The symmetric solutions for boundary value problems of second-order singular differential equation

Li Xiguang

Abstract—In this paper, by constructing a special operator and using fixed point index theorem of cone, we get the sufficient conditions for symmetric positive solution of a class of nonlinear singular boundary value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Banach space, cone, fixed point index, singular differential equation, p-Laplace operator, symmetric solutions.

1. INTRODUCTION

The boundary value problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-Newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method (see [1-4]). On the other hand, the study for the symmetric and multiple solutions to this problem is more and more active (see [5-6]). In paper [5], Sun study the problem

\[ \begin{cases} (u')'' + a(t)(f(t, uu(t)) = 0, t \in (0, 1) \\ (u(0) = \alpha u(\eta) = u(1), \end{cases} \]

where \( \alpha \in (0, 1), \eta \in (0, \frac{1}{2}) \), by using spectrum theory, Sun get the existence of symmetric and multiple solution. But when \( p \neq 2 \), \( \phi_p(u) \) is nonlinear, so the method of the paper [5] is not suitable to p-Laplace operator. In paper [6], Tian and Liu study the problem

\[ \begin{cases} (\phi_p(u'))' + a(t)(f(t, uu(t)) = 0, t \in (0, 1) \\ (u(0) = \alpha u(\eta) = u(1), \end{cases} \]

where \( \phi(s) \) is p-Laplace operator. Motivated by paper [5,6], we consider the existence of solution for the following problems:

\[ \begin{cases} (\phi_p(u'))' + h_1(t)f(u, v) = 0, \\ (\phi_p(v'))' + h_2(t)g(u) = 0, \\ (u(0) = \gamma u(\eta) = u(1), \\ v(0) = \gamma v(\eta) = v(1), \end{cases} \]

where \( t \in (0, 1), \gamma \in (0, 1), \eta \in (0, \frac{1}{2}], \phi(s) \) is a p-Laplace operator, i.e. \( \phi_p(s) = |s|^{p'-2}s, p > 1 \). Obviously, if \( \frac{1}{p} + \frac{1}{q} = 1 \), then \( (\phi_p)^{-1} = \phi_q \).

Compare with above paper, our method is different. By constructing a new operator, and using fixed point index theorem, we get the sufficient condition of the existence of symmetric solution, which improved and generalized the result of paper [5,6,7].

In this paper, we always suppose that the following conditions hold:

\[ \begin{align*} (H_1) & f \in C([0, +\infty) \times [0, +\infty), [0, +\infty)), g \in C([0, +\infty), [0, +\infty)). \\ (H_2) & h_i \in C((0, 1), [0, +\infty)), h_i(t) = h_i(1 - t), t \in (0, 1), for any subinterval of (0, 1), h_i(t) \neq 0, and \\ & \int_0^1 h_i(t)dt < +\infty(i = 1, 2). \\ (H_3) & There exists \alpha \in (0, 1], such that \liminf_{u \to +\infty} \frac{g(u)}{u} = +\infty and \liminf_{v \to +\infty} \frac{f(u, v)}{v^{(p-1)/p}} > 0 hold uniformly to u \in R^+. \\ (H_4) & There exists \beta \in (0, +\infty), such that \limsup_{u \to -0^+} \frac{g(u)}{u} = 0 and \limsup_{v \to -0^+} \frac{f(u, v)}{v^{(p-1)/p}} < +\infty hold uniformly to u \in R^+. \\ (H_5) & There exists n \in (0, 1], such that \liminf_{u \to -0^+} \frac{g(u)}{u} = +\infty and \liminf_{v \to -0^+} \frac{f(u, v)}{v^{(p-1)/p}} > 0 hold uniformly to u \in R^+. \\ (H_6) & f(u, v) and g(u) are nondecreasing with respect to u and v, and there exists R > 0, such that \\ & \frac{\gamma}{1 - \gamma} \int_0^\frac{\gamma}{1 - \gamma} \phi_p(k_1(s))dsf(R, \gamma) - \int_0^\frac{\gamma}{1 - \gamma} \phi_p(k_1(s))ds \times g(R) < R, where k_1(s) = \int_s^{s+1} h_i(\tau)d\tau, i = 1, 2. \end{align*} \]

For convenience, we list the following definitions and lemmas:

**Definition 1.1** If \( u(t) = u(1 - t), t \in [0, 1] \), we call \( u(t) \) is symmetric in \([0, 1]\).

**Definition 1.2** If \((u, v)\) is a positive solution of problem (1), and \( u, v \) is symmetric in \([0, 1]\), we call \((u, v)\) is symmetric positive solution of problem (1).

