A new approximate procedure based on He’s variational iteration method for solving nonlinear hyperbolic wave equations

Jinfeng Wang, Yang Liu and Hong Li

Abstract—In this article, we propose a new approximate procedure based on He’s variational iteration method for solving nonlinear hyperbolic equations. We introduce two transformations \( q = u_t \) and \( \sigma = u_x \) and formulate a first-order system of equations. We can obtain the approximation solution for the scalar unknown \( u \), time derivative \( q = u_t \) and space derivative \( \sigma = u_x \), simultaneously. Finally, some examples are provided to illustrate the effectiveness of our method.

Keywords—Hyperbolic wave equation; Nonlinear; He’s variational iteration method; Transformations

I. INTRODUCTION

In this article, we consider the following hyperbolic wave equation

\[
u_{tt} - u_{xx} + \mathfrak{N}u = f(x, t),
\]

where \( \mathfrak{N} \) is a nonlinear operator.

The hyperbolic wave equations are a high-order partial differential equations with mixed partial derivative with respect to time and space, which describe heat and mass transfer, reaction diffusion and nerve conduction, and other physical phenomena. In recent years, a lot of researchers have studied and proposed many numerical methods for second-order hyperbolic wave equations, such as finite element methods [1], [2], [3], mixed finite element methods [4], [5], [6], [7], [8], [9], [10], [11], the reduced finite volume element formulation based on POD method [12], He’s variational iteration method [13], [14], [15], [16], [17], [18], [19], He’s homotopy perturbation method [16] and Adomian decomposition method [20], [21].

In 1997, He [22] proposed the variational iteration method (VIM) for some nonlinear partial differential equations. From then on, He’s VIM has been applied to solve many linear and nonlinear differential equations [23], [24], [25], [26], [27], [28]. In [13], [14], [15], [16], [17], [18], [19], He’s variational iteration method were studied and analyzed for second-order hyperbolic wave equations.

In this article, our aim is to propose a new approximate procedure based on He’s variational iteration method (VIM) to find approximate solutions for the second-order hyperbolic equations. We introduce two transformations \( q = u_t \) and \( \sigma = u_x \) and formulate a first-order system of equations, which has three equalities: correction functional, integral equation and differential equation. Our method can obtain the approximation solution for the scalar unknown \( u \), time derivative \( q = u_t \) and space derivative \( \sigma = u_x \), simultaneously.

II. NEW PROCEDURE BASED ON VIM

Introducing the two auxiliary variables

\[
q = u_t \quad \text{and} \quad \sigma = u_x,
\]

the equation (1) can be rewritten as the following first-order system

\[
\begin{align*}
(a) \quad q_t - \sigma_x + \mathfrak{N}u &= f(x, t), \\
(b) \quad u_t - q &= 0, \\
(c) \quad \sigma &= u_x.
\end{align*}
\]

According to the variational iteration method, we can construct the following correction functional for equation (3a)

\[
q_{n+1}(x, t) = q_n(x, t) + \int_0^t \lambda(\xi) \frac{\partial q_n(x, \xi)}{\partial \xi} - \frac{\partial \sigma_n(x, \xi)}{\partial x} + \mathfrak{N}u_n(x, \xi) - f(x, \xi) d\xi,
\]

and the following two equalities

\[
u_{n+1}(x, t) = \int_0^t q_{n+1}(x, \xi) d\xi + u_n(x, 0),
\]

and

\[
\sigma_{n+1}(x, t) = \frac{\partial u_{n+1}(x, t)}{\partial x}.
\]

Choosing the function \( u_0(x, t) \) with functions \( \sigma_0(x, t) = u_{0x}(x, t) \), \( q_0(x, t) = u_{0t}(x, t) \), we can obtain the exact solution by

\[
\begin{align*}
u(x, t) &= \lim_{n \to \infty} u_n(x, t), \\
q(x, t) &= \lim_{n \to \infty} q_n(x, t), \\
\sigma(x, t) &= \lim_{n \to \infty} \sigma_n(x, t).
\end{align*}
\]
III. NUMERICAL EXAMPLE

In this section, we will provide some examples to illustrate the effectiveness of our method.

