Dense chaos in coupled map lattices

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Abstract—This paper is mainly concerned with a kind of coupled map lattices (CMLs). New definitions of dense \( \delta \) -chaos and dense chaos (which is a special case of dense \( \delta \) -chaos with \( \delta = 0 \)) in discrete spatiotemporal systems are given and sufficient conditions for these systems to be densely chaotic or densely \( \delta \) -chaotic are derived.

Keywords—Discrete spatiotemporal systems; coupled map lattices; dense \( \delta \) -chaos; Li-Yorke pairs.

I. INTRODUCTION

CHAOTIC properties of a dynamical system are ardently discussed since the introduction of the term chaos in 1975 by Li and Yorke[1]. Let \( I \) be a closed interval on real line. If a dynamical system \((I, f)\) has an uncountable set \( S \subset I \) in which \((x, y)\) is a Li-Yorke pair for \( \forall x, y \in S : x \neq y \) (the definition of Li-Yorke pairs will be seen follow), then \((I, f)\) is said to be chaotic in the sense of Li-Yorke. While, the definition of chaos in the sense of Li-Yorke is inconveniently in engineering applications. In 1989, R. L. Devaney[2] stated a definition of chaos, known as Devaney chaos today. A map \( f \) is said to be chaotic in the sense of Devaney on \( I \) if \( f \) is transitive on \( I \), the set of periodic points of \( f \) is dense in \( I \) and \( f \) has sensitive dependence on initial conditions. Then, in 1992, Banks[3] proved that if \( f : (X, d) \to (X, d) \) is transitive and has dense periodic points then \( f \) has sensitive dependence on initial conditions (where \( X \) is a compact metric space with no isolated point). This causes that Devaney’s chaoticity is preserved under topological conjugation on generally infinite metric space. And then, in 1992, Lubomir Snoha[4] say that \( f \) is densely chaotic if the set of Li-Yorke pairs is dense in \( I \times I \). In 2005, the definition of densely \( \delta \) -chaotic is given by Schweizer and Smital[5]. That is, \( f \) is densely \( \delta \) -chaotic if the set of Li-Yorke pairs modulus \( \delta \) is dense in \( I \times I \).

The coupled map lattices (CMLs) as spatiotemporal chaotic systems were proposed in 1983 by Kaneko[6]. Since it is a simple model with most essential features of spatiotemporal chaos, the CMLs have been extensively studied in the fields of bifurcation and chaos, pattern formation, physical biology and engineering, cryptography, thermodynamics and chaotic dynamics. (see, for instance, [7-12] and some references therein.)

In this paper, we consider the CML of the form
\[
x_{m+1,n} = (1 - \varepsilon)f(x_{m,n}) + \varepsilon[f(x_{m,n-1}) + f(x_{m,n+1})],
\]
where \( f : \mathbb{R} \to \mathbb{R} \) (\( \mathbb{R} \) denotes the real numbers) is a function, \( m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), \( n \in \mathbb{Z} = \{-1, 0, 1, \ldots\} \), and \( \varepsilon \in [0, 1] \) is a constant.

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Let \( \mathbb{N}_t = \{t, t + 1, \ldots\} \) with integer \( t \in \mathbb{Z} \) and denote \( \Omega = \{(0, n) | n \in \mathbb{Z}\} = \{\ldots, (0, -1), (0, 0), (0, 1), \ldots\} \).

For any sequence \( \phi = \{\phi_{m,n}\} \) defined on \( \Omega \), it is easy to construct by induction a double-indexed sequence \( x = \{x_{m,n} | m = 0, 1, 2, \ldots; n = \ldots, -1, 0, 1, \ldots\} \) that equals the initial condition \( \phi \) on \( \Omega \) and satisfies system (1) on \( \mathbb{N}_1 \times \mathbb{Z} \).

