The positive solution for singular eigenvalue problem of one-dimensional p-Laplace operator

Lv Yuhua

Abstract—In this paper, by constructing a special cone and using fixed point theorem and fixed point index theorem of cone, we get the existence of positive solution for a class of singular eigenvalue value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Cone, fixed point index, eigenvalue problem, p-Laplace operator, positive solutions.

I. INTRODUCTION

The eigenvalue problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method(see[1-9]). In paper [10], Wang and Ge study for the following problem been obtained by a variety of method(see[1-9]). In paper [10], they are widely applied in studying for non-newtonian fluid

\[ \phi(t') = \phi(t) + a(t)f(t,u(t)) = 0, \quad t \in (0,1) \]

By using fixed point theorem of cone, they get the existence of multiple positive solution. Motivated by paper [4,6,10], we consider the following conditions:

\[ (\text{H}_1) \quad h(t) \text{ is nonnegative measurable function in } (0,1), \quad h(t) \text{ may be singular at } t = 0,1, \quad \alpha > 0, \beta > 0, \gamma > 0, \delta > 0, f(x) \text{ is nonnegative continuous function in } (0,+,\infty), \quad f \text{ is sup-linear and sub-linear at } 0 \text{ and } \infty. \]

We first list the following conditions:

\[ (\text{H}_2) \quad \phi(t) \in C([0,+,\infty],[0,+,\infty)) \text{ and } f(0) = 0; \text{ for } u > 0, f(u) > 0; \]

\[ (\text{H}_3) \lim_{x \to 0} \frac{f(x)}{x^{p-1}} = a, \text{ where } a \in [0,+,\infty]; \]

\[ (\text{H}_4) \lim_{x \to +\infty} \frac{f(x)}{x^{p-1}} = +\infty; (f \text{ is sup-linear at } x = +\infty.); \]

\[ (\text{H}_5) \lim_{x \to 0} \frac{f(x)}{x^{p-1}} = 0; (f \text{ is sub-linear at } x = 0). \]

\[ (\text{H}_6) \lim_{x \to +\infty} \frac{f(x)}{x^{p-1}} = 0. (f \text{ is sub-linear at } x = +\infty). \]

For the sake of convenience, we list the following definitions and lemmas:

Definition 1.1 If \( x \in C[0,1] \cap C^1(0,1) \) and satisfy (1), \( \varphi_p(x'(t)) \) is absolutely continuous in \( (0,1) \), \( -(\varphi_p(x'(t)))' = -\lambda h(t)f(x(t)) \) hold almost everywhere in \( (0,1) \), we call \( x \) is positive solution for problem (1).

Definition 1.2 Let \( E \) be a real Banach space, if \( K \) is a nonempty convex closed set in \( E \), and satisfy the following conditions:

\[ (1) \quad x(t) \geq \|x(t)|t(1-t), \forall t \in [0,1]; \]

\[ (2) \quad x(t) \geq \psi(t), \forall t \in [0,1]. \]

Lemma 1.2 Suppose \( H_5, H_6 \) hold, then there exists \( R > 0 \), such that \( \int_{1}^{R} \frac{f(t)}{t^{p-1}} = \max_{t>0} \frac{f(t)}{t^{p-1}} = C^p \).

Lemma 1.3 Suppose \( H_5, H_6 \) hold, then there exists \( L > 0 \), such that \( \int_{1}^{R} \frac{f(t)}{t^{p-1}} = \max_{t>0} \frac{f(t)}{t^{p-1}} = C^p \).

Lemma 1.4 (see [11]) Let \( E \) be Banach space, \( K \) is a cone in \( E \), for \( r > 0 \), we define \( K_r = \{x \in K : \|x\| \leq r\}. \) Suppose \( \sup_{u \in \partial K_r} u = \{x \in K : \|x\| = r\}, \) we have \( Tx \neq x. \)

II. CONCLUSION

Theorem 2.1 If conditions \((H_1), (H_2), (H_3), (H_4)\) hold, and \( a = +\infty. \)

(a) If there exists \( \lambda^* > 0 \) such that \( \frac{f(\cdot)}{\|x\|^{p-1}} + \max \{\frac{\beta}{\alpha}, \frac{\delta}{\gamma}\} \cdot \int_{1}^{\|x\|} \psi(t) \cdot \int_{0}^{\frac{\lambda^*}{t}} h(t)dt \leq 1 \)
where $\psi_p(t) = \|t\|_{\frac{1}{p}} \text{sgn}(t)$ is converse function of $\varphi_p$, $R \in (0, R]$ is the maximum point of $f$ in $(0, R]$, then for $0 < \lambda < \lambda^*$, Problem (1) has two positive solutions $x_1(t), x_2(t)$, and satisfy $0 < \|x_1\| < R < \|x_2\|$.

