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Bifurcations for a FitzHugh-Nagumo model with time delays

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Abstract—In this paper, a FitzHugh-Nagumo model with time delays is investigated. The linear stability of the equilibrium and the existence of Hopf bifurcation with delay τ is investigated. By applying Nyquist criterion, the length of delay is estimated for which stability continues to hold. Numerical simulations for justifying the theoretical results are illustrated. Finally, main conclusions are given.

Keywords—FitzHugh-Nagumo model; Time delay; Stability; Hopf bifurcation.

I. INTRODUCTION

RECENTLY substantial efforts have been made in FitzHugh-Nagumo system which is frequently used(e.g., in brain research, and to some extent in modelling cardiac movements etc), for example, Medvedev and Kopell [1] studied synchronization and transient dynamics of electrically coupled FitzHugh-Nagumo oscillators, Kostova et al.[4] investigated the stability and bifurcation of a revised FitzHugh-Nagumo model, etc. In 2009, Ringkvist and Zhou[6] introduced the following two-stage predator-prey interaction model with a generic functional response:

$$\begin{cases} \dot{x}(t) = -Cy(t) - Ax(t)[x(t) - B][x(t) - \lambda] + I, \\ \dot{y}(t) = \varepsilon[x(t) - \delta y(t)], \end{cases} \tag{1}$$

where $A, B, C, \delta, \varepsilon, \lambda$ are non-zero parameters and I is an external force, commonly referred to as the magnitude of the stimulating current,which can be a function of t. In details, one can see [6]. Nothing that most practical implementations of feedback have inherent delays, we incorporate time delay into model (1) as follows:

$$\begin{cases} \dot{x}(t) = -Cy(t-\tau) - Ax(t)(x(t) - B)(x(t) - \lambda) + I, \\ \dot{y}(t) = \varepsilon[x(t) - \delta y(t-\tau)]. \end{cases} \tag{2}$$

It is well known that delay may have very complicated impact on the dynamics of a system. To obtain a deep and clear understanding of dynamics of FitzHugh-Nagumo system with delay, in this paper, we will study the Hopf bifurcation of model (2). Choosing the delay τ as the bifurcation parameter, we shall investigated the effect of the delay τ on the dynamics of system (2).

The remainder of the paper is organized as follows. In Section 2, we discuss the stability of the equilibrium and the existence of Hopf bifurcations occurring at the equilibrium. In Section 3, the length of delay is estimated for which stability continues to hold by means of Nyquist criterion. In Section 4, numerical simulations are carried out to validate our main results. Some main conclusions are drawn in Section 5.

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II. STABILITY OF THE EQUILIBRIUM AND LOCAL HOPF BIFURCATIONS

First, we assume that system (2) has a equilibrium $E_0(x^*, y^*)$, where $x^* = \delta y^*$ and y^* satisfies the following equation:

$$A\delta^3 y^3 - A\delta^2(B+\lambda)y^2 + (C - AB\lambda\delta)y + I = 0. \tag{3}$$

Let $\bar{x}(t) = x(t) - x^*, \bar{y}(t) = y(t) - y^*$ and still write $\bar{x}(t)$ and $\bar{y}(t)$ as $x(t)$ and $y(t)$ respectively, then the linearization of Eq. (2) at $E_0(x^*, y^*)$ takes the form

$$\begin{cases} \dot{x}(t) = -A[x^*(2x^* - \lambda - B) + (x^* - B)(x^* - \lambda)]x(t) \\ \qquad - Cy(t-\tau), \\ \dot{y}(t) = \varepsilon x(t) - \varepsilon\delta y(t-\tau). \end{cases} \tag{4}$$

The characteristic equation of system (4) takes the form

$$\lambda^2 + m_1\lambda + m_0 + (n_1\lambda + n_0)e^{-\lambda\tau} = 0, \tag{5}$$

where

$$\begin{aligned} m_0 &= C\varepsilon, m_1 = A[x^*(2x^* - \lambda - B) + (x^* - B)(x^* - \lambda)], \\ n_0 &= C\varepsilon + \varepsilon\delta a_1, n_1 = \varepsilon\delta. \end{aligned}$$

The stability of the positive equilibrium of system (2) depends on the locations of the roots of the characteristic equation (5), When all roots of Eq.(5) locate in the left half of complex plane, the trivial solution is stable, otherwise, it is instable.

