Cubic B-spline collocation method for numerical solution of the Benjamin-Bona-Mahony-Burgers equation

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Abstract—In this paper, numerical solutions of the nonlinear Benjamin-Bona-Mahony-Burgers (BBMB) equation are obtained by a method based on collocation of cubic B-splines. Applying the Von-Neumann stability analysis, the proposed method is shown to be unconditionally stable. The method is applied on some test examples, and the numerical results have been compared with the exact solutions. The $L_\infty$ and $L_2$ in the solutions show the efficiency of the method computationally.

Keywords—Benjamin-Bona-Mahony-Burgers equation; Cubic B-spline; Collocation method; Finite difference.

I. INTRODUCTION

In this paper we consider the solution of the BBMB equation

$$u_t - u_{xx} - \alpha u_{x} + \beta u_{x} + uu_x = 0, \quad x \in [a, b], \quad t \in [0, T],$$

with the initial condition

$$u(x, 0) = f(x), \quad x \in [a, b],$$

and boundary conditions

$$u(a, t) = u(b, t) = 0,$$

where $\alpha$ and $\beta$ are constants.

BBMB equations play a dominant role in many branches of science and engineering [1]. For $\alpha = 0$, Eq. (1) is called the Benjamin-Bona-Mahony (BBM) equation. In the past several years, many different methods have been used to estimate the solution of the BBMB equation and the BBM equation, for example, see [2-6].

The paper is organized as follows. In Section 2, cubic B-spline collocation method is explained. In Section 3, we develop an algorithm for the numerical solution of the BBMB equation. Section 4, is devoted to stability analysis of the method. In Section 5, examples are presented. A summary is given at the end of the paper in Section 6. Note that we have computed the numerical results by Mathematica-7 programming.

II. CUBIC B-SPLINE COLLOCATION METHOD

The interval $[a, b]$ is partitioned into a mesh of uniform length $\Delta x = \frac{b - a}{N}$ by the knots $x_{i}, i = 0, 1, \ldots, N$ such that $a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$. Our numerical treatment for BBMB equation using the collocation method with cubic B-spline is to find an approximate solution $U_N(x, t)$ to the exact solution $u(x, t)$ in the form

$$U_N(x, t) = \sum_{i=-3}^{N-1} c_i(t) B_i(x),$$

where $c_i(t)$ are time-dependent quantities to be determined from the boundary conditions and collocation form of the differential equations. Also $B_i(x)$ are the cubic B-spline basis functions at knots, given by [7,8]

$$B_i(x) = \left\{ \begin{array}{ll}
\frac{(x-x_0)^3}{6h^3}, & x \in [x_0, x_{i+1}], \\
\frac{1}{6h^3}, & x \in [x_{i+1}, x_{i+2}], \\
\frac{h^3 + 3h^2(x - x_{i+1})^2 + 3h(x - x_{i+1})}{6h^3}, & x \in [x_{i+1}, x_0], \\
\frac{h^3 + 3h(x_{i+3} - x)^2 + 3h(x_{i+3} - x)}{6h^3}, & x \in [x_0, x_{i+3}], \\
\frac{(x_{i+4} - x)^3}{6h^3}, & x \in [x_{i+3}, x_{i+4}], \\
0, & \text{otherwise}. \end{array} \right. \quad (5)$$

The values of $B_i(x)$ and its derivatives may be tabulated as in Table 1. The values of $U_i$ and its space derivatives at the knots $x_i$ can be obtained as

$$U_i = \frac{1}{6} (c_{i-3} + 4c_{i-2} + c_{i-1}),$$

$$U_i' = \frac{1}{2h} (c_{i-1} - c_{i-3}),$$

$$U_i'' = \frac{1}{h^2} (c_{i-3} - 2c_{i-2} + c_{i-1}).$$

III. CONSTRUCTION OF THE METHOD

To apply the proposed method, discretizing the time derivative in the usual finite difference way. Using the finite difference method, we can write

$$\frac{u_{i}^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_i^{n+1} + u_{i+1}^{n+1}}{2} +$$
\[
\beta \frac{u_{n+1} + u_n}{2} + (u_{n+1})^n + (u_n)^n = 0.
\]