In fact, from (1), for any \( n \in \mathbb{Z} \), one can calculate a sequence \( x_1 = \{x_{1,n}\}_{n=-\infty}^{\infty} = \{(\ldots, x_{1,-1}, x_{1,0}, x_{1,1}, \ldots) \} \) by using the initial condition \( \phi \). Then, by induction, for any \( m \in \mathbb{N}_0 \), one can calculate a sequence \( x_m = \{x_{m,n}\}_{n=-\infty}^{\infty} \), so as to obtain \( x = \{x_{m,n} | m = 0, 1, 2, \ldots; n = \ldots, -1, 0, 1, \ldots\} \) satisfying system (1), which is said to be a solution of system (1) with initial condition \( \phi \).

In 2003, Chen and Liu[13] initiated the study of chaos in the sense of Li-Yorke for a certain type of discrete spatiotemporal systems by using a method similar to the discussion in 1D discrete systems. Then, in 2005, Chen et al. [14] showed one kind of close relationship between a 2D discrete system and an infinite-dimensional discrete system, thus introducing a new definition of chaos for 2D systems in the sense of Devaney. And in 2007, a definition about chaos in the sense of Li-Yorke in discrete spatiotemporal systems is given by Tian and Chen [15]. Along the same line, this paper introduced definitions about dense \( \delta \) -chaos and dense chaos in discrete spatiotemporal systems. Some sufficient conditions for system (1) to be densely \( \delta \) -chaotic or densely chaotic is derived.

II. DENSE \( \delta \) -CHAOS IN METRIC SPACES

A metric (or distance) on a set \( X \) is a function \( d : X \times X \to \mathbb{R}^+ = [0, \infty) \) with the following properties:

(1) \( d(x, y) \geq 0 \) for all \( x, y \in X \) with \( d(x, y) = 0 \) if and only if \( x = y \);

(2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(3) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \), where \( d(x, y) \) is called the distance between \( x \) and \( y \). The pair \((X, d)\) is called a metric space.

Definition 2.1 Let \((X, d)\) and \((Y, d)\) be two metric spaces, and \( h : X \to Y \) be an one-to-one and onto map. If there exist two positive constants \( \alpha, \beta > 0 \) such that:

\[\alpha d(x, y) \leq d(h(x), h(y)) \leq \beta d(x, y), \quad \text{for all} \; x, y \in X,\]

\[\text{where} \; h = h(x) \text{ and} \; \bar{y} = h(y), \quad \text{then the metric space} \; (X, d) \text{ is said to be equivalent to the metric space} \; (Y, d) \text{ (with respect to the map} \; h). \]

If \( X = Y \) and \( h(x) = x \) for all \( x \in X \), then \( d \) is said to be equivalent to \( d \).

Referring to the definitions of dense chaos and dense \( \delta \) -chaos for a real compact interval \( I \) in [4,5], definitions of dense chaos and dense \( \delta \) -chaos defined on a metric space are naturally generalized as follows.
Lyke chaos is also concerned with Li-Yorke pairs. However, let

\[ (a, b) = \limsup_{n \to \infty} d(f^n(x), f^n(y)) > \delta, \]

where \( \delta \) is a constant. By the similar argument, the sufficiency is follows immediately.

**Remark** For some constants \( \delta > 0 \), if \( f \) is densely \( \delta \)-chaotic on \( (I, d_1) \), \( f \) may not be densely \( \delta \)-chaotic on \( (I, d_2) \) even though \( d_1 \) is equivalent to \( d_2 \). In fact \( f \) is densely \( \alpha \delta \)-chaotic on \( (I, d_2) \) (where \( \alpha \) is a positive constant).