For $\tau = 0$, (5) becomes

$$\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0. \tag{6}$$

It is easy to see that a set of necessary and sufficient conditions that all roots of (6) have a negative real part is given in the following form:

$$\text{(H1)} \quad m_1 + n_1 > 0, \; m_0 + n_0 > 0.$$

For $\omega > 0, i\omega$ is a root of (5), then

$$-\omega^2 + im_1\omega + m_0 + (in_1\omega + n_0)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} n_0\cos\omega\tau + n_1\omega\sin\omega\tau = \omega^2 - m_0, \\ n_1\omega\cos\omega\tau - n_0\sin\omega\tau = -m_1\omega \end{cases} \tag{7}$$

which leads to

$$\omega^4+(m_1^2-n_1^2-2m_0)\omega^2+m_0^2-n_0^2=0. \tag{8}$$

In the sequel, we consider three cases.
(a) If the condition

$$\text{(H2) } m_1^2-n_1^2-2m_0>0, m_0^2-n_0^2>0 \text{ or}$$
$$\Delta=(m_1^2-n_1^2-2m_0)^2-4(m_0^2-n_0^2)<0$$

holds, then Eq.(8) has no positive root.
(b) If the condition

$$\text{(H3) } m_1^2-n_1^2-2m_0<0, m_0^2-n_0^2>0,$$
$$\Delta=(m_1^2-n_1^2-2m_0)^2-4(m_0^2-n_0^2)>0$$

holds, then Eq.(8) has two positive roots

$$\omega_\pm = \frac{\sqrt{2}}{2}\left[-m_1^2+n_1^2+2m_0 \right. \tag{9}$$
$$\left. \pm\sqrt{(m_1^2-n_1^2-2m_0)^2-4(m_0^2-n_0^2)}\right]^{\frac{1}{2}}. \tag{10}$$

(c) If

$$\text{(H4) } m_0^2-n_0^2<0 \text{ or } m_1^2-n_1^2-2m_0<0 \text{ and}$$
$$\Delta=(m_1^2-n_1^2-2m_0)^2-4(m_0^2-n_0^2)=0$$

holds, then Eq.(8) has only one positive root ω_+.

Without loss of generality, we assume that (8) has two positive roots denoted by $\omega_\pm$. It follows from (7) that

$$\tau_j^\pm=\frac{1}{\omega_\pm}\left[\arccos\frac{n_0(\omega_\pm^2-m_0)-m_1n_1\omega_\pm^2}{n_0^2+n_1^2\omega_\pm^2}+2j\pi\right], \tag{11}$$

$(j=0,1,2,\cdots)$ at which Eq.(5) has a pair of purely imaginary roots $\pm i\omega_\pm$. Let $\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$ be the root of Eq.(5) such that $\alpha(\tau_j^\pm)=0, \omega(\tau_j^\pm)=\omega_\pm$. Due to functional differential equation theory, for every $\tau_j^\pm, j=0,1,2,\cdots$, there exists $\varepsilon>0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau-\tau_j^\pm|<\varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (5) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{(2\lambda+m_1)e^{\lambda\tau}}{\lambda(n_1\lambda+n_0)}+\frac{n_1}{\lambda(n_1\lambda+n_0)}-\frac{\tau}{\lambda},$$