**Definition 1.3** If \( u(\lambda_1 + (1 - \lambda)t_2) \geq \lambda u(t_1) + (1 - \lambda)u(t_2) \), we call \( u(t) \) is concave in \([0, 1]\).

Let \( E = C[0, 1] \), define the norm \( ||u|| = \max_{t \in [0, 1]} |u(t)| \), obviously \((E, ||||)\) is a Banach space.

Let \( K = \{ u \in E | u(t) > 0, u(t) \) is a symmetric concave function, \( t \in [0, 1] \}, then \( K \) is a cone in \( E \). By \((H_1), (H_2)\), the solution of problem (1) is equivalent to the solution of system of equation (2).
\[
\begin{align*}
\phi(t) &= \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
&\quad \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
&\quad \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau,
\end{align*}
\]

where
\[
\begin{align*}
v(t) &= \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
&\quad \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
&\quad \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau.
\end{align*}
\]

Obviously \( Tu \in E \), it is easy to show if \( T \) has fixed point \( u \), then by (4), problem (1) has a solution \( u, v \).

**Lemma 1.1** Let \((H_1), (H_2)\), then \( T : K \rightarrow K \) is completely continuous.

**Proof** \( \forall u \in K \), by \((H_1), (H_2)\), we can get \((Tu)(t) \geq 0, t \in [0, 1]\).

\[
v'(t) = \begin{cases} 
\phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau), & 0 \leq t \leq \frac{1}{2} \\
-\phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau), & \frac{1}{2} \leq t \leq 1,
\end{cases}
\]
correspondingly \((\phi_{p}(v')) = -h_{2}(t) g(u) \leq 0, 0 < t < 1\), so \( v \) is concave in \([0, 1]\).

Next we show \( v \) is symmetric in \([0, 1]\).

When \( t \in \left[ \frac{1}{2}, 1 \right] \), so
\[
v(1 - t) = \int_{1-t}^{1} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau = v(t).
\]

Similarly, we have \( (Tu)'(t) \) is symmetric in \([0, 1]\).

\[
(Tu)'(t) = \begin{cases} 
\phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau), & 0 \leq t \leq \frac{1}{2} \\
-\phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau), & \frac{1}{2} \leq t \leq 1,
\end{cases}
\]
so \((\phi_{p}(Tu))' = -h_{2}(t) f(u, v) \leq 0, 0 < t < 1\), i.e. \( Tu \) is concave in \([0, 1]\).

Next we show \( Tu \) is symmetric in \([0, 1]\), when \( t \in \left[ \frac{1}{2}, 1 \right], 0 < t \leq 1\), so
\[
(Tu)(1 - t) = \int_{1-t}^{1} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau = (Tu)(t).
\]

Similarly, we have \((Tu)(1 - t) = (Tu)(t) \), so \( Tu \) is concave in \([0, 1]\), so \( TK \subset K \). On the other hand, let \( D \) is a arbitrary bounded set of \( K \), then there exist constant \( c > 0 \), such that \( D \subset \{ u \in K ||u|| \leq c \} \). Let \( b = \max_{u \in [a, c]} g(u) \), so \( \forall u \in D \), we have
\[
||v|| = \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau.
\]
\[
\begin{align*}
\int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau &\leq c \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau \leq a. \\
\end{align*}
\]

Let \( L = \max_{u \in [a, c]} f(u, v) \), so \( \forall u \in D \), we have
\[
||Tu|| = \frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau + \\
\frac{1}{\gamma} \int_{0}^{t} \phi_{q}(\frac{1}{\gamma} h_{2}(\tau) g(\phi(\tau)) d\tau) d\tau.
\]
\[ \|Tu\| = \max \{ \|\phi_q(\int_0^\frac{1}{2} h_1(\tau)f(u(\tau),v(\tau))d\tau)\|, \|\phi_q(\int_1^\frac{1}{2} h_1(\tau)f(u(\tau),v(\tau))d\tau)\| \} \]
\[ \leq L^{n-1} \phi_q(\int_0^\frac{1}{2} h_1(\tau)d\tau). \]

By Arzela-Ascoli theorem, we know \( TD \) is compact set. By Lebesgue dominated convergence theorem, it is easy to show \( T \) is continuous in \( K \), so \( T : K \rightarrow K \) is completely continuous.