(b) There exists $\lambda^{**}$, when $\lambda > \lambda^{**}$, the problem (1) has no positive solution.

(a) **Proof.** For any $x \in K$, we have $x'\prime(0) \geq 0, x'(1) \leq 0$, so there exists a constant $\sigma(= \sigma_{x})$ such that $x'(\sigma) = 0$, we define $T_\lambda : K \to E$ as follows

$$(T_\lambda x)(t) = \begin{cases} 
\psi_p(\frac{\beta}{\alpha} \int_0^\sigma \lambda h(r)f(u(r))dr) + \\
\int_0^t \psi_p(\lambda h(r)f(u(r))dr)ds, 0 \leq t \leq \sigma, \\
\psi_p(\frac{\delta}{\gamma} \int_\sigma^1 \lambda h(r)f(u(r))dr) + \\
\int_\sigma^1 \psi_p(\lambda h(r)f(u(r))dr)ds, \sigma \leq t \leq 1,
\end{cases}$$

By the definition of $T_\lambda$, we know $\forall \sigma \in K, T_\lambda x \in C^1[0,1]$ is nonnegative and satisfy the boundary condition, furthermore, $(T_\lambda x)'(t) = \begin{cases} 
\psi_p(\int_0^\sigma \lambda h(r)f(u(r))dr) > 0, \\
0 \leq t \leq \sigma, \\
\psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr) < 0, \\
\sigma \leq t \leq 1,
\end{cases}$

is continuous and non-increasing in $[0,1]$, and $(T_\lambda x)'(\sigma) = 0$, so $(T_\lambda x)(\sigma)$ is the maximum value of $T_\lambda x$ in $[0,1]$. Since $(T_\lambda x)'$ is continuous and non-increasing in $[0,1]$, we have $T_\lambda x \in K$, this imply $T_\lambda K \subseteq K$, furthermore, $-\varphi(T_\lambda x'(t))' = \lambda h(t)f(x(t))$, so the fixed point of $T_\lambda$ in $K$ is solution for problem (1).

Similar to the method of [4,5], we know $T_\lambda : K \to K$ is completely continuous.

By $(H_1), \forall \sigma > 0$, we have $0 < \int_\varepsilon^{1-\varepsilon} h(t)dt < +\infty$, and when $\varepsilon \leq x \leq 1 - \varepsilon$, $y(x) = \int_\varepsilon^x \psi_p(\int_\sigma^1 h(r)dr)ds + \int_\varepsilon^x \psi_p(\int_\sigma^1 h(r)dr)ds$ is nonnegative continuous.

Let $P \equiv \min_{\varepsilon \leq x \leq 1-\varepsilon} y(x) > 0$, by $(H_3)$ and $a = \infty$, i.e. $\lim_{x \to 0^+} \frac{f(x)}{x} = \infty$, we know there exists $0 < r' < R$, such that when $0 \leq x \leq r'$, $f(x) \geq (Mx)^{p-1}$, where $M = 2(\lambda^\frac{1}{p} - \varepsilon^2 P)$, for $x \in \partial K_{r'} = \{x \in K : \|x\| = r'\}$, we have

$$\|T_\lambda x\| \geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr)ds + \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr)ds \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}(\int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 h(r)dr)ds + \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 h(r)dr)ds) \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}y(\sigma) \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}P \geq 2r' = 2\|x\|, \sigma \in [\varepsilon, 1-\varepsilon]$$

$$\|T_\lambda x\| \geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr)ds + \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr)ds \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}(\int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 h(r)dr)ds + \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 h(r)dr)ds) \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}y(\sigma) \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}P \geq 2r' = 2\|x\|, \sigma \in [\varepsilon, 1-\varepsilon]$$

so for $x \in \partial K_{r'}$, we have $\|T_\lambda x\| \geq \|x\|$, by lemma 1.3, $\sigma > 1 - \varepsilon$, $i(T_\lambda, K_{r'}, K) = 0$.