which, together with (7), leads to

$$\begin{aligned}
\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^\pm} &= \mathrm{Re}\left\{\frac{(2\lambda+a_1)e^{\lambda\tau}}{\lambda(n_1\lambda+n_0)}\right\}_{\tau=\tau_j^\pm} \\
&\quad +\mathrm{Re}\left\{\frac{n_1}{\lambda(n_1\lambda+n_0)}\right\}_{\tau=\tau_j^\pm} \\
&= \mathrm{Re}\left\{\frac{m_1\cos\omega_\pm\tau_j^\pm-2\omega_\pm\sin\omega_\pm\tau_j^\pm}{-n_1\omega_\pm^2+in_0\omega_\pm}\right. \\
&\quad \left.+\frac{i(2\omega_\pm\cos\omega_\pm\tau_j^\pm+m_1\sin\omega_\pm\tau_j^\pm)}{-n_1\omega_\pm^2+in_0\omega_\pm}\right\} \\
&\quad +\mathrm{Re}\left\{\frac{n_1}{-n_1\omega_\pm^2+in_0\omega_\pm}\right\} \\
&= \frac{1}{\Lambda}\{m_1\omega_\pm(n_0\sin\omega_\pm\tau_j^\pm-n_1\omega_\pm\cos\omega_\pm\tau_j^\pm) \\
&\quad +2\omega_\pm^2(m_1\cos\omega_\pm\tau_j^\pm+n_1\omega_\pm\sin\omega_\pm\tau_j^\pm)-n_1^2\omega_\pm^2\} \\
&= \frac{1}{\Lambda}\{m_1^2\omega_\pm^2+2\omega_\pm^4-2m_0\omega_\pm^2-n_1^2\omega_\pm^2\} \\
&= \frac{\omega_\pm^2}{\Lambda}\{2\omega_\pm^2+m_1^2-n_1^2-2m_0\} \\
&= \frac{\omega_\pm^2}{\Lambda}\{-m_1^2+n_1^2+2m_0\pm\sqrt{\Delta}+m_1^2-n_1^2-2m_0\} \\
&= \frac{\omega_\pm^2}{\Lambda}\{\pm\sqrt{\Delta}\},
\end{aligned}$$

where

$$\Lambda=n_1^2\omega_\pm^4+n_0^2\omega_\pm^2>0,\ \sqrt{\Delta}=(n_1^2-m_1^2+2m_0)^2-4(m_0^2-n_0^2).$$

Thus, if $\Delta\neq 0$, we obtain

$$\begin{aligned}
\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j^+}\right\} &= \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^+}\right\} \\
&= \mathrm{sign}\left\{\frac{\omega_\pm^2}{\Lambda}\{\sqrt{\Delta}\}\right\}=1>0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j^-}\right\} &= \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j^-}\right\} \\
&= \mathrm{sign}\left\{\frac{\omega_\pm^2}{\Lambda}\{-\sqrt{\Delta}\}\right\}=-1<0.
\end{aligned}$$

According to above analysis and the results of Ruan and Wei[7], Yang[5] and Hale[3], we have the following result.

Theorem 2.1. *Let $\tau_j^\pm(j=0,1,2,\cdots)$ be defined by (10) and $\tau_0=\min\{\tau_0^+,\tau_0^-\}$. If $(H1),(H2),(H3)$ hold, then the equilibrium $E_0(x^*,y^*)$ of system (2) is asymptotically stable for $\tau\in[0,\tau_0)$. If $(H1),(H2),(H4)$ hold, system (2) undergoes a Hopf bifurcation at the equilibrium $E_0(x^*,y^*)$ when $\tau=\tau_j^\pm, j=0,1,2,\cdots$.*

III. Estimation of the Length of Delay to Preserve Stability

In the present section, we will obtain an estimation τ_+ for the length of the delay τ which preserves the stability of the equilibrium $E_0(x^*,y^*)$, i.e., $E_0(x^*,y^*)$ is asymptotically stable if $\tau<\tau_+$.

We consider system (4) in $C([-\tau,\infty),R^2)$ with the initial values

$$x(\xi)=\varphi_1(\xi),\ y(\xi)=\varphi_2(\xi),\ \varphi_i(0)\geq 0, i=1,2,\xi\in[-\tau,0].$$

Taking Laplace transform of system (4), we obtain

$$\left\{\begin{array}{l} [s+m_1]\tilde{x}=-Ce^{-s\tau}M(s)-C\tilde{y}+\varphi_1(0), \\ s\tilde{y}=\varepsilon\tilde{x}-\varepsilon\delta(M(s)+\tilde{y})e^{-s\tau}+\varphi_2(0), \end{array}\right. \tag{12}$$

where $\tilde{x},\tilde{y}$ are the Laplace transform of $x(t),y(t)$, respectively, and $M(s)=\int_{-\tau}^0 e^{-s\tau}y(t)dt$.