**Lemma 1.3.** Let \( K \) be a cone of \( E \) in Banach space, \( \Omega_1 \) and \( \Omega_2 \) are open subsets in \( E \), \( \theta \in \Omega_1, \Omega_2 \subset \Omega_2 \), and \( T : K \cap (\Omega_2 \setminus \Omega_1) \rightarrow K \) is a completely continuous operator, and satisfy one of the following conditions:

1. \( \|Tx\| \leq \|x\|, \forall x \in K \cap \partial \Omega_1, \|Tx\| \geq x, \forall x \in K \cap \Omega_2. \)
2. \( \|Tx\| \geq x, \forall x \in K \cap \partial \Omega_1, \|Tx\| \leq x, \forall x \in K \cap \Omega_2. \)

Then \( A \) has at least one fixed point in \( K \cap (\Omega_2 \setminus \Omega_1). \)

**Lemma 1.4.** Let \( K \) be a cone of \( E \) in Banach space, \( K_r = \{ x \in K \mid x \leq r \} \), suppose \( A : K_r \rightarrow K \) is a completely continuous, and satisfy \( Tx \neq x, \forall x \in K_r, \)

1. If \( \|Tx\| \leq x, \forall x \in \partial K_r, \) then \( i(T, K_r, K) = 1. \)
2. If \( \|Tx\| \geq x, \forall x \in K_r, \) then \( i(T, K_r, K) = 0. \)

II. Conclusion

**Theorem 2.1.** Suppose \((H_1)-(H_4)\) hold, then problem (1) has at least one positive solution.

**Proof.** By \((H_3)\), there exist \( \nu \) and a sufficient large number \( M > 0, \) such that

\[ f(u, v) \geq \nu^{p-1} v^{(p-1)\alpha}, \forall u \in R^+, v > M, \]

\[ g(u) \geq C_0^{p-1} u^{\frac{\alpha}{p-1}}, \forall u > M, \]

where \( C_0 = \max \{ \frac{1}{\nu}, \frac{1}{\nu^{p-1}} \} \)

\( L^{n-1} \phi_q(\int_0^\frac{1}{2} h_1(\tau)d\tau). \)

By Lemma 2, \( \min_{t \in [\epsilon, 1-\epsilon]} u(t) \geq e^2 \|u\| = e^2 N = M + 1, \) by (3)-(6) and the symmetric property, for any \( t \in [\epsilon, 1-\epsilon] \)

\( v(t) = \int_0^t \phi_q(\int_0^s h_2(\tau)(u(\tau),v(\tau))d\tau)ds, \)

so \( \|Tu\| > \|u\|, \forall u \in K \cap K_N, \) by lemma 1.4, we can get

\[ i(T, K \cap K_N, K) = 0. \]

On the other hand, by the second limit of \( H_4, \) there exists a sufficient small number \( r_1 \in (0, 1) \) such that

\( C_1^{p-1} = \sup_{\|u\| \leq r_1, v(0) \in (0, r_1)} \int_0^\frac{1}{2} \phi_q(\int_0^s h_2(\tau)(u(\tau),v(\tau))d\tau)ds < +\infty. \)

Let \( \epsilon = \min \{ \frac{r_1(1-\gamma)}{\nu^{p-1}}, \frac{1}{\nu^{p-1}} \} \)

\( \int_0^\frac{1}{2} \phi_q(\int_0^s h_1(\tau)(u(\tau),v(\tau))d\tau)ds \)

