Some new inequalities for eigenvalues of the Hadamard product and the Fan product of matrices

Jing Li and Guang Zhou

Abstract—Let $A$ and $B$ be nonnegative matrices. A new upper bound on the spectral radius $\rho(A \circ B)$ is obtained. Meanwhile, a new lower bound on the smallest eigenvalue $q(A \circ B)$ for the Hadamard product of $B$ and $A^{-1}$ of two nonsingular $M$-matrices $A$ and $B$ are given. Some results of comparison are also given in theory. To illustrate our results, numerical examples are considered.

Keywords—Hadamard product, Fan product, nonnegative matrix, $M$-matrix, Spectral radius, Minimum eigenvalue, 1-path cover.

I. INTRODUCTION

Let $A$ and $B$ be nonnegative matrices. A new upper bound on the spectral radius $\rho(A \circ B)$ is obtained. Meanwhile, a new lower bound on the smallest eigenvalue $q(B \circ A^{-1})$ for the Hadamard product of $B$ and $A^{-1}$ of two nonsingular $M$-matrices $A$ and $B$ are given. Some results of comparison are also given in theory. To illustrate our results, numerical examples are considered.

I. INTRODUCTION

Let $A = (a_{ij})$ be an $n \times n$ matrix with all diagonal entries being nonzero throughout. For any $i, j, k \in N$, denote

$$R_i = \sum_{k \neq i} |a_{ik}|$$

$$d_i = \frac{R_i}{a_{ii}}$$

$$r_{ji} = \frac{|a_{ji}|}{\sum_{k \neq j} |a_{jk}|}, \quad j \neq i$$

$$r_i = \max_{j \neq i} (r_{ji})$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq i} |a_{jk}| r_i}{|a_{jj}|}, \quad j \neq i$$

$$s_j = \max_{i \neq j} (s_{ji})$$

Denote the set of all simple circuits in the digraph $\Gamma_A$ by $\Psi(A)$. A circuit of length $k$ in $\Gamma_A$ is an ordered sequence $\gamma = (i_1 \ldots i_k)$ where $i_1 \ldots i_k \in N$ are all distinct, and $i_k + 1 = i_1$. The set $\{i_1 \ldots i_k\}$ is called the support of $\gamma$ and is denoted by $\gamma$. The length of the circuit $\gamma$ is denoted by $|\gamma|$. $\eta$ is the greatest common divisor of 2 and $s$, $\tau = \frac{\tau}{\eta}$. $E(A) = \{a_{ij}|a_{ij} \neq 0, i, j \in N\}$ is the set of directed edge of $\Gamma(A)$. We say $\{c_{i_1}, c_{i_2}, c_{i_3} \ldots c_{i_k} \}$ is the $1$-path cover; $\{c_{i_1}, c_{i_2}, c_{i_3} \ldots c_{i_{\tau}} \}$ is the $\tau$-path cover; $\{c_{i_1}, c_{i_2}, c_{i_3} \ldots c_{i_{\eta}} \}$ is the $\eta$-path cover. The certain $\eta$-path cover of $\gamma$ recorded as $p^\eta(\gamma)$. When $s$ is an odd positive number, the odd and even $\eta$-path cover is the same, namely, only one $\eta$-path cover contains all the directed edge of $\gamma$. We denote $p^\eta(\gamma) = \bigcup_{\eta \in \Psi(A)} \gamma$ is a $1$-path cover of $\Gamma(A)$. For any $i, j \in N$, denote $\alpha = \{i \in N|i \in \gamma \in \Psi(A)\}$, $\Theta_A = \{a_{ii}|i \in N \setminus \alpha\}$, $A^\circ = \begin{bmatrix} A_{i_1i_1} & A_{i_1i_2} & \cdots & A_{i_1i_m} \\ A_{i_2i_1} & A_{i_2i_2} & \cdots & A_{i_2i_m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i_mi_1} & A_{i_mi_2} & \cdots & A_{i_mi_m} \end{bmatrix}$, $\{i_1, i_2, \ldots, i_m\} = \alpha$, $m^\eta_\gamma(\Delta) = \max_{\gamma \in \Psi(A)} \max_{\gamma \in \Psi(A)} \max_{r_\Delta(\gamma)} \max_{\gamma \in \Psi(A)} \max_{r_\Delta(\gamma)} \max_{\gamma \in \Psi(A)}$, $M^\eta_\gamma(\Delta) = \max_{\gamma \in \Psi(A)} \min_{\gamma \in \Psi(A)} r_\Delta(\gamma) \max_{\gamma \in \Psi(A)}$. $r_\Delta(\gamma)$ denotes the real roots of the equation

$$\prod_{i \in \gamma}(x - a_{ii}) = \prod_{i \in \gamma} R_i(\Delta^\circ),$$

Jing Li and Guang Zhou are with the School of Mathematics Science, University Electronic Science and Technology of China, Chengdu 611731, PR China.