Solving (11) for $\tilde{x}$ leads to $\tilde{x} = \frac{K(s,\tau)}{J(s)}$, where

$$\begin{aligned} K(s,\tau) &= C[\varepsilon\delta e^{-s\tau}M(s) - \varphi_2(0)] \\ &\quad - [Ce^{-s\tau}M(s) + \varphi_1(0)][\varepsilon\delta e^{-s\tau} + s], \\ J(s) &= [s + m_1)(s + \varepsilon\delta e^{-s\tau}) + Cs. \end{aligned}$$

Following along the lines of Freedman and Rao[2] and using the Nyquist criterion, we obtain that the conditions for local asymptotic stability of $E_0(x^*, y_2^*)$ are given by

$$\text{Im}\{J(i\omega_0)\} > 0, \tag{13}$$

$$\text{Re}\{J(i\omega_0)\} = 0, \tag{14}$$

where Im$\{J(i\omega_0)\}$ and Re$\{J(i\omega_0)\}$ are the imaginary part and real part of $J(i\omega_0)$, respectively and ω_0 is the small positive root of (13).

By (12) and (13), we have

$$m_1\omega_0 > n_0 \sin\omega_0\tau - n_1\omega_0\cos\omega_0\tau, \tag{15}$$

$$\omega_0^2 - m_0 = n_0\cos\omega_0\tau + n_1\omega_0\sin\omega_0\tau. \tag{16}$$

It follows from (15) that

$$\omega_0^2 - |n_1|\omega_0 - (m_0 + |n_0|) \le 0 \tag{17}$$

which leads to $\omega_0 \le \omega_+$, where $\omega_+ = \frac{|n_1| + \sqrt{n_1^2 + 4(m_0 + |n_0|)}}{2}$. By (15), we have

$$\omega_0 = \frac{n_0\cos\omega_0\tau + n_1\omega_0\sin\omega_0\tau + m_0}{\omega_0}. \tag{18}$$

Substituting (17) into (14) and rearranging, we get

$$\begin{aligned} &(m_1n_0 + n_1\omega_0^2)(1 - \cos\omega_0\tau) + (n_0 - n_1m_1)\omega_0\sin\omega_0\tau \\ &< m_1m_0 + m_1n_0 + n_1\omega_0^2. \end{aligned} \tag{19}$$

Using the bounds

$$\begin{aligned} (m_1n_0 + n_1\omega_0^2)(1 - \cos\omega_0\tau) &= 2(m_1n_0 + n_1\omega_0^2)\sin^2\left(\frac{\omega_0\tau}{2}\right) \\ &\le \frac{1}{2}(|m_1n_0| + |n_1|\omega_+^2)\omega_+^2\tau^2 \end{aligned}$$

and

$$(n_0 - n_1m_1)\omega_0\sin\omega_0\tau \le |(n_0 - n_1m_1)|\omega_+^2\tau,$$

we obtain from (3.8)

$$K_1\tau^2 + K_2\tau \le K_3,$$

where

$$\begin{aligned} K_1 &= \frac{1}{2}(|m_1n_0| + |n_1|\omega_+^2)\omega_+^2, \\ K_2 &= |(n_0 - n_1m_1)|\omega_+^2, \\ K_3 &= m_1m_0 + m_1n_0 + |n_1|\omega_+^2. \end{aligned}$$

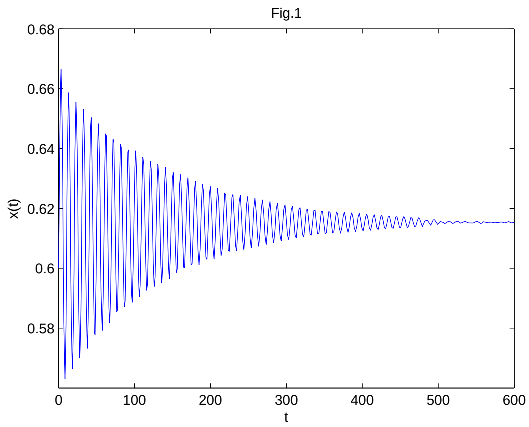
It is obvious to see that if $\tau < \tau_+ = \frac{-K_2 + \sqrt{K_2^2 + 4K_1K_3}}{2K_1}$, the stability of $E_0(x^*, y^*)$ of system (2) is preserved.