\( \frac{C_1(1-\gamma)}{\nu^{p-1}} \int_0^\frac{1}{2} \phi_q(\int_0^s h_1(\tau)(u(\tau),v(\tau))d\tau)ds < +\infty, \) by the first limit of \( H_4, \) there exist a sufficient small number \( r_2 \in (0, 1) \) such that

\[ g(u) \leq e^{p-1} u^{\frac{\alpha}{p-1}}, \forall u \in [0, r_2]. \]
Take $r = \min\{r_1, r_2\}$, by (9), we can get
\[
v(t) = \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds + \int_0^t \phi_q(\int_s^t h_1(\tau)f(u(\tau), v(\tau))d\tau)ds + \int_0^t \phi_q(\int_s^t h_1(\tau)d\tau)ds
\]
\[
\leq \varepsilon^{-\frac{1}{q}}\int_0^t \phi_q(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds
\]
\[
\leq \varepsilon^{-\frac{1}{q}}\int_0^t \phi_q(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds
\]
\[
\leq \frac{\varepsilon}{1+\frac{1}{q}} < r_1, \forall u \in K \cap \partial K_r, s \in [0, 1].
\]
By (8), we can get
\[
||Tu|| \leq \int_0^t \phi_q(\int_s^t h_1(\tau)f(u(\tau), v(\tau))d\tau)ds + \int_0^t \phi_q(\int_s^t h_1(\tau)d\tau)ds
\]
\[
\leq \varepsilon^{-\frac{1}{q}}\int_0^t \phi_q(\int_s^t h_1(\tau)f(u(\tau), v(\tau))d\tau)ds
\]
\[
\leq \frac{\varepsilon}{1+\frac{1}{q}} < r_1, \forall u \in K \cap \partial K_r, t \in [0, 1].
\]
So $||Tu|| \leq ||u||, \forall u \in K \cap \partial K_r$, by lemma 1.4, we get
\[
i(T, K \cap K_r, K) = 1.
\]
By lemma 1.5, $T$ has at least one fixed point in $K \cap (K_N \setminus K_r)$, so problem (1) has at least a system positive solution.

**Theorem 2.2** Suppose (H1), (H2), (H3), (H4), (H6) hold, then problem (1) has at least two systems positive solutions.

**Proof** By (H2), there exists $\mu > 0$ and a sufficient small number $\xi \in (0, 1)$, such that
\[
f(u, v) \geq \mu^{p-1}v^{\nu(p-1)}, \forall u \in R^+, 0 \leq v \leq \xi,
\]
\[
g(u) \geq (C_2u)^{\frac{p-1}{\mu}}, \forall 0 \leq u \leq \xi,
\]
where
\[
C_2 = 2\left(\frac{\mu^2(1-q)}{1-q}\right)^{\frac{1}{q}} \int_0^\xi \phi_q(1_k(s))ds \int_0^\xi (\phi_q(1_k(s)))^\mu ds
\]
since $g \in C(R^+, R^+), g(0) \equiv 0$, so there exists $\sigma \in (0, \xi)$ such that $\forall u \in [0, \sigma]$, we have
\[
g(u) \leq \left(\frac{1}{1-\gamma}\right)\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds^{-1},
\]
this imply
\[
v(t) \leq \frac{1}{1-\gamma}\int_0^t \phi_q(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds
\]
\[
\leq \xi, \forall u \in K \cap \partial K_r.
\]
By using Jensen inequality, $0 < q \leq 1$, and (11)-(13), we can get
\[
(Tu)(\frac{1}{q}) \geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
||Tu|| \geq ||u||, \forall u \in K \cap \partial K_r.
\]
By (7),(14), we have
\[
\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
||u||, \forall u \in K \cap \partial K_r.
\]
So for any $u \in K \cap K_r$, by lemma 1.4, we can get
\[
i(T, K \cap K_r, K) = 0.
\]
We can choose $N > R > \sigma$, such that (7),(14) hold together. On the other hand by (3),(4) and $H_6$ we can get
\[
(Tu)(t) \leq \frac{1}{1-\gamma}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
\geq \frac{\mu^q}{1-q}\int_0^\xi \phi_q(\int_s^\xi h_1(\tau)d\tau)ds
\]
\[
||u||, \forall u \in K \cap \partial K_r.
\]
So for any $u \in K \cap K_r$, by lemma 1.4, we can get
\[
i(T, K \cap K_r, K) = 1.
\]
By (7),(14),(15), we have
\[
i(T, K \cap K_r, K) = 1
\]
\[
i(T, K \cap K_r, K) = -1.
\]
So $T$ have at least two fixed points in $K \cap (K_N \setminus K_r)$ and $K \cap (K_R \setminus K_r)$, by (4), problem (1) has at least two system solutions.

**REFERENCES**


