The relationship of eigenvalues between backward MPSD and Jacobi iterative matrices

Zhuan-de Wang, Hou-biao Li and Zhong-xi Gao

Abstract—In this paper, the backward MPSD (Modified Pre-conditioned Simultaneous Displacement) iterative matrix is firstly proposed. The relationship of eigenvalues between the backward MPSD iterative matrix and backward Jacobi iterative matrix for block \( p \)-cyclic case is obtained, which improves and refines the results in the corresponding references.

Keywords—Backward MPSD iterative matrix, Jacobi iterative matrix, eigenvalue, \( p \)-cyclic matrix

I. INTRODUCTION

To solve the equations

\[ Ax = b, \]

where \( A = [a_{ij}] \) is a given \( n \times n \) complex matrix and nonsingular, \( n \geq 2 \), which is partitioned in the form

\[ A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,p} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & \cdots & A_{p,p} \end{bmatrix}, \]

iterative methods are considered.

Let \( A = D - C_L - C_U \) where \( D = \text{diag}(A) \) is a block diagonal matrix obtained from \( A \) and nonsingular, \(-C_L\) and \(-C_U\) are strictly lower and upper triangular matrices obtained from \( A \), respectively. We also let \( L = D^{-1}C_L, U = D^{-1}C_U \). The equation (1) becomes the equivalent one

\[ (I - L - U)x = D^{-1}Ax = D^{-1}b. \]

The Jacobi iterative matrix is

\[ B = L + U = I - D^{-1}A. \]

The MPSD (Modified Pre-conditioned Simultaneous Displacement) iterative method is studied in [2-5]. Here, we give the backward MPSD iterative matrix as follows:

\[ \tilde{S}_{\tau, \omega_1, \omega_2} = (I - \omega_1U)^{-1}(I - \omega_2L)^{-1}[(1 - \tau)L + (\tau - \omega_1)U + (\tau - \omega_2)L + \omega_1\omega_2LU], \]

with special values of \( \omega_1, \omega_2 \) and \( \tau \), we have

(1) When \( \omega_1 = 0, \omega_2 = 0 \) and \( \tau = 1 \), we obtain the Jacobi iterative method;
(2) When \( \omega_1 = 0, \omega_2 = 0 \) and \( \tau = \omega \), we obtain the backward JOR iterative method;
(3) When \( \omega_1 = 1, \omega_2 = 0 \) and \( \tau = 1 \), we obtain the backward G-S iterative method;
(4) When \( \omega_1 = \omega, \omega_2 = 0 \) and \( \tau = \omega \), we obtain the backward SOR iterative method;
(5) When \( \omega_1 = \omega, \omega_2 = 0 \) and \( \tau = \alpha \), we obtain the backward AOR iterative method;
(6) When \( \omega_1 = \omega, \omega_2 = \omega \) and \( \tau = \omega(2 - \omega) \), we obtain the backward SSOR iterative method;
(7) When \( \omega_1 = \omega, \omega_2 = \omega \) and \( \tau = \omega \), we obtain the backward EMA iterative method;
(8) When \( \omega_1 = \omega, \omega_2 = \omega \) and \( \tau = \alpha \), we obtain the backward PSD iterative method;
(9) When \( \omega_1 = \omega, \omega_2 = \omega \) and \( \tau = 1 \), we obtain the backward PJ iterative method.

If \( A \) has the following block form

\[ A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & \cdots & 0 \\ 0 & A_{2,2} & A_{2,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & A_{p-1,p} & 0 \\ A_{p,1} & 0 & \cdots & \cdots & A_{p,p} \end{bmatrix}, \]

then \( A \) is called a \( p \)-cyclic matrix [1]. Such matrices naturally arise, e.g., for \( p = 2 \) in the discretization of second-order elliptic or parabolic PDEs by finite differences, finite element or collocation methods, for \( p = 3 \) in the case of large scale least-squares problems, and for any \( p \geq 2 \) in the case of Markov chain analysis. The \( p \)-cyclic matrix is considered in many papers [1, 6-13]. The eigenvalue relationship between the SOR iterative matrix and the Jacobi iterative matrix for \( p \)-cyclic case is studied in Theorem 4.5 in [1], and the eigenvalue relationship between the USAOR iterative matrix and the Jacobi iterative matrix for the \( p \)-cyclic case is studied in [6]. In the following we will consider the eigenvalue relationship between the backward MPSD iterative matrix and the Jacobi iterative matrix for the \( p \)-cyclic case.