### III. A NEW DEFINITION OF DENSE CHAOS IN SPATIOTEMPORAL SYSTEMS

In this section, new definitions of dense chaos and dense \( \delta \)-chaos in discrete spatiotemporal systems are introduced. Let \( \mathbb{R}_\infty^\infty \) be a set of (bi-directional) 1D real sequences, i.e.,

\[ \mathbb{R}_\infty^\infty = \{(a_n)_{n=-\infty}^\infty : a_n \in \mathbb{R}, n \in \mathbb{Z}\}. \]

Obviously, several different metrics can be defined on \( \mathbb{R}_\infty^\infty \). For example, for two sequences

\[ x_1 = (x_{1,n})_{n=-\infty}^\infty \in \mathbb{R}_\infty^\infty, \]
\[ x_2 = (x_{2,n})_{n=-\infty}^\infty \in \mathbb{R}_\infty^\infty, \]

one may defines

\[ d_1(x_1, x_2) = \sum_{n=-\infty}^\infty |x_{1,n} - x_{2,n}|, \]
\[ d_2(x_1, x_2) = \sup_{n \in \mathbb{Z}} \{|x_{1,n} - x_{2,n}| : n \in \mathbb{Z}\}; \]
\[ d_3(x_1, x_2) = \sum_{n=-\infty}^\infty \frac{1}{2^n} |x_{1,n} - x_{2,n}|. \]

Then, it is easy to prove that \( d_1(i = 1, 2, 3, 4) \) are metrics on \( \mathbb{R}_\infty^\infty \). Moreover, \( d_1 \) is not equivalent to \( d_j (i \neq j, i, j = 1, 2, 3, 4) \).

Let \( I \) be a subset of \( \mathbb{R} \) and denote

\[ I_\infty^\infty = \{(a_n)_{n=-\infty}^\infty : a_n \in \mathbb{R}, I \subseteq \mathbb{R}, n \in \mathbb{Z}\}. \]

It is obvious that \( (I_\infty^\infty, d) \) is a metric subspace of \( (\mathbb{R}_\infty^\infty, d) \).

Let \( f : I \to I \) be a function and

\[ x = (x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}) \]

be a solution of system (1) with initial condition

\[ \phi = (\phi_n = \phi_0, n \in \mathbb{N}_0) \]

for all \( n \in \mathbb{N}_0 \).

And denote

\[ x_m = (x_{m,n})_{n=-\infty}^\infty = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots) \]

for all \( m \in \{0, 1, 2, \ldots\} \).

Let

\[ x_{m+1} = (x_{m+1,n})_{n=-\infty}^\infty = (\cdots, x_{m+1,-1}, x_{m+1,0}, x_{m+1,1}, \cdots) \]

\[ = F(x_m), \]

where \( F(x) = (\cdots, x_{n+1}, \cdots) \) and

\[ x_{m+1,n} = (1 - \varepsilon)f(x_{m,n}) + \frac{\varepsilon}{2}[f(x_{m,n-1}) + f(x_{m,n+1})], \]

\( m \in \mathbb{N}_0, n \in \mathbb{Z} \).

Then, one can see that system (1) is equivalent to a system in the form of

\[ x_{m+1} = F(x_m), x_m \in I_\infty^\infty \subseteq \mathbb{R}_\infty^\infty, m = 0, 1, 2, \cdots \].

(6)

The map \( F \) of system (6) is said to be induced by system (1). And \( (f, F) \) is a pair of maps associated with the two systems (1) and (6).

Obviously, a double-indexed sequence \( \{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\} \) is a solution of system (1) if and only if the sequence \( \{x_{m,n}\}_{m=0}^\infty \) is a solution of system (6), where

\[ x_m = (x_{m,n})_{n=-\infty}^\infty, m \in \mathbb{N}_0. \]
Definition 3.1 Let $(I_{\infty}^2, d)$ be a metric space. For any $(x, y)$ in $I_{\infty}^2 \times I_{\infty}^2$, $\forall \epsilon > 0$, a point $(x_1, y_1)$ in $I_{\infty}^2 \times I_{\infty}^2$ is said to be in $\epsilon$-neighborhood of $(x, y)$, if $d((x_1, y_1), (x, y)) < \epsilon$.

The $\epsilon$-neighborhood of $(x, y)$ is denoted by $B((x, y), \epsilon)$.