On the other hand, for $x \in \partial K_{r'}$, we have

$$\|T_\lambda x\| \geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr)ds + \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 \lambda h(r)f(u(r))dr)ds \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}(\int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 h(r)dr)ds + \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\sigma^1 h(r)dr)ds) \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}y(\varepsilon) \geq \lambda^\frac{1}{p} M^{\frac{1}{p}} \varepsilon^{2r'}P \geq 2r' = 2\|x\|, \varepsilon \in [\varepsilon, 1-\varepsilon]$$

so for $x \in \partial K_{r'}$, we have $\|T_\lambda x\| \geq \|x\|$, by lemma 1.3, $i(T_\lambda, K_{r'}, K) = 0$. (3)
\[\|T_\lambda x\| \leq \psi(p \int_0^1 \lambda h(r)f(u(r))dr) + \|x\|\]
\[\max \{\psi(p \int_0^1 \lambda h(r)f(u(r))dr), \psi(\delta \int_0^1 \lambda h(r)f(u(r))dr)\}\]
\[\leq \lambda \frac{1}{x} \left(1 + \max \left\{\frac{\beta}{\gamma}, \frac{\delta}{\gamma} \right\}\right) \times \psi(p \int_0^1 \lambda h(r)f(u(r))dr)
\]
\[= \lambda \frac{1}{x} \left(1 + \max \left\{\frac{\beta}{\gamma}, \frac{\delta}{\gamma} \right\}\right) \times \psi(\delta \int_0^1 \lambda h(r)f(u(r))dr)
\]
\[\leq \lambda \frac{1}{x} \left(1 + \max \left\{\frac{\beta}{\gamma}, \frac{\delta}{\gamma} \right\}\right) \times \psi(\delta \int_0^1 \lambda h(r)f(u(r))dr)
\]
\[\leq \lambda \frac{1}{x} \left(1 + \max \left\{\frac{\beta}{\gamma}, \frac{\delta}{\gamma} \right\}\right) \times \psi(\delta \int_0^1 \lambda h(r)f(u(r))dr)
\]
by lemma 1.3.

(i) \(T_\lambda x, K_R, K = 1\).

by (2),(3),(4) and the additivity of fixed point index
\[i(T_\lambda, K_R \setminus \bar{K}) = -1, i(T_\lambda, K_R \setminus \bar{K}_\epsilon) = 1\]

So \(T_\lambda\) has fixed point \(x\) in \(K_R \setminus \bar{K}_\epsilon\) and \(x\) in \(K_R \setminus \bar{K}_\epsilon\).

Next we show \(\lambda_\epsilon > 0\) when \(\lambda > \lambda_\epsilon\), so \(\||T_\lambda x|| = ||x||\).

Since \(x\) satisfy (1), we have

(b) Proof Suppose there exists a subsequence \(\{\lambda_n\}\), and \(\lambda_n > \lambda_\epsilon\) such that for any \(n\), problem (1) has a positive solution \(x_n \in K\), by \(H_3, \forall \rho > 0\), we have \(f(x) \geq Cx^{p-1}\), where
\[C = \frac{1}{\rho}, \text{ when } \sigma \leq \epsilon, \text{ by lemma 1.1, we have}
\]
\[\|x_n\| \geq \int_0^{1-x} \psi(p \int_0^1 \lambda_n h(r)f(u(r))dr) + \|x_n\|\]
\[\geq \lambda_n \frac{1}{x} \int_0^{1-x} \psi(p \int_0^1 h(r)C(u_n)\lambda_n h(r)f(u(r))dr) + \|x_n\|\]
\[\geq (\lambda_n C) \frac{1}{x} \int_0^{1-x} \psi(p \int_0^1 h(r)C(u_n)\lambda_n h(r)f(u(r))dr) + \|x_n\|\]
\[\geq (\lambda_n C) \frac{1}{x} \int_0^{1-x} \psi(p \int_0^1 h(r)C(u_n)\lambda_n h(r)f(u(r))dr) + \|x_n\|\]
so
\[1 \geq nC \frac{2^{(p-1)} \lambda_n h(r)f(u(r))dr) + \|x_n\|}{\lambda_n h(r)f(u(r))dr) + \|x_n\|}\]

Since \(n\) is sufficiently large, so we get a contradiction.

When \(\sigma > 1 - \epsilon\) and \(\sigma \in [\epsilon, 1 - \epsilon]\), we can get the similar result.

So there exists \(\lambda^*_\epsilon\), when \(\lambda > \lambda^*_\epsilon\), problem (1) has no positive solution, the proof is finished.