(9)

The nonlinear term in Eq. (9) can be approximated by using the following formula [9]:

\[
(u_{n+1})^n = u_{n+1} u_n^n + u_n^n u_{n+1} - (u_n)^n.
\]

(10)

Substituting the approximate solution \( u \) for \( u \) and putting the values of the nodal values \( U \) and its derivatives using Eqs. (6)-(8) at the knots in Eq. (9) yield the following difference equations.

\[
\alpha c_{i-3} + \beta c_{i-2} + \gamma c_{i-1} = \delta c_i + \epsilon c_{i+1} + \phi c_{i+2}, \quad i = 0, 1, \ldots, N.
\]

(11)

where

\[
\begin{align*}
\alpha &= \frac{\Delta x}{h}, \\
\beta &= \frac{\Delta x}{h} + \frac{\Delta x}{y}, \\
\gamma &= \frac{\Delta x}{h} + \frac{\Delta x}{z}, \\
\delta &= 1 + \frac{\Delta x}{h} - \frac{\Delta x}{y}, \\
\epsilon &= 4 - \frac{2\Delta x}{y}, \\
\phi &= 1 + \frac{\Delta x}{h} + \frac{\Delta x}{y}.
\end{align*}
\]

with \( x = 1 + \frac{\Delta t u_0}{2}, \ y = \frac{\Delta t u_0}{2} + \frac{\Delta t u_0}{2}, \ z = -1 - \frac{\Delta t u_0}{2}, \ w = -1 + \frac{\Delta t u_0}{2}, \ v = -\frac{\Delta t u_0}{2}.
\]

The system (11) consists of \( N + 1 \) linear equations in \( N + 3 \) unknowns \( \{c_{-3}, c_{-2}, \ldots, c_{N-2}, c_{N-1}\} \). To obtain a unique solution for \( C = \{c_{-3}, \ldots, c_{N-1}\} \), we must use the boundary conditions. From the boundary conditions and Table 1, we can write

\[
\begin{align*}
\frac{1}{6}(c_{-3} + 4c_{-2} + c_{-1}) &= 0, \\
\frac{1}{6}(c_{N-3} + 4c_{N-2} + c_{N-1}) &= 0.
\end{align*}
\]

(13)

(14)

Associating (13) and (14) with (11) we obtain a \((N + 3) \times (N + 3)\) system of equations in the following form

\[
AC = Q,
\]

(15)

where

\[
A = \begin{pmatrix}
\frac{1}{\gamma} & \frac{1}{\delta} & \frac{1}{\epsilon} & \cdots & 0 \\
\frac{1}{\delta} & \frac{1}{\epsilon} & \frac{1}{\phi} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\]

(16)

\[
C = (c_{-3}, c_{-2}, \ldots, c_{N-2}, c_{N-1})^T.
\]

(17)

with \( x = -1 - \frac{\Delta t u_0}{2}, \ y = \frac{\Delta t u_0}{2} + \frac{\Delta t u_0}{2}, \ z = -1 - \frac{\Delta t u_0}{2}. \)

with \( \dot{x} = -1 - \frac{\Delta t u_0}{2}, \ \dot{y} = \frac{\Delta t u_0}{2} + \frac{\Delta t u_0}{2}, \ \dot{z} = -1 - \frac{\Delta t u_0}{2}. \)

Dividing both sides of (20) by \( \exp((i-2)\Delta t) \), we can write:

\[
\xi^{n+1} \left( a \exp((i)\lambda k) + \beta \exp((-1)\lambda k) \right) = \xi^n \left( d \exp((i)\lambda k) + \epsilon \exp((-2)\lambda k) \right),
\]

(22)

Eq. (22) can be rewritten in a simple form as:

\[
\xi = \frac{X - \lambda Y}{X_1 + \lambda Y},
\]