II. PRELIMINARY

If \( A \) is a \( p \)-cyclic matrix, then

\[ D = \begin{bmatrix} A_{1,1} & \cdots & \cdots & \cdots & A_{1,p} \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ A_{p,1} & \cdots & \cdots & \cdots & A_{p,p} \end{bmatrix}, \]

\[ C_L = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ -A_{p,1} & \cdots & \cdots & \cdots & 0 \\ -A_{1,1} & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \]

\[ C_U = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & -A_{p,1} \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ -A_{p-1,1} & \cdots & \cdots & \cdots & 0 \end{bmatrix}. \]
and
\[ L = D^{-1}C_L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & B_{1,1} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{p,1} & & & 0 \end{bmatrix}, \]
\[ U = D^{-1}C_U = \begin{bmatrix} 0 & B_{1,2} & \cdots & B_{2,3} \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ B_{p,1} & & & 0 \end{bmatrix}. \]

Let \( \lambda \) be the eigenvalue of \( \tilde{S}_{\tau,\omega_1,\omega_2} \), \( x \) be the corresponding eigenvector. Then
\[ \tilde{S}_{\tau,\omega_1,\omega_2}x = \lambda x, \]
equivalently,
\[ (I - \omega_1U)(I - \omega_2L)^{-1}[(1 - \tau)I + (\tau - \omega_1)L + (\tau - \omega_2)L + \omega_1\omega_2LU]x = \lambda x, \]
or
\[ [(1 - \tau)I + (\tau - \omega_1)U + (\tau - \omega_2)L + \omega_1\omega_2LU]x = \lambda(I - \omega_2L)(I - \omega_1U)x, \]
that is to say,
\[ \begin{bmatrix} (1 - \tau)I & (\tau - \omega_1)B_{1,2} \\ (\tau - \omega_2)B_{p,1} & \omega_1\omega_2B_{p,1}B_{1,2} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} \end{bmatrix}, \]
where
\[ LU = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & B_{1,1} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{p,1} \end{bmatrix}. \]
If \( \mu \) is an eigenvalue of \( B \) and \( x \) is the corresponding eigenvector, that is,
\[ \begin{bmatrix} 0 & B_{1,2} & \cdots & B_{2,3} \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ B_{p,1} & & & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} = \lambda \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix}, \]
with \( \lambda + \tau - 1 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (4).

**Proof.** Let \( \lambda \) be the eigenvalue of \( \tilde{S}_{\tau,\omega_1,\omega_2} \), \( x \) be the corresponding eigenvector. By (2), we have
\[ \begin{bmatrix} \begin{bmatrix} \eta x_1 \\ \eta x_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \xi_1B_{1,2}x_2, \\ \xi_2B_{1,2}x_3 \end{bmatrix} \end{bmatrix}, \]
\[ \begin{bmatrix} \begin{bmatrix} \eta x_{p-1} \\ \eta x_p \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \xi_1B_{p-1,p}x_{p-1} + \xi_2B_{p-1,p}x_1, \\ \xi_2B_{p-1,p}x_{p-1} + \xi_1B_{p-1,p}x_1 \end{bmatrix} \end{bmatrix}. \]

From the first \( n - 2 \) equations, we have
\[ \eta^2x_1 = \xi_1^2B_{1,2}B_{2,3}x_2 \]
\[ \eta^2x_2 = \xi_1^2B_{1,2}B_{2,3}x_3 \]
\[ \vdots \]
\[ \eta^{p-2}x_1 = \xi_1^{p-2}B_{1,2}B_{2,3} \cdots B_{n-2,1}x_1, \]
and from the last two equations and the first equation, we have
\[ \eta^2x_{p-1} = \xi_1B_{p-1,p}x_p \]
\[ = \xi_1B_{p-1,p}[(1 - \lambda)\omega_1\omega_2B_{p,1}B_{1,2}x_2 + \xi_2B_{p,1}x_1], \]
\[ = \xi_1\xi_2B_{p-1,p}x_{p-1} + (1 - \lambda)\omega_1\omega_2B_{p-1,p}B_{p,1} \]
\[ = \xi_1\xi_2B_{p-1,p}x_{p-1} + \xi_2B_{p-1,p}x_1, \]
\[ = \eta(1 - \lambda)\omega_1\omega_2 + \xi_2B_{p-1,p}x_{p-1} + \xi_2B_{p-1,p}x_1. \]