Email address: zhoguang@163.com.
which greater than \( \max_{i \in \gamma} \{a_{ii}\} \).

II. MAIN RESULTS

For convenience, we give some lemmas which are useful for obtaining the main results.

**Lemma 2.1 [1]**. Let \( A \in \mathbb{R}^{n \times n} \) be an irreducible nonnegative matrix. Then
1) \( A \) has a positive real eigenvalue equals to its spectral radius;
2) To \( \rho(A) \) there corresponds an eigenvector \( x > 0 \).

**Lemma 2.2 [2]**. Let \( A, B \in \mathbb{R}^{n \times n} \). If \( E, F \) are diagonal matrices of order \( n \), then
\[
E(A \circ B)F = (EAF) \circ (BF) = (AF) \circ (EB) = A \circ (EAF).
\]
and
\[
E(A \ast B)F = (EAF) \ast (BF) = (AF) \ast (EB) = A \ast (EAF).
\]

**Lemma 2.3 [1]**. Let \( A \in \mathbb{R}^{n \times n} \), with \( n \geq 2 \). Then, if \( \lambda \) is an eigenvalue of \( A \), there is a pair \( (i, j) \) of positive integers with \( i \neq j, (1 \leq i, j \leq n) \) such that
\[
|\lambda - a_{ii}| \leq |\lambda - a_{jj}| \leq R_i R_j.
\]

**Lemma 2.4 [2]**. Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \), we have
\[
\beta_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j} |a_{jk}| r_k}{a_{jj}}, \quad \text{for all } j \neq i.
\]

**Lemma 2.5 [2]**. Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \), we have
\[
\beta_{ji} \leq s_j \beta_{ii}, \quad \text{for all } j \neq i.
\]

**Lemma 2.6 [3]**. Let \( A = (a_{ij}) \in \mathbb{M}_n \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \), we have

**Theorem 2.1 [5]**. Let \( A = (a_{ij}) \in \mathbb{M}_n \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \),\( B = (b_{ij}) \in \mathbb{M}_n \), we have
\[
q(B \circ A^{-1} - b_{ii}) \geq q(B) \min \beta_{ii}.
\]

**Theorem 2.2 [6]**. Let \( A = (a_{ij}) \in \mathbb{M}_n \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \), \( B = (b_{ij}) \in \mathbb{M}_n \), we have
\[
q(B \circ A^{-1} - b_{ii}) \geq \frac{1 - \rho(J_A \rho(J_B))}{1 + \rho^2(J_B)} \min b_{ii}. 
\]

**Theorem 2.3 [7]**. Let \( A = (a_{ij}) \in \mathbb{M}_n \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \), \( B = (b_{ij}) \in \mathbb{M}_n \), we have
\[
q(B \circ A^{-1}) \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\}. \tag{3}
\]

**Theorem 2.4**. Let \( A = (a_{ij}) \in \mathbb{M}_n \) be a strictly row diagonally dominant \( M \)-matrix. Then, for \( A^{-1} = (\beta_{ij}) \), \( B = (b_{ij}) \in \mathbb{M}_n \), we have
\[
q(B \circ A^{-1}) \geq \min_{1 \leq i \neq j \leq n} \left\{ b_{ii} \beta_{ii} + b_{jj} \beta_{jj} - \left( b_{ii} \beta_{ii} - b_{jj} \beta_{jj} \right)^2 \right\}. \tag{4}
\]

Proof: If \( A \) is irreducible, then \( 0 < s_i < 1 \), for any \( i \in N \). Since \( q(B \circ A^{-1}) \) is an eigenvalue of \( B \circ A^{-1} \). From Lemma 2.2 and Lemma 2.5, \( q(B \circ A^{-1}) = q(D^{-1}(B \circ A^{-1})D) = q(D(B^T \circ (A^{-1})^T)D^{-1}) \). Let \( D = (s_1, s_2, \ldots, s_n) > 0 \)
\[
R_i(B \circ A^{-1}) = R_i(D^{-1}(B \circ A^{-1})D) = R_i(D(B^T \circ (A^{-1})^T)D^{-1}) = \sum_{j \neq i} b_{ij} \beta_{ji} \frac{s_i}{s_j} \leq \sum_{j \neq i} s_j |b_{ij}| |s_i| \beta_{ii} \leq \sum_{j \neq i} s_i |b_{ij}| s_j |\beta_{ii}| = s_i \beta_{ii} \sum_{j \neq i} b_{ij}.
\]