Definition 3.2 Let $I$ be a subset of $\mathbb{R}$, $f : I \rightarrow I$ is a function and $F : I_{\infty} \rightarrow I_{\infty}$ is a map on $(I_{\infty}^2, d)$ induced by system (1). If the map $F$ is chaotic on $I_{\infty}^2$, i.e., system (6) is chaotic on $(I_{\infty}^2, d)$, then system (1) is said to be chaotic on $(I_{\infty}^2, d)$.

In particular, if $F$ is densely chaotic on $(I_{\infty}^2, d)$, then system (1) is said to be densely chaotic on $(I_{\infty}^2, d)$. If $F$ is densely $\delta$-chaotic on $(I_{\infty}^2, d)$, then system (1) is said to be densely $\delta$-chaotic on $(I_{\infty}^2, d)$.

The following conclusion is easy to check.

Theorem 3.1 Assume that $I \subset \mathbb{R}$, $f : I \rightarrow I$ is a function, $I_{\infty}^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \in I, n \in \mathbb{Z}, m \in \mathbb{N}_0\}$, $F : I_{\infty}^2 \rightarrow I_{\infty}^2$ is a map induced by system (1) with the function $f$. $d_1, d_2$ are two metrics in $I_{\infty}^2$ and $d_1$ is equivalent to $d_2$. Then $F$ is densely chaotic on $(I_{\infty}^2, d_1)$ if and only if $F$ is densely chaotic on $(I_{\infty}^2, d_2)$.

IV. MAIN RESULTS

In this section, we will consider that system (1) is densely $\delta$-chaotic or not (i.e., $F$ is densely $\delta$-chaotic or not) if $F$ is densely $\delta$-chaotic.

Theorem 4.1 Assume that $I \subset \mathbb{R}$, and define the distance in $I$ with $d(a, b) = |a - b|$ for any two points $a, b \in I$ (where $| \cdot |$ denotes modulus). $f : I \rightarrow I$ is a function. Let $\Delta_{\infty} = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1, x_2, n \in I, n \in \mathbb{Z}, m \in \mathbb{N}_0\}$, $F : \Delta_{\infty} \rightarrow \Delta_{\infty}$ is a map induced by system (1) with the function $f$. If the function $f$ is densely $\delta$-chaotic (where $\delta \geq 0$ is a constant), then system (1) is densely $\delta$-chaotic on $(\Delta_{\infty}, d_1)$.

Where $d_1$ is defined by (2).

Proof: If $(a, b) \in I \times I \neq (a \neq b)$ is a Li-Yorke pair with modulus $\delta$ of $f$, i.e.,

$$\limsup_{n \rightarrow \infty} |f^k(a) - f^k(b)| \geq \delta$$

and

$$\liminf_{n \rightarrow \infty} |f^k(a) - f^k(b)| = 0.$$

In the following, we show that $(x^*, y^*) \in I_{\infty}^2 \times I_{\infty}^2$ is a Li-Yorke pair with modulus $\delta$ of $F$, where

$$x^* = \{x_n = a\}_{n=-\infty}^\infty$$

and

$$y^* = \{y_n = b\}_{n=-\infty}^\infty.$$

In fact, $F(x^*) = \{f(x^*_n)\}_{n=-\infty}^\infty$ and $F(y^*) = \{f(y^*_n)\}_{n=-\infty}^\infty$.

Then

$$d_1(F^k(x^*), F^k(y^*)) = \{\{f^k(x^*_m)\}_{n=-\infty}^\infty, \{f^k(y^*_n)\}_{n=-\infty}^\infty\} = \limsup_{n \rightarrow \infty} |f^k(a) - f^k(b)| = \frac{3}{2} |f^k(a) - f^k(b)|.$$

The following results are straightforward.

$$\limsup_{n \rightarrow \infty} d_1(F^k(x^*), F^k(y^*)) = 3 \limsup_{n \rightarrow \infty} |f^k(a) - f^k(b)| > 3 \delta > \delta.$$
Li-Yorke pair of system (1) with the function
\( x \) of Definition (12ZA098).

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REFERENCES