Combining (6) with (7), we have
\[ \eta^2x_1 = \xi_1^2\xi_2^2B_{1,2}B_{2,3} \cdots B_{n-2,1}x_1, \]
\[ = \xi_1^2\xi_2^2B_{1,2}B_{2,3} \cdots B_{n-2,1}[(1 - \lambda)\omega_1\omega_2 + \xi_2B_{p,1}x_1], \]
\[ = \xi_1^2\xi_2^2[(1 - \lambda)\omega_1\omega_2 + \xi_2B_{p,1}x_1], \]
\[ = \eta^2\xi_1^2\xi_2^2[(1 - \lambda)\omega_1\omega_2 + \xi_2B_{p,1}x_1], \]
\[ = \eta^2\xi_1^2\xi_2^2x_1. \]

Assuming that \( \lambda + \tau - 1 \neq 0 \).
If \( x_1 = 0 \), then, by (5), we have \( x_p = x_{p-1} = \cdots = x_2 = x_1 = 0 \). But \( x \) is an eigenvector, so \( x_1 \neq 0 \). By (8), we know that
\[ \eta^0 \]
\[ = \xi_1^0\xi_2^0[(1 - \lambda)\omega_1\omega_2 + \xi_2B_{p,1}x_1], \]
is an eigenvalue of \( B_{1,2}B_{2,3} \cdots B_{n-2,1}B_{p-1,p}B_{p,1} \) and \( x_1 \) is the corresponding eigenvector.

If \( \mu \) is an eigenvalue of \( B \) and \( x \) is the corresponding eigenvector, by (3) we obtain
\[ B_{1,2}x_2 = \mu x_1, \]
\[ B_{2,3}x_3 = \mu x_2, \]
\[ \vdots \]
\[ B_{p-1,p}x_{p-1} = \mu x_{p-1}, \]
\[ B_{p,1}x_1 = \mu x_p. \]
From the above equations, we have
\[ B_{1,2}B_{2,3}B_{3,4}x_4 = \mu^3 x_1, \]
\[ \vdots \]
\[ B_{1,2}B_{2,3}B_{3,4} \cdots B_{p-1,p}x_p = \mu^{p-1}x_1. \]

So,
\[ \begin{cases} \mu B_{1,2}B_{2,3}B_{3,4} \cdots B_{p-1,p}x_p = \mu^p x_1, \\ B_{1,2}B_{2,3}B_{3,4} \cdots B_{p-1,p}B_{p,1}x_1 = \mu x_1. \end{cases} \] (9)

Hence, \( \mu^p \) is an eigenvalue of \( B_{1,2}B_{2,3}B_{3,4} \cdots B_{p-1,p} \).

Combining (8) with (9), we obtain that
\[ \eta^p = \frac{\mu^p}{\xi_1^{p-2} \left[ \eta(1 - \lambda)\omega_1\omega_2 + \xi_1 \xi_2 \right]}, \]
i.e.,
\[ \mu^p \left( \xi_1^{p-2} \left[ \eta(1 - \lambda)\omega_1\omega_2 + \xi_1 \xi_2 \right] \right) = \eta^p. \] (10)

So, if \( \lambda \) is an eigenvalue of \( \tilde{S}_{\omega_1,\omega_2} \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (10). Conversely, if \( \mu \) is an eigenvalue of \( B \) and \( \lambda \) satisfies (10), we can easily prove that \( \lambda \) is an eigenvalue of \( \tilde{S}_{\omega_1,\omega_2} \). Thus, we proved Theorem 3.1. 

With the special values of \( \omega_1, \omega_2 \), and \( \tau \), we have several corollaries, which improve and refine the results in the corresponding references.

**Corollary 3.1** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \) and \( \lambda \) satisfies
\[ \mu^p \omega^p = (\lambda + \omega - 1)^p. \] (11)

Then \( \lambda \) is an eigenvalue of the backward JOR iterative matrix \( S_{\omega,0,0} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{\omega,0,0} \) with \( \lambda + \omega - 1 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (11).

**Corollary 3.2** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \) is an eigenvalue of \( B \), \( \mu \neq 0 \), and \( \lambda \) satisfies
\[ \mu^p = \lambda. \] (12)

Then \( \lambda \) is an eigenvalue of the backward Gauss-Seidel iterative matrix \( S_{1,1,0} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{1,1,0} \) for which \( \lambda \neq 0 \), then there must exist an eigenvalue \( \mu \) of \( B \) satisfying (12).

**Corollary 3.3** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \), and \( \lambda \) satisfies
\[ \lambda^p - \mu^p \omega^p = (\lambda + \omega - 1)^p. \] (13)

Then \( \lambda \) is an eigenvalue of the backward SOR iterative matrix \( S_{\omega,\omega,0} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{\omega,\omega,0} \) with \( \lambda + \omega - 1 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (13).