Thus, by Lemma 2.3, there exists a pair \( (i, j) \) of positive integers with \( i \neq j, (1 \leq i, j \leq n) \) such that
\[
|q(B \circ A^{-1} - b_{ii})| |q(B \circ A^{-1})| - b_{jj} |b_{jj}| \leq s_i \beta_{ii} \sum_{j \neq i} |b_{ij}| s_j |\beta_{jj}| = s_i \beta_{ii} \sum_{j \neq i} b_{ij}.
\]

From the above inequality and \( 0 \leq q(B \circ A^{-1}) \leq a_{ii} b_{ii}, \forall i \in N \), we have
\[
q(B \circ A^{-1} - b_{ii}) |q(B \circ A^{-1})| - b_{jj} |b_{jj}| \leq s_i \beta_{ii} \sum_{j \neq i} b_{ij} |s_j |\beta_{jj}| = s_i \beta_{ii} \sum_{j \neq i} b_{ij}.
\]

Thus, from (5), we have
\[
q(B \circ A^{-1}) \geq \min_{1 \neq j \leq n} \left\{ b_{ii} \beta_{ii} + b_{jj} \beta_{jj} - \left( b_{ii} \beta_{ii} - b_{jj} \beta_{jj} \right)^2 \right\} \geq \min_{1 \neq j \leq n} \left\{ b_{ii} \beta_{ii} + b_{jj} \beta_{jj} - \left( b_{ii} \beta_{ii} - b_{jj} \beta_{jj} \right)^2 \right\} + 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ij}| s_j |\beta_{jj}| = 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ij}| s_j |\beta_{jj}|.
\]

If \( A \) is reducible, it is well known that a matrix in \( \mathbb{Z}_n \) is a nonsingular \( M \)-matrix if and only if all its leading principle minors are positive. If we denote by \( D = (d_{ij}) \) the \( n \times n \) permutation matrix with \( d_{12} = d_{23} = \cdots = d_{n-1n} = d_{n1} = 1 \),
the remaining $d_{ij}$ zero, then $A - tD$ is irreducible nonsingular $M$-matrices for any chosen positive real number $t$, sufficiently small such that all the leading principle minors of $A - tD$ is positive. Now we substitute $A - tD$ for $A$ in the previous case, and then letting $t \to 0$, the result follows by continuity. 

**Remark 2.1** We next give a simple comparison between the lower bound in (4) and the lower bound in (3). Without loss of generality, for $i \neq j$, assume that

$$b_{ii}\beta_{ii} - s_{ii}\beta_{ii} \sum_{j \neq i} |b_{ij}| \leq b_{jj}\beta_{jj} - s_{jj}\beta_{jj} \sum_{j \neq i} |b_{ij}| \quad (6)$$

Thus, we can write (6) equivalently as

$$s_{jj}\beta_{jj} \sum_{j \neq i} |b_{ij}| \leq b_{jj}\beta_{jj} - b_{ii}\beta_{ii} + s_{ii}\beta_{ii} \sum_{j \neq i} |b_{ij}|$$

From (4) and the above inequality, we get

$$b_{ii}\beta_{ii} + b_{jj}\beta_{jj} = \left[ (b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4s_{ii}s_{jj}\beta_{ii}\beta_{jj} \sum_{j \neq i} |b_{ij}| \right]^{\frac{1}{2}}$$

$$\geq b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[ (b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4(b_{jj}\beta_{jj} - b_{ii}\beta_{ii})s_{ii}\beta_{ii} \sum_{j \neq i} |b_{ij}| \right]$$

$$+ \left( 2s_{ii}\beta_{ii} \sum_{j \neq i} |b_{ij}| \right)^{\frac{1}{2}}$$

$$= 2b_{ii}\beta_{ii} - 2s_{ii}\beta_{ii} \sum_{j \neq i} |b_{ij}|$$

From Lemma 2.6, we have

$$q(B \circ A^{-1}) \geq \min_{j \neq i} \left\{ \left( \frac{b_{ii}\beta_{ii}}{b_{jj}\beta_{jj}} \right)^{\frac{1}{2}} \right\}$$

$$= \min_{j \neq i} \left\{ \frac{b_{ii}\beta_{ii}}{b_{jj}\beta_{jj}} \sum_{j \neq i} |b_{ij}| \right\}$$

Hence, the bound (4) is sharper than the bound (3).