(23)
where

\[ X = \left( \frac{1}{b} + \frac{i}{\Delta t} \right) \cos(kh) + \left( \frac{1}{3} - \frac{i}{\Delta t} \right), \]

\[ X_1 = \left( \frac{1}{b} + \frac{i}{\Delta t} \right) \cos(kh) + \left( \frac{1}{3} + \frac{i}{\Delta t} \right), \]

\[ Y = \left( \frac{\Delta t}{2b} \right) \sin(kh). \]

X and X_1 can be rewritten in the form:

\[ X_1 = \left( \frac{1}{b} - \frac{i}{\Delta t} \right) \cos(kh) + \left( \frac{1}{3} + \frac{i}{\Delta t} \right) + \frac{\Delta t}{2b}(1 - \cos(kh)), \]

\[ X = \left( \frac{1}{b} - \frac{i}{\Delta t} \right) \cos(kh) + \left( \frac{1}{3} - \frac{i}{\Delta t} \right) - \frac{\Delta t}{2b}(1 - \cos(kh)). \]

We note that \( X \leq X_1 \), so \( |\xi|^2 = \xi^2 \leq 1 \). Therefore, the linearized numerical scheme for the BBMB equation is unconditionally stable.

V. NUMERICAL EXAMPLES

We now obtain the numerical solutions of the BBMB equation for two problems. To show the efficiency of the present method for our problem in comparison with the exact solution, we report \( L_\infty \) and \( L_2 \) using formulae

\[ L_\infty = \max_i \left| U(x_i, t) - u(x_i, t) \right|, \]

\[ L_2 = (h \sum_i \left| U(x_i, t) - u(x_i, t) \right|^2)^{1/2}, \]

where \( U \) is numerical solution and \( u \) denotes analytical solution.

**Example 1.** Consider the BBMB equation with \( \alpha = 0 \) and \( \beta = 1 \) in the interval \([-40, 60]\), with the exact solution \( u(x, t) = 3\csech^2(k(x-vt-x_0)) \). We have taken \( c = 0.03, v = 1, x_0 = 0 \) and \( k = \frac{4.71001}{\sqrt{3}} \). The initial condition is taken from the exact solution. Table 2 gives a comparison between the \( L_\infty \) and \( L_2 \) found by our method in different times and different values of \( N \) with \( \Delta t = 0.1 \). Also Table 3 gives comparison of absolute errors found by present method with \( \Delta t = 0.01 \) and \( N = 300 \).

**Example 2.** As a last study we consider here a numerical solution of the BBMB in the interval \([-12, 12]\) with \( \alpha = 1 \), \( \beta = 1 \) and initial condition \( u(x, 0) = \csech^2(\frac{x}{4}) \). Tables 4 and 5 give numerical results with \( \Delta t = 0.01 \) and \( N = 200 \). Also Fig 3 shows approximate solution graphs.

**TABLE II**

<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
<th>N</th>
<th>( L_2 \times 10^4 )</th>
<th>( L_\infty \times 10^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>present method</td>
<td>1</td>
<td>100</td>
<td>0.902614</td>
<td>0.328588</td>
</tr>
<tr>
<td>present method</td>
<td>10</td>
<td>100</td>
<td>8.14052</td>
<td>1.76978</td>
</tr>
<tr>
<td>present method</td>
<td>20</td>
<td>100</td>
<td>16.2506</td>
<td>3.53644</td>
</tr>
<tr>
<td>present method</td>
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<td>300</td>
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</tr>
<tr>
<td>present method</td>
<td>10</td>
<td>300</td>
<td>4.71001</td>
<td>1.77131</td>
</tr>
<tr>
<td>present method</td>
<td>20</td>
<td>300</td>
<td>9.40151</td>
<td>3.54203</td>
</tr>
<tr>
<td>method in [12]</td>
<td>1</td>
<td>1000</td>
<td>14.45</td>
<td>3.996</td>
</tr>
</tbody>
</table>