**Corollary 3.4** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \) and \( \lambda \) satisfies
\[ \mu^p \alpha (\alpha - \omega + \lambda \omega)^{p-1} = (\lambda + \alpha - 1)^p. \] (14)

Then \( \lambda \) is an eigenvalue of the backward AOR iterative matrix \( S_{\alpha,\omega,0} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{\alpha,\omega,0} \) with \( \lambda + \alpha - 1 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (14).

**Corollary 3.5** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \) and \( \lambda \) satisfies
\[ \left( \lambda - (\omega - 1)^2 \right)^p = \mu^p \left( (\alpha + \omega - 1)^2 (1 - \omega)^2 + (\lambda - \omega)^2 \right). \] (15)

Then \( \lambda \) is an eigenvalue of the backward SSOR iterative matrix \( S_{(2-\omega),\omega,\omega} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{(2-\omega),\omega,\omega} \) for which \( \lambda + (\omega - 1)^2 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (15).

**Corollary 3.6** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \) and \( \lambda \) satisfies
\[ (\lambda + \omega - 1)^p = \mu^p \left( (\alpha + \omega + \lambda \omega)^2 - (1 - \omega)^2 \right). \] (16)

Then \( \lambda \) is an eigenvalue of the backward EMA iterative matrix \( S_{\omega,\omega,\omega} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{\omega,\omega,\omega} \) with \( \lambda + \omega - 1 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (16).

**Corollary 3.7** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \) and \( \lambda \) satisfies
\[ (\lambda + \alpha - 1)^p = \mu^p \left( (\alpha + \omega + \lambda \omega)^2 - (1 - \omega)^2 \right). \] (17)

Then \( \lambda \) is an eigenvalue of the backward PSD iterative matrix \( S_{\omega,\omega,\omega} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{\omega,\omega,\omega} \) with \( \lambda + \alpha - 1 \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (17).

**Corollary 3.8** Let \( A \) be a \( p \)-cyclic matrix, \( B \) be the corresponding Jacobi iterative matrix. If \( \mu \neq 0 \) is an eigenvalue of \( B \) and \( \lambda \) satisfies
\[ \mu^p \left( (1 - \omega + \lambda \omega)^2 - (1 - \omega + \lambda \omega)^2 \right). \] (18)

Then \( \lambda \) is an eigenvalue of the backward PJ iterative matrix \( S_{\omega,1,1} \). Conversely, if \( \lambda \) is an eigenvalue of \( S_{\omega,1,1} \) for which \( \lambda \neq 0 \), then there exists an eigenvalue \( \mu \) of \( B \) satisfying (18).

**IV. NUMERICAL EXAMPLE**

**Example 4.1** Let the coefficient matrix \( A \) of (1) be
\[
A = \begin{bmatrix}
1 & 0 & -0.125 & -0.125 & 0 & 0 \\
0 & 1 & -0.125 & -0.125 & 0 & 0 \\
0 & 0 & 1 & 0 & -0.125 & -0.125 \\
-0.125 & -0.125 & 0 & 0 & 1 & 0 \\
-0.125 & -0.125 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

It is obvious that \( A \) is 3-cyclic matrix. By calculation, we obtain \( \mu = \frac{1}{4} \) is an eigenvalue of the Jacobi matrix \( B \).
Gauss-Seidel iteration is triple that of the Jacobi iteration. So, the asymptotic rate of convergence of the backward Gauss-Seidel iteration is triple that of the Jacobi iteration.

(2) With $\omega_2 = 0$, $\omega_1 = 1$, $\tau = 1$, we obtain the backward AOR iterative method. By calculation, $\lambda = \frac{669}{3814}$ is an eigenvalue of the backward AOR matrix. Meanwhile the equation $\mu^3 \alpha + \omega_3 \lambda - \omega_2 \lambda^2 - 2 \lambda - 1 = 0$, and $\lambda = \frac{669}{3814}$ is just the root of it.

(3) The numerical results between other iterative method and the Jacobi iterative method is analogous to the above, and is omitted.

V. Conclusion
The eigenvalue relationship is vital for the convergence of iterative methods. In this paper, the backward MPSD iterative matrix is proposed, and the relationship of eigenvalues between backward MPSD and Jacobi iterative matrices for $p$-cyclic case is obtained, which is useful to some issues such as Markov Chains, etc. These results involve some special iterative methods which are proposed in the references.

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