), \lambda = \max\{|a_{ij}|, |a_{ji}|\} \), and \( \lambda_{max} = \max\{|q_{ij}|, |q_{ji}|\}, i = 1, 2, \ldots, n \).

Proof: In order to proceed with proof of the existence and uniqueness, we consider the following mapping associated with system (1).
\[
H(x) = -Cx + Af(x) + Bf(x) + Df(x) + u(x)
\]
Next, we multiply both sides by \((x - y)^T\), we get
\[
(x - y)^T(H(x) - H(y)) = -(x - y)^TC(x - y) + (x - y)^TBf(x - f(y)) + (x - y)^TD\sigma f(x - f(y))
\]
We can write the following inequalities
\[
\sum_{i=1}^{n} c_i(x_i - y_i)^2 \leq \sum_{i=1}^{n} q_i(x_i - y_i)^2 \leq \sum_{i=1}^{n} c_i(x_i - y_i)^2
\]
We also get that
\[
(x - y)^Tf(x - f(y)) \leq \ell_M \|A\|_2 \|x - y\|_2 + \ell_M \|Q\|_2 \|x - y\|_2
\]
\[
\begin{align*}
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} (x_i - y_i) (f_j(x_j) - f_j(y_j)) & \\
\leq & \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_i - y_i| |f_j(x_j) - f_j(y_j)| \\
\leq & \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \|x_i - y_i\|_1 \\
\leq & \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \left( \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \right)^2 \\
\leq & \frac{1}{2} \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \sqrt{\frac{\|b\|_1}{\|b\|_\infty}} (x_i - y_i)^2 \\
+ & \frac{1}{2} \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \sqrt{\frac{\|b\|_1}{\|b\|_\infty}} (x_i - y_i)^2 \\
\leq & \frac{1}{2} \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \sqrt{\frac{\|b\|_1}{\|b\|_\infty}} (x_i - y_i)^2 \\
\leq & \frac{1}{2} \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \sqrt{\frac{\|b\|_1}{\|b\|_\infty}} (x_i - y_i)^2 \\
\leq & \frac{1}{2} \ell M \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \sqrt{\frac{\|b\|_1}{\|b\|_\infty}} (x_i - y_i)^2 \\
\leq & \frac{1}{2} \ell M \sqrt{||B||_1 ||B||_\infty} \| x - y \|_2 \quad (10)
\end{align*}
\]

Then, using (5)-(8) in (4) yields
\[
(x - y)^T (H(x) - H(y)) \leq -c_m \| x - y \|_2^2 + \ell M \| Q \|_2 \| x - y \|_2 + \ell M \sqrt{||B||_1 \|B\|_\infty} \| x - y \|_2 = -c_m \ell M \| Q \|_2 \| x - y \|_2 + \ell M \sqrt{||B||_1 \|B\|_\infty} \| x - y \|_2
\]

Thus, from which we can conclude that if \( x \neq y \), then \( H(x) \neq H(y) \).

Let \( y = 0 \) in (9), we have
\[
(x - y)^T (H(x) - H(0)) \leq -\epsilon \| x \|_2^2
\]

Taking the absolute value of both sides, we have
\[
|x^T (H(x) - H(0))| \geq \epsilon \| x \|_2^2
\]

Since,
\[
| x^T (H(x) - H(0)) | \leq \| x \|_\infty \| H(x) - H(0) \|_1
\]

\[
\| x \|_\infty \| H(x) - H(0) \|_1 \geq \epsilon \| x \|_2^2
\]

And, \( \| x \|_\infty \leq \| x \|_2 \), so
\[
\| H(x) - H(0) \|_1 \geq \epsilon \| x \|_2
\]

Then, we have
\[
\| H(x) \|_1 + \| H(0) \|_1 \geq \epsilon \| x \|_2
\]

\[
\| H(x) \|_1 \geq \epsilon \| x \|_2 - \| H(0) \|_1
\]

Since, \( \| H(0) \|_1 \) is finite, we can get the conclusion that if \( \| x \|_\infty \rightarrow \infty \), that \( \| H(x) \|_1 \rightarrow \infty \). By Lemma 1, it is easy to know that \( H(x) \) is homeomorphism of \( R^n \). So, we have completed the proof of the existence and uniqueness of the equilibrium point for the neural networks defined by (1).

2. Stability of equilibrium point.

In the section 2, we have completed the proof of the existence and uniqueness of the equilibrium point for the networks defined by (1). Next, we will prove stability of equilibrium point for the system (1). We will first simplify system (1) as considering the stability of \( x^* \) point for the system (1). We will first simplify system (1) as considering the stability of \( \hat{x}(t) \) and \( \hat{y}(t) \) for the system (1). We will first simplify system (1) as considering the stability of \( \hat{x}(t) \) and \( \hat{y}(t) \) for the system (1). We will first simplify system (1) as considering the stability of \( \hat{x}(t) \) and \( \hat{y}(t) \) for the system (1). We will first simplify system (1) as considering the stability of \( \hat{x}(t) \) and \( \hat{y}(t) \) for the system (1). We will first simplify system (1) as considering the stability of \( \hat{x}(t) \) and \( \hat{y}(t) \) for the system (1).
system (10) is obtained as follows:

\[
V_1(z(t)) = -\sum_{i=1}^{n} c_i z_i^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i(t) g_j(z_j(t)) + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} z_i(t) g_j(z_j(t) - \tau_{ij}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} z_i(t) f_{i-\sigma}^t g_j(z_j(s)) ds + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_M |b_{ij} - \beta z_i^2(t) - z_j^2(t) - \tau_{ij}| \]