**Theorem 2.2** If \((H_1), (H_3), (H_4)\) hold , and \(0 < a < +\infty\), if there exists \(\lambda^*_\epsilon > 0\) and \(\lambda^*_\epsilon = \frac{1}{\rho} \left(1 + \max \left\{\frac{\beta}{\gamma}, \frac{\delta}{\gamma} \right\}\right)\)

\(\psi(p \int_0^1 h(t)dt) \leq 1\), where \(\psi(t) = |t| \frac{\rho}{2^{(p-1)}} \text{sgn}(t)\) is converse function of \(\varphi\), so for \(0 < \lambda < \lambda^*_\epsilon\), problem (1) has a positive solution.

Proof Take \(\epsilon > 0\), such that \(\left(\frac{\lambda^*_\epsilon}{\rho} + \max \left\{\frac{\beta}{\gamma}, \frac{\delta}{\gamma} \right\}\right)(a + \epsilon) \frac{\rho}{2^{(p-1)}} \psi(p \int_0^1 h(t)dt) < 1\), by \(H_3, \therefore \exists \eta > 0\) such that when \(0 \leq x < \eta\), \(f(x) \leq Cx^{p-1}(a + \epsilon)\).

So, \(\lambda < \lambda^*_\epsilon\), problem (1) has a positive solution.

By \(H_4, \exists \alpha > 0\), such that when \(x \geq \alpha, f(x) \geq (Mx)^{p-1}\), choose \(\mu > \max \left\{\alpha, \eta\right\}\), by the similar method with theorem 2.1, we can show when \(x \in \partial K_\rho, \||T_\lambda x|| \geq ||x||\), so if we define
\( \Omega_1 = \{ x \in K : \| x \| < \eta \}, \Omega_2 = \{ x \in K : \| x \| < \mu \}, \)

by lemma 1.4, \( T_\lambda \) has at least one fixed point \( x \in K \), and \( \mu > \| x \| > \eta \), the proof is finished.

**Corollary** In condition \( H_3 \), let \( a = 0 \) , then \( \forall \lambda > 0 \), problem (1) has at least one positive solution.

**Theorem 2.3** If \( H_1, H_2, H_5, H_6 \) hold, then

(a)\( \forall \varepsilon \in (0, \frac{1}{2}) \), there exists \( \lambda_\varepsilon = \lambda_\varepsilon (\varepsilon > 0) \), such that for all \( \lambda > \lambda_\varepsilon \), problem (1) has at least two \( x_1, x_2 \) and \( 0 < \| x_1 \| < L < \| x_2 \| \).

(b)\( \text{If there exist } \lambda_{**} > 0 \text{ such that } \lambda_{**} = \max(1 + \max(\frac{\beta}{\alpha}, \frac{\gamma}{\gamma - \alpha}), \frac{1}{\gamma - \alpha}) \int_0^1 h(t)dt < 1, \text{ then for all } \lambda < \lambda_{**}, \text{ problem (1) has no positive solution, where } C' = \frac{C}{\max(\alpha, \gamma - \alpha)} \).

(a) **Proof** For any \( 0 < \varepsilon < \frac{1}{2}, \forall x \in K \) and \( \| x \| = L \),

let \( \lambda = \frac{1}{(e^\varepsilon Q)^{p-1}}, \) where \( Q = \min_{\varepsilon \leq s \leq 1-\varepsilon} y(s) > 0 \), then for \( \lambda > \lambda_{**} \) we have

\[
\| T_\lambda x \| \geq \int_{1-\varepsilon}^1 \psi_p(\int_s^1 \lambda h(r)f(u(r))dr)ds + \int_s^1 \psi_p(\int_s^1 \lambda h(r)f(u(r))dr)ds \\
> (\lambda \nu)\int_{1-\varepsilon}^1 \psi_p(\int_s^1 h(r)dr)ds \\
= (\lambda \nu)\int_{1-\varepsilon}^1 \epsilon^2 QL \\
> 2L = 2\| x \|, \sigma \in [\varepsilon, 1-\varepsilon],
\]

so for \( x \in \partial K_L \), we have \( \| T_\lambda x \| > \| x \| \).