IV. Numerical Examples

In this section, we will prove some numerical results of system (2) to illustrate our results obtained in Section 2. We consider the system (2) with $A = B = C = 1, \lambda = -0.04, I = 0.05, \varepsilon = 0.2, \delta = 3$. That is,

$$\begin{cases} \dot{x}(t) = -y(t-\tau) - x(t)(x(t) - 1)(x(t) + 0.04) + 0.05, \\ \dot{y}(t) = 0.2(x(t) - 3y(t-\tau)), \end{cases} \tag{1}$$

which has a positive equilibrium $E_0(0.75, 0.25)$ and satisfies the conditions indicated in Theorem 2.1. When $\tau = 0$, the positive equilibrium $E_0(0.75, 0.25)$ is asymptotically stable. The positive equilibrium $E_0(0.75, 0.25)$ is stable when $\tau < \tau_0 \approx 1.66$ which is illustrated by the computer simulations(see Figs.1-4). When τ passes through the critical value τ_0, the positive equilibrium $E_0(0.75, 0.25)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_0(0.75, 0.25)$ (see Figs.5-8).



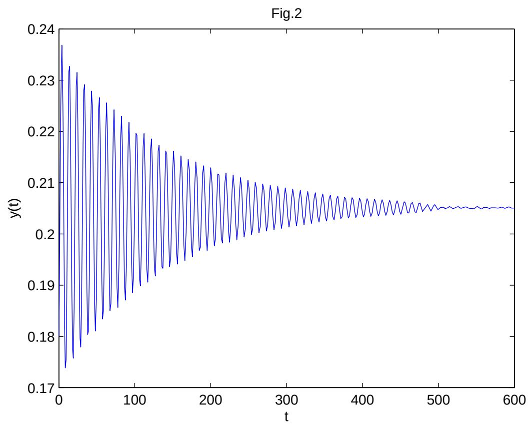


Fig.3

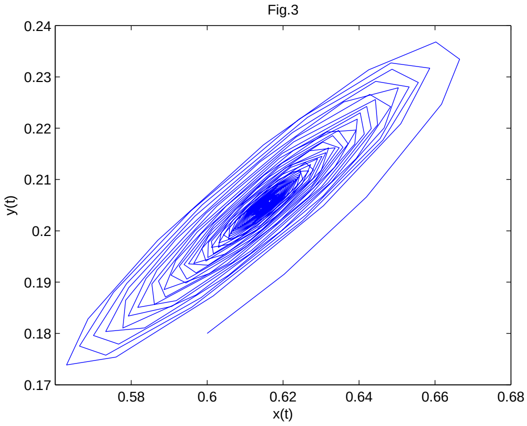
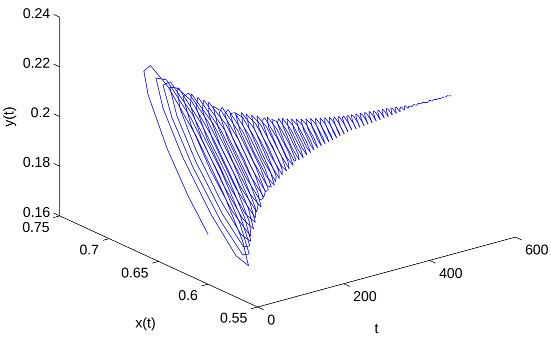


Fig.4



Figs.1-4 When $\tau = 1.65 < \tau_0 \approx 1.66$. The positive equilibrium $E_0(0.75, 0.25)$ of system (19) is asymptotically stable. The initial value is (0.6,0.18).