**TABLE III**

<table>
<thead>
<tr>
<th>( x/t )</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>-30</td>
<td>2.12681 \times 10^{-4}</td>
<td>1.99925 \times 10^{-4}</td>
<td>5.42065 \times 10^{-4}</td>
</tr>
<tr>
<td>-20</td>
<td>1.06721 \times 10^{-3}</td>
<td>1.00969 \times 10^{-3}</td>
<td>2.72086 \times 10^{-3}</td>
</tr>
<tr>
<td>-10</td>
<td>3.57975 \times 10^{-3}</td>
<td>3.52499 \times 10^{-3}</td>
<td>1.00584 \times 10^{-3}</td>
</tr>
<tr>
<td>0</td>
<td>7.45779 \times 10^{-3}</td>
<td>7.40873 \times 10^{-3}</td>
<td>5.59997 \times 10^{-3}</td>
</tr>
<tr>
<td>10</td>
<td>3.9186 \times 10^{-3}</td>
<td>4.06787 \times 10^{-3}</td>
<td>1.25094 \times 10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>1.36982 \times 10^{-3}</td>
<td>1.50618 \times 10^{-3}</td>
<td>5.09114 \times 10^{-4}</td>
</tr>
<tr>
<td>30</td>
<td>2.81202 \times 10^{-4}</td>
<td>3.12619 \times 10^{-4}</td>
<td>1.07576 \times 10^{-4}</td>
</tr>
</tbody>
</table>

**TABLE IV**

<table>
<thead>
<tr>
<th>( x/t )</th>
<th>0.2</th>
<th>0.5</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12</td>
<td>3.333 \times 10^{-4}</td>
<td>-2.233 \times 10^{-4}</td>
<td>-3.333 \times 10^{-4}</td>
</tr>
<tr>
<td>-10</td>
<td>0.023913</td>
<td>0.019827</td>
<td>0.0179501</td>
</tr>
<tr>
<td>-5</td>
<td>0.256278</td>
<td>0.224742</td>
<td>0.206391</td>
</tr>
<tr>
<td>0</td>
<td>0.978102</td>
<td>0.933352</td>
<td>0.897596</td>
</tr>
<tr>
<td>5</td>
<td>0.319376</td>
<td>0.380993</td>
<td>0.42342</td>
</tr>
<tr>
<td>10</td>
<td>0.0304198</td>
<td>0.0397963</td>
<td>0.0472631</td>
</tr>
<tr>
<td>12</td>
<td>2 \times 10^{-10}</td>
<td>6.66666 \times 10^{-10}</td>
<td>-2.66667 \times 10^{-10}</td>
</tr>
</tbody>
</table>
Fig. 3. Approximate solution graphs of Example 2 for $x \in [-12, 12]$ with $\Delta t = 0.01$ and $N = 200$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t$</th>
<th>$t = 1$</th>
<th>$t = 1.5$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12</td>
<td>0.01543</td>
<td>1.333333 $\times 10^{-11}$</td>
<td>-1.26667 $\times 10^{-10}$</td>
<td>8.66667 $\times 10^{-11}$</td>
</tr>
<tr>
<td>-10</td>
<td>0.01934</td>
<td>0.0119344</td>
<td>0.00916508</td>
<td>0.00916508</td>
</tr>
<tr>
<td>-6</td>
<td>0.149239</td>
<td>0.123215</td>
<td>0.123215</td>
<td>0.123215</td>
</tr>
<tr>
<td>0</td>
<td>0.733537</td>
<td>0.631526</td>
<td>0.631526</td>
<td>0.631526</td>
</tr>
<tr>
<td>1</td>
<td>0.589817</td>
<td>0.676809</td>
<td>0.676809</td>
<td>0.676809</td>
</tr>
<tr>
<td>10</td>
<td>0.088659</td>
<td>0.12528</td>
<td>0.12528</td>
<td>0.12528</td>
</tr>
<tr>
<td>12</td>
<td>-1.13333 $\times 10^{-9}$</td>
<td>-3.3333 $\times 10^{-10}$</td>
<td>-1.01048 $\times 10^{-16}$</td>
<td>-1.01048 $\times 10^{-16}$</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

The cubic B-spline collocation method is used to solve the Benjamin-Bona-Mahony-Burgers (BBMB) equation. The stability analysis of the method is shown to be unconditionally stable. The numerical results given in the previous section demonstrate the good accuracy and stability of the proposed scheme in this research.

REFERENCES