(11)

Then, we can write the following inequalities:

\[
-\sum_{i=1}^{n} c_i z_i^2(t) \leq -\sum_{i=1}^{n} c_i z_i^2(t) = -\sum_{i=1}^{n} c_i \| z(t) \|_2^2
\]

(12)

and

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i(t) g_j(z_j(t)) = z^T(t) A g(z(t)) \leq \|z(t)\|_2 \|A_2\| \| g(z(t)) \|_2 \\
\leq \ell_M \|A_2\| \|z(t)\|_2^2
\]

(13)

and

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} z_i(t) g_j(z_j(t) - \tau_{ij}) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} z_i(t) \| g_j(z_j(t) - \tau_{ij}) \| \\
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_M |b_{ij}| |z_j(t) - \tau_{ij}| \| z_j(t) - \tau_{ij} \| \\
\leq \frac{1}{2} \ell_M \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \| \beta z_i^2(t) - z_j^2(t) + \frac{1}{2} z_j^2(t) \|
\]

Thus, we can get

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} z_i(t) g_j(z_j(t) - \tau_{ij}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_M |b_{ij}| \| \frac{1}{2} \beta z_i^2(t) - z_j^2(t) + \frac{1}{2} z_j^2(t) \|
\]

(14)

Let \( \beta = \sqrt{\frac{\parallel M \parallel_2}{\parallel M \parallel_1}} \), we have

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} z_i(t) g_j(z_j(t) - \tau_{ij}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_M |b_{ij}| \| \beta z_i^2(t) - z_j^2(t) + \frac{1}{2} z_j^2(t) \|
\]

\[
\leq \ell_M \sqrt{\| B \|_1 \| B \|_\infty} \|z(t)\|_2^2
\]

From which we can observe that \( \dot{V}(z(t)) < 0 \), for all \( z(t) \neq 0 \). Assume that \( z(t) = 0 \), we can get \( \dot{V}(z(t)) < 0 \) easily. If \( z(t) = 0 \) and \( z_j(t) = \tau_{ij} = 0 \) for all \( i, j \), implying that \( \dot{V}(z(t)) = 0 \), and \( V(z(t)) \) is 0 otherwise. It is easy to prove that \( V(z(t)) \) is radially unbound since \( V(z(t)) \to \infty \) as \( \|z(t)\| \to \infty \). So, from Lyapunov theorems, we can conclude that the system (3) or equivalently the equilibrium point of system (1) is globally asymptotically stable.

IV. EXAMPLES

Example 1. According to the model (1), giving the following matrices:

\[
A = \begin{bmatrix}
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
\end{bmatrix}, \quad \bar{A} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 2 & 2 & 4 \\
1 & 2 & 2 & 4 \\
1 & 2 & 2 & 4 \\
1 & 2 & 2 & 4 \\
\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
-1 & -2 & -2 & -4 \\
-1 & -2 & -2 & -4 \\
-1 & -2 & -2 & -4 \\
-1 & -2 & -2 & -4 \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1, \quad c_1 = c_2 = c_3 = c_4 = 1, \quad \sigma = 1.
\]

We can calculate the follow matrices:

\[
A^* = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad A_0 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
We can calculate the following matrices:

\[ A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 & 2 & 4 \\
1 & 2 & 2 & 4 \\
1 & 2 & 2 & 4 \\
1 & 2 & 2 & 4
\end{bmatrix}, \]

\[ D^* = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \]

from which we can directly calculate the norms: \( \| A^* \|_2 = 4 \), \( \| A_2 \|_2 = 4 \), \( \| A^*_2 \|_2 + \| \hat{A}_r \|_2 + \| A^*_2 \|_2 = 4 \), \( \| \hat{A} \|_2 = 4 \). So \( \| q \|_2 = 3 \). We can also get that \( \| B \|_1 = 16 \), \( \| \hat{B} \|_\infty = 4 \), \( \lambda_{\text{max}}(D^*) = 4 \).

Firstly, we apply the result of Theorem 1 to the neural network employing the network parameters of this example, we get

\[ \varepsilon = c_m - \| \hat{M} \|_\infty - \lambda_{\text{max}}(D^*) = c_m - 11 > 0 \]

Hence, \( c_m > 11 \) proved to be a sufficient condition for robust stability of the neural network parameters of this example.

V. CONCLUSION

In this letter, a improved delay-dependent global robust exponential stability criterion for uncertain stochastic discrete-time neural networks with time-varying delay is proposed. A suitable Lyapunov functional has been proposed to derive some less conservative delay-dependent stability criteria by using the free-weighting matrices method and the convex combination theorem. Finally, two numerical examples have been given to demonstrate the effectiveness of the proposed method.

REFERENCES

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