Fig.5

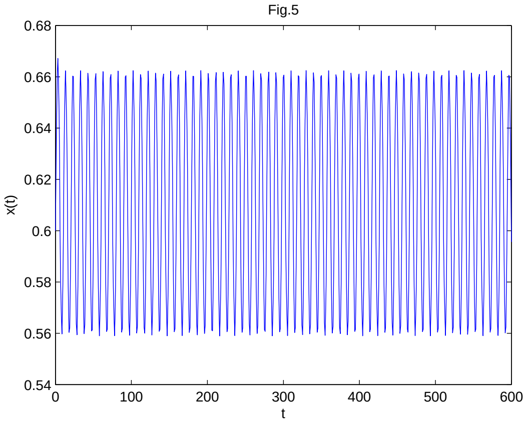


Fig.6

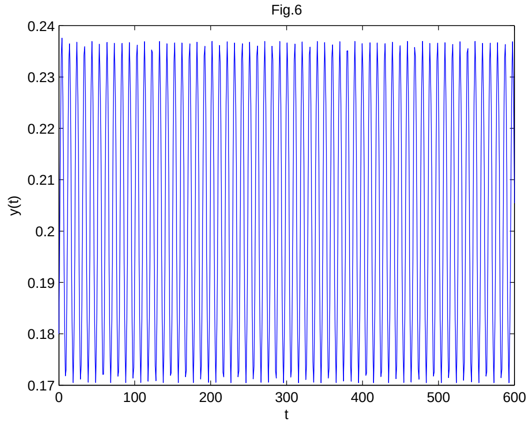


Fig.7

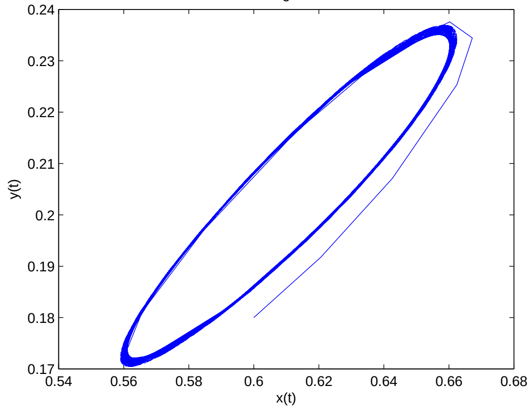
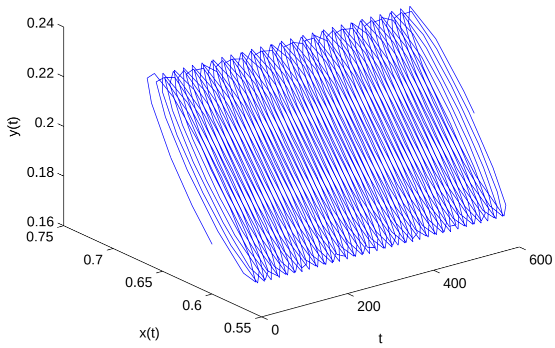


Fig.8



Figs.5-8 When $\tau = 1.676 > \tau_0 \approx 1.66$. Hopf bifurcation of system (19) occurs from the positive equilibrium $E_0(0.75, 0.25)$. The initial value is (0.6,0.18).

V. Conclusions

In this paper, the local stability of the equilibrium $E_0(x^*, y^*)$ and local Hopf bifurcation in a FitzHugh-Nagumo model with time delay are investigated. It is showed that if the conditions $(\mathrm{H1}) - (\mathrm{H3})$ hold, the equilibrium $E_0(x^*, y^*)$ of system (2) is asymptotically stable for all $\tau \in [0, \tau_0)$. If the conditions $(\mathrm{H1}), (\mathrm{H2})$ and $(\mathrm{H4})$ hold, as the delay τ increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur at the equilibrium $E_0(x^*, y^*)$, i.e., a family of periodic orbits bifurcates from the the positive equilibrium $E_0(x^*, y^*)$. Moreover, the length of delay preserving the stability of the equilibrium $E_0(x^*, y^*)$ is estimated. Some numerical simulations are performed to verify our theoretical results found.

VI. Conclusions

In this paper, we have investigated the dynamical behaviors of a nonlinear delay population model. It is shown that under a certain condition, there exists a critical value τ_0 of the delay τ for the stability of the population system. If $\tau \in [0, \tau_0)$, the positive equilibrium of the population system is asymptotically stable which means that the size of the population will keep in a steady state. When the delay τ passes through some critical values $\tau = \tau_k, k = 0, 1, 2, \cdots$, the positive equilibrium of the population system loses its stability and a Hopf bifurcation will occur. Moreover, the existence of global Hopf bifurcation are established.

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