**Theorem 2.5** If $A = (a_{ij}) \in R^{n \times n}$, $B = (b_{ij}) \in R^{n \times n}$, are two nonnegative matrices, then

$$\rho(A \circ B) \leq \max_{\gamma \in \Sigma(\rho(A \circ B))} \min_{i \neq j} r_{A \circ B}^{\gamma}(\gamma), \max_{\Theta_{A \circ B}}$$

$r_{A \circ B}(\gamma)$ denotes the real roots of the equation $\prod_{i \neq j} (x - a_{ij}b_{ij}) = \prod_{i \in \gamma} B(A \circ B)^{\circ}$ which greater than $\max_{i \in \gamma} a_{ii}b_{ii}$.

**Proof:** From Lemma 2.7 it is easy to obtain the desired result.

**Theorem 2.6** Let $A = (a_{ij}) \in M_n$ and $B = (b_{ij}) \in M_n$. Then

$$q(A \circ B) \geq \min_{i \neq j} \left\{ \left( \frac{a_{ii}b_{ij} + a_{jj}b_{ij} - (a_{ii}b_{ij} - a_{jj}b_{ij})}{4a_{ii}b_{ij} - q(B)(b_{ij} - q(B))} \right)^{\frac{1}{2}} \right\}$$

where $\alpha_i = \max_{k \neq i} |a_{ki}|, \forall i \in N$.

**Proof:** If $A \circ B$ is irreducible, then $A$ and $B$ are irreducible. Since, $A - q(A)I$ and $B - q(B)I$ are singular irreducible $M$-matrices. Then

$$a_{ii} - q(A) > 0, \forall i \in N$$

and

$$b_{ii} - q(B) > 0, \forall i \in N$$

Since $A = (a_{ij}), B = (b_{ij})$ are irreducible nonsingular $M$-matrices, there exists two positive vectors $u, v$ such that $Au = q(A)u, Bv = q(B)v$. Thus, we have

$$a_{ii} - \sum_{j \neq i} \frac{|a_{ij}|}{u_i} = q(A)$$

or equivalently,

$$\sum_{j \neq i} |a_{ij}| u_j = |a_{ii} - q(A)| u_i$$

and

$$b_{ii} - \sum_{j \neq i} \frac{|b_{ij}|}{v_i} = q(B)$$

or equivalently,

$$\sum_{j \neq i} |b_{ij}| v_j = |b_{ii} - q(B)| v_i$$

For convenience, let denote $\alpha_i = \max_{k \neq i} |a_{ki}|, \forall i \in N$. Since $B$ is an irreducible matrix, $\alpha_j > 0, \forall j \in N$. Define a positive diagonal matrix $Z = diag(z_1, \ldots, z_n)$, where

$$z_i = \frac{v_i}{\alpha_i} > 0, \forall i \in N$$

By Lemma 2.2, we have

$$q(A \circ B) = q(Z^{-1}(A \circ B)Z) = q(A \circ (Z^{-1}BZ))$$

so we have

$$R_i(Z^{-1}(A \circ B)Z) = R_i(A \circ BZ)$$

$$\leq \sum_{j \neq i} |b_{ij}| v_j \alpha_j$$

According to Lemma 2.3, there exists a pair $(i, j)$ of positive integers with $i \neq j (1 \leq i, j \leq n)$, such that

$$|q(A \circ B) - a_{ii}b_{ii}||q(A \circ B) - a_{jj}b_{jj}| \leq (b_{ii} - q(B))\alpha_i(b_{jj} - q(B))\alpha_j$$

From the above inequality and $0 \leq q(A \circ B) \leq a_{ii}b_{ii}, \forall i \in N$, we have

$$q(A \circ B) \leq a_{ii}b_{ii}q(A \circ B) - a_{jj}b_{jj} \leq \alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B))$$
If \( A \times B \) is reducible. It is well known that a matrix in \( Z_n \) is a nonsingular M-matrix if and only if all its leading principal minors are positive. If we denote by \( D = (d_{ij}) \) the \( n \times n \) permutation matrix with \( d_{12} = d_{23} = \cdots = d_{n-1n} = d_{nn} = 1 \), the remaining \( d_{ij} \) zero, then both \( A - tD \) and \( B - tD \) are irreducible nonsingular M-matrices for any chosen positive real number \( t \), sufficiently small such that all the leading principal minors of both \( A - tD \) and \( B - tD \) are positive. Now we substitute \( A - tD \) and \( B - tD \) for \( A \) and \( B \), respectively in the previous case, and then letting \( t \to 0 \), the result follows by continuity.