For the same \( \lambda \), choose \( \varepsilon > 0 \) such that \( \epsilon'(\lambda \nu)^{-1} + \max(\frac{\beta}{\alpha}, \frac{\gamma}{\gamma - \alpha})\psi_p(\int_0^1 h(t)dt) < 1 \), by \( H_5 \), there exists \( 0 < l < L \), such that when \( 0 \leq x \leq l \), \( f(x) \leq (\epsilon')^{-p} \),

\[
\| T_\lambda x \| \leq \psi_p\int_0^1 \lambda h(r)f(u(r))drds + \max\{\psi_p\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr, \\
\psi_p\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr\} \\
\leq \lambda \nu \int_0^1 \lambda h(r)f(u(r))dr \\
\leq (\lambda \nu)\int_0^1 \epsilon^2 QL \\
\leq L = \| x \|.
\]

We define a new function \( \tilde{f}(x) = \max_{0 \leq x \leq x} f(s) \), so \( \tilde{f}(x) \) is nondecreasing monotonously, by \( (H_6) \) \( \lim_{x \to \infty} \frac{\tilde{f}(x)}{x^{p-1}} = 0 \), we can get \( \lim_{x \to \infty} \frac{\tilde{f}(x)}{x^{p-1}} = 0 \), for the same \( \epsilon' > 0 \), there exists \( S > 0 \) such that when \( x \leq S, \tilde{f}(x) \leq (\epsilon')^{-p} \), choose \( L' = \max\{L, S\} \), so for \( x \in K_L \), we have

\[
\| T_\lambda x \| \leq \psi_p\int_0^1 \lambda h(r)f(u(r))drds + \max\{\psi_p\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr, \\
\psi_p\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr\} \\
\leq \lambda \nu \int_0^1 \lambda h(r)f(u(r))dr \\
\leq (\lambda \nu)\int_0^1 \epsilon^2 QL \\
\leq L' = \| x \|.
\]

We define \( \Omega_1 = \{ x \in K : \| x \| < L \}, \Omega_2 = \{ x \in K : \| x \| < L' \} \), by lemma 1.4, \( T_\lambda \) has at least two fixed points \( x_1(t), x_2(t) \) in \( K \), and satisfy \( l \leq \| x_1 \| < L < \| x_2 \| < L' \).

Similarly to the proof of theorem 2.1, \( T_\lambda \) has no fixed point in \( \partial K_L \), so \( x_1(t) \neq x_2(t) \), the proof is finished.

(b) **Proof** Suppose there exists a subsequence \( \lambda_n < \lambda_{**} \) and \( \lambda_n \in (0, \frac{1}{n}) \) such that for \( \forall \eta \) problem (1) has a positive solution \( x_n \in K \), since \( x > 0 \), \( f(x) < (C'x)^{p-1} \), where...
\[ C' = \frac{f(L)}{\ell} \text{, we have} \]

\[ \|x_{\lambda n}\| \leq \psi_p \left( \int_0^1 \lambda_n h(r) f(x_{\lambda n}(r)) dr \right) ds + \max \{ \psi_p, \frac{\mu}{\alpha} \} \int_0^1 \lambda h(r) f(x_{\lambda n}(r)) dr \]

\[ \leq \lambda_n^{\frac{1}{\alpha}} \left( 1 + \max \left\{ \frac{\beta}{\alpha}, \frac{\delta}{\gamma} \right\} \right) \psi_p \left( \int_0^1 h(r) dr \right) \]

i.e.

\[ 1 \leq \lambda_n^{\frac{1}{\alpha}} \left( 1 + \max \left\{ \frac{\beta}{\alpha}, \frac{\delta}{\gamma} \right\} \right) C' \psi_p \left( \int_0^1 h(r) dr \right). \]  

(7)

since \( \lambda_n \downarrow \lambda_* \) so \( \lambda_n^{\frac{1}{\alpha}} \to \lambda_*^{\frac{1}{\alpha}} \), and \( \lambda_n^{\frac{1}{\alpha}} \to \lambda_*^{\frac{1}{\alpha}} \),

is contradiction, the proof is finished.

**Example 1**

\[ \{ -\varphi_p(x'(t))' = \lambda(1-t)^{p_1} t^{p_2} (cx^{q_1}(t) + x^{q_2}(t)), \quad t \in (0, 1), \]

\[ x(0) = x(1) = 0, \]

where \( \lambda \) is a positive parameter, \( c \in \mathbb{R}^+ \cup \{0\} \), \( -1 < p_1 < 0, -1 < p_2 < 0, 0 < q_1 < 0, p_1 < q_2 < 0 \).