Example 1: Use the new procedure to solve the second hyperbolic equation with initial and boundary condition

\[
\begin{align*}
&u_{tt} - u_{xx} = 0, \quad 0 < x < \pi, \ t > 0, \\
&u(0, t) = 0, \ u(\pi, t) = 0, \\
&u(x, 0) = 0. \ u_t(x, 0) = \sin x.
\end{align*}
\]

From (4)-(6), we can obtain \( \lambda = -1 \). With the given initial values, we can choose \( u_0(x, t) = t + \sin x, q_0(x, t) = \sin x, \sigma_0(x, t) = \cos x \). Using (4)-(6), we can obtain the following successive approximations

\[
\begin{align*}
&u_0(x, t) = t + \sin x, \\
&q_0(x, t) = 1, \\
&\sigma_0(x, t) = \cos x, \\
&q_1(x, t) = 1 - \frac{t^2}{2}, \\
&u_1(x, t) = t + \sin x - \frac{1}{3!}t^3, \\
&\sigma_1(x, t) = \cos x, \\
&q_2(x, t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!}, \\
&u_2(x, t) = t + \sin x - \frac{1}{3!}t^3 + \frac{1}{5!}t^5, \\
&\sigma_2(x, t) = \cos x, \\
&\ldots.
\end{align*}
\]

Using Taylor series for \( \sin t \) and \( \cos t \) and (7), we obtain the exact solution

\[
\begin{align*}
&q(x, t) = \cos t, \\
&u(x, t) = \sin x + \sin t, \\
&\sigma(x, t) = \cos x.
\end{align*}
\]

Example 3: Use the new procedure to solve the nonlinear inhomogeneous Klein-Gordon equation (20), P380 with initial condition

\[
\begin{align*}
&u_{tt} - u_{xx} - u + u^2 = xt + x^2 t^2, \\
u(x, 0) = 1, \ u_t(x, 0) = x.
\end{align*}
\]

From (4)-(6), we can obtain \( \lambda = -1 \). With the given initial values, we can choose \( u_0(x, t) = 1 + xt, q_0(x, t) = x, \sigma_0(x, t) = t \). Using (4)-(6), we can obtain the following successive approximations

\[
\begin{align*}
&u_0(x, t) = 1 + xt, \\
&q_0(x, t) = x, \\
&\sigma_0(x, t) = t, \\
&q_1(x, t) = x, \\
&u_1(x, t) = 1 + xt, \\
&\sigma_1(x, t) = t.
\end{align*}
\]

From (15), we know that the exact solution

\[
\begin{align*}
&u(x, t) = 1 + xt, \\
&q(x, t) = x, \\
&\sigma(x, t) = t.
\end{align*}
\]
TABLE I
COMPARISON BETWEEN THE EXACT SOLUTION WITH VIM SOLUTION \{u_1(x,t), q_1(x,t), \sigma_1(x,t)\}.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(t)</th>
<th>(u(x,t))</th>
<th>(q(x,t))</th>
<th>(\sigma(x,t))</th>
</tr>
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<tr>
<td>0.2</td>
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<td>0.099335</td>
<td>0.490033</td>
<td>0.172053</td>
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<td></td>
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<td>0.797662</td>
<td>0.194709</td>
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<td>0.17205</td>
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</table>

IV. CONCLUDING REMARKS

In this article, we propose a new approximate procedure based on He’s Variational iteration method for second-order hyperbolic wave equations. We split the hyperbolic wave equation (1) into a first-order system (3) of equations by introducing two transformations \(q = u_t\) and \(\sigma = u_x\) and formulate a new iteration system (4)-(6). Our procedure can obtain the approximation solution for the scalar unknown \(u\), time derivative \(q = u_t\) and space derivative \(\sigma = u_x\), simultaneously. We choose some examples to show the effectiveness of our method.

Fig. 1. The surface of \(u_1(x,t)\)

Fig. 2. The surface of \(u(x,t)\)

Fig. 3. The surface of \(q_1(x,t)\)

Fig. 4. The surface of \(q(x,t)\)

Fig. 5. The surface of \(\sigma_1(x,t)\)
Fig. 6. The surface of $\sigma(x,t)$

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