**Theorem 2.7** Let \( A = (a_{ij}) \in M_n \) and \( B = (b_{ij}) \in M_n \). Then

\[
q(A \times B) \geq \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[ (a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{1/2} \right\}
\]

where \( \beta_i = \max_{k \neq i}\{ |b_{ki}| \}, \forall i \in N \).

According to Theorem 2.6 and Theorem 2.7, it is easy to obtain the following corollary.

**Corollary 2.1** If \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are two \( n \times n \) nonsingular M-matrices, then

\[
q(A \times B) \geq \min_{\forall i, j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[ (a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{1/2} \right\}.
\]

where \( \alpha_i = \max_{k \neq i}\{ |a_{ki}| \} \) and \( \beta_i = \max_{k \neq i}\{ |b_{ki}| \}, \forall i \in N \).

**Corollary 2.2** If \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are two \( n \times n \) nonsingular M-matrices, then

\[
|\text{det}(A \times B)| \geq |q(A \times B)|^n \geq \min_{\forall i, j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[ (a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{1/2} \right\}^n,
\]

and

\[
|\text{det}(A \times B)| \geq |q(A \times B)|^n \geq \min_{\forall i, j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[ (a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\beta_i\beta_j(a_{ii} - q(A))(a_{jj} - q(A)) \right]^{1/2} \right\}^n.
\]

### III. Numerical Examples

**Example 3.1**

Let \( A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}, \)

\[
A = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}.
\]

By calculation, we have \( q(B \circ A^{-1}) = 0.2148 \). By the inequality (1), we get

\[
q(B \circ A^{-1}) \geq 0.07
\]

By the inequality (2), we get

\[
q(B \circ A^{-1}) \geq 0.052
\]

By the inequality (3), we get

\[
q(B \circ A^{-1}) \geq 0.075
\]

By Theorem 2.4, we have

\[
q(B \circ A^{-1}) \geq 0.1729.
\]

**Example 3.2**

Let \( A = \begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \)

It is easy to calculate that \( \rho(A \circ B) = \rho(A) = 8.1801 \). If we use Gersgorin theorem and Brauer theorem, we have

\[
\rho(A \circ B) \leq 9.
\]

and

\[
\rho(A \circ B) \leq 9.
\]

If we take \( p^i(A) = \{ e_1, e_2, e_3, e_4, e_5 \} \), \( r_{A \circ B}(1,2) = r_{A \circ B}(1,4) = r_{A \circ B}(2,5) = 8.3166, r_{A \circ B}(1,5) = 9, r_{A \circ B}(2,4) = 4, r_{A \circ B}(3,4) = 6, r_{A \circ B}(1,3) = r_{A \circ B}(3,5) = 8.5616. \)

From Theorem 2.5 we get

\[
\rho(A \circ B) \leq M_{n+1}'(A \circ B) = \max \left\{ \min_{\forall \gamma \in q(\theta_{A \circ B})} \rho(A \circ B), \max \Theta_{A \circ B} \right\} = 8.3166.
\]

**Example 3.3**

Let \( A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \)

By calculation, we have

\[
q(A \times B) = 6
\]

By Theorem 2.6, we get

\[
q(A \times B) = 6.
\]
IV. CONCLUSIONS

In this paper, we give some inequalities for the spectral radius of the Hadamard product of two nonnegative matrices. These bounds improve some existing results and numerical examples illustrate that our results are superior.

REFERENCES


Jing Li was born in Henan Province, China, in 1988. She received the B.S. degree from Huabei Normal University in 2011. She is currently pursuing the M.S. degree from University of Electronic Science and Technology of China. Her research interests are numerical algebra and matrix analysis.

Guang Zhou was born in Anhui Province, China, in 1987. He received the B.S. degree from Fuyang University in 2011. He is currently pursuing the M.S. degree from University of Electronic Science and Technology of China. His research interests include chaos synchronization, switch and delay dynamic systems.