We consider the following two cases:

(1) When \( 0 < q_1 < p_1 < p_2 < q_2 < c > 0, 0 < R = \bar{R} = \left( \frac{b - 1}{q_2 - p_1 + 1} \right)^{\frac{1}{q_2 - p_1 + 1}}, L = \frac{1}{\left( q_2 - p_1 + 1 \right)^{q_2 - p_1 + 1}}, \lambda = \lambda_* \]

\[ \left( \frac{1}{1 - \lambda_*} - 1 \right) \times (\beta(p_1 + 1, p_2 + 1))^{(1 - \alpha)\frac{q_2 - p_1 + 1}{2(q_1 - 1)}}. \]

By theorem 2.1, if \( \lambda \in (0, \lambda^*) \), then problem (1) has at least two positive solutions \( x_1, x_2 \) satisfy \( 0 < \|x_1\| < R < \|x_2\| \), there exists \( \lambda^* \) sufficient large, when \( \lambda > \lambda^* \), problem (1) has no positive solution.

(2) \( q_1 = p_1 - 1, c \geq 0, \frac{1}{\alpha} < c > 0, \) then \( h(t) = (1-t)^{p_1} t^{p_2}, \) and \( f(x) = cx^{q_1}(t) + x^{q_2}(t) \) satisfy all the conditions of theorem 2.2, and \( \beta = 0, \lambda = 0. \) Let \( \lambda^* = (c\beta(p_1 + 1, p_2 + 1))^{(1 - \alpha)\frac{q_2 - p_1 + 1}{2(q_1 - 1)}} \) where \( \beta = \beta \) function, for \( 0 < \lambda < \lambda^* \), problem (1) has at least one positive solution.

If \( c > 0 \), by corollary, for each \( \lambda > 0 \), problem (1) has at least one positive solution.

**Example 2**

\[ \{ -\varphi_p(x'(t))' = \lambda h(t) (e^{\varphi(t)} - 1), \quad t \in (0, 1), \]

\[ x(0) = x(1) = 0, \]

where \( \lambda \) is positive parameter, \( h(t) \) is same as above, we consider three cases:

(1) \( p > 2, \) let \( \lambda^* = \left( \frac{f(R)}{R^{p-1}} \right) (\beta(p_1 + 1, p_2 + 1))^{-1} \), where \( R = \bar{R} = (p_2 - p_1, p_2 - 1) \) is the only zero point of function \( \chi(x) = e^{\varphi(x-p_1+1)} + p_1 - 1 \), by theorem 2.1, when \( \lambda \in (0, \lambda^*) \), problem (1) has at least two solutions, and \( 0 < \|x_1\| < R < \|x_2\| \), there exists \( \lambda^* \) sufficient large, when \( \lambda > \lambda^* \), problem (1) has no solution.

(2) \( p = 2, \) let \( \lambda^* = (\beta(p_1 + 1, p_2 + 1))^{-1} \), where \( \beta \) is function. by theorem 2.2, for \( 0 < \lambda < \lambda^* \), problem (1) has at least one positive solution.

(3) \( 1 < p < 2, \) in this case \( a = 0, \) by corollary, for each \( \lambda > 0 \), problem (1) has at least one positive solution.

**Example 3**

\[ \{ -\varphi_p(x'(t))' = \lambda t^{-\sigma} e^{-\varepsilon t} \quad t \in (0, 1), \]

\[ x(0) = x(1) = 0 \]

where \( 0 < \alpha < 1, \) \( p - 1 < q \).

By theorem 2.3, for \( \varepsilon \in (0, \bar{\varepsilon}) \), let \( \lambda_*(\varepsilon) = \left( \frac{2}{\varepsilon^2 q R} \right)^{p-1} \), for \( 0 < \lambda > \lambda_* \), problem (1) has at least two positive solutions, and \( 0 < \|x_1\| < q + p - 1 < q + \|x_2\| \), there exists \( \lambda_* \) sufficient small, when \( \lambda < \lambda_* \), problem (1) no solution. Specially, \( p = 2, \) let \( \lambda_*(\varepsilon) = \left( \frac{2}{\varepsilon^2 q R} \right)^{p-1}, v = e^{2(q-1)}(q-1)(1-e)^{2(q-1)}, \) and \( Q = \left( \frac{1}{R} + \frac{1}{2} \right)^{2(q-1)} - 2 \frac{1}{R} \), we can get for \( 0 < \lambda > \lambda_* \), problem (1) has at least two positive solutions, and \( 0 < \|x_1\| < q - 1 < \|x_2\| \).

**REFERENCES**


