The symmetric solutions for three-point singular boundary value problems of differential equation

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Abstract—In this paper, by constructing a special operator and using fixed point index theorem of cone, we get the sufficient conditions for symmetric positive solution of a class of nonlinear singular boundary value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Banach space, cone, fixed point index, singular differential equation, p-Laplace operator, symmetric solutions.

I. INTRODUCTION

THE boundary value problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method (see[1-4]). On the other hand, the study for the symmetric and multiple solutions to this problem is more and more active (see[5-6]). In paper [5], Sun study for the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
(u')' + a(t)f(t,u(t)) = 0, \\
u(0) = \alpha u(\eta) = u(1),
\end{array} \right. \\
\end{align*}
\]

where \( \alpha \in (0,1), \eta \in (0,1) \), by using spectrum theory, Sun get the existence of symmetric and multiple solution. But when \( p \neq 2 \), \( \phi_p(u) \) is nonlinear, so the method of the paper [5] is not suitable to p-Laplace operator. In paper [6], Tian and Liu study for the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
(\phi_p(u'))' + a(t)f(t,u(t)) = 0, \\
u(0) = \alpha u(\eta) = u(1),
\end{array} \right. \\
\end{align*}
\]

where \( \phi(s) \) is p-Laplace operator. Motivated by paper [5,6], we consider the existence of solution for the following problems:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\phi_p(u'))' + h_1(t) f(u,v) = 0, \\
(\phi_p(v'))' + h_2(t) g(u,v) = 0, \\
u(0) = \alpha u(\eta) = u(1), \\
v(0) = \gamma v(\eta) = v(1),
\end{array} \right. \\
\end{align*}
\]

where \( t \in (0,1), \gamma \in (0,1), \eta \in (0,\frac{1}{2}] \), \( \phi(s) \) is a p-Laplace operator, i.e. \( \phi_p(s) = |s|^{p-2}s, p > 1 \). Obviously, if \( p = 2 \), then \( \phi_p^{-1} = \phi_q \).

Compare with above paper, our method is different. By constructing a new operator, and using fixed point index theorem, we get the sufficient condition of the existence of symmetric solution, which improved and generalized the result of paper [5,6,7].

In this paper, we always suppose that the following conditions hold:

\( (H_1) \) \( f \in C([0,\infty) \times [0,\infty), [0,\infty)), g \in C([0,\infty), [0,\infty)) \).

\( (H_2) \) \( h_i \in C([0,1), [0,\infty)), h_i(t) = h_i(1-t), t \in (0,1), \) for any subinterval of \((0,1), h_i(t) \neq 0, \) and \( \int_0^1 h_i(t)dt < +\infty (i = 1,2) \).

\( (H_3) \) There exists \( \alpha \in (0,1), \) such that \( \liminf_{u \to +\infty} \frac{g(u)}{u^{\alpha}} = +\infty \) and \( \liminf_{u \to +\infty} \frac{f(u,v)}{v^{(p-1)n}} > 0 \) hold uniformly to \( u \in R^+ \).

\( (H_4) \) There exists \( \beta \in (0,\infty), \) such that \( \limsup_{u \to +\infty} \frac{g(u)}{u^{\beta}} = 0 \) and \( \limsup_{v \to +\infty} \frac{f(u,v)}{v^{(p-1)n}} < +\infty \) hold uniformly to \( u \in R^+ \).

\( (H_5) \) There exists \( n \in (0,1), \) such that \( \liminf_{u \to +\infty} \frac{g(u)}{u^{n}} = +\infty \) and \( \liminf_{v \to +\infty} \frac{f(u,v)}{v^{(p-1)n}} > 0 \) hold uniformly to \( u \in R^+ \).

\( (H_6) \) \( f(u,v) \) and \( g(u) \) are nondecreasing with respect to \( u \) and \( v, \) and there exists \( R > 0, \) such that \( \int_0^{R} \int_0^{R} \phi_p(k_1(s))dsf[R, \frac{\gamma}{1-\gamma}] \int_0^{R} \phi_p(k_1(s))ds \times g(R) < R, \) where \( k_1(s) = \int_0^s h_i(\tau)d\tau, i = 1,2 \).

For convenience, we list the following definitions and lemmas:

**Definition 1.1** If \( u(t) = u(1-t), t \in [0,1], \) we call \( u(t) \) is symmetric in \([0,1]\).

**Definition 1.2** If \( (u,v) \) is a positive solution of problem (1), and \( u, v \) is symmetric in \([0,1], \) we call \( (u,v) \) is symmetric positive solution of problem (1).

**Definition 1.3** If \( u(\lambda t_1 + (1-\lambda)t_2) \geq \lambda u(t_1) + (1-\lambda)u(t_2), \) we call \( u(t) \) is concave in \([0,1].\)

Let \( E = C[0,1], \) define the norm \( ||u|| = \max_{t \in [0,1]} |u(t)|, \) obviously \((E, ||.||) \) is a Banach space.

Let \( K = \{ u \in E | u(t) > 0, u(t) \) is a symmetric concave function, \( t \in [0,1], \) then \( K \) is a cone in \( E. \) By \((H_1),(H_2),\) the solution of problem (1) is equivalent to the solution of system of equation (2).
\[
\begin{align*}
\{v(t) & = \begin{cases} 
\int_{0}^{t} \phi_0(\int_{s}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds, \\
0 \leq t \leq 1,
\end{cases} \\
\{u(t) & = \begin{cases} 
\int_{0}^{t} \phi_0(\int_{s}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds,
\end{cases}
\end{align*}
\]

We define \( T : K \to E \):

\[
(Tu)(t) = \begin{cases} 
\int_{0}^{t} \phi_0(\int_{s}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds, \\
0 \leq t \leq 1,
\end{cases}
\]

where

\[
\begin{align*}
v(t) & = \begin{cases} 
\int_{0}^{t} \phi_0(\int_{s}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds, \\
0 \leq t \leq 1,
\end{cases}
\end{align*}
\]

Obviously \( Tu \in E \), it is easy to show if \( T \) has fixed point \( u \), then by (4), problem (1) has a solution \((u, v)\).

**Lemma 1.1.** Let \((H_1), (H_2)\), then \( T : K \to K \) is completely continuous.

**Proof.** \( \forall u \in K \), by \((H_1), (H_2)\), we can get \((Tu)(t) \geq 0, t \in [0, 1]\).

\[
v'(t) = \begin{cases} 
\phi_0(\int_{t}^{1} h_2(\tau) g(u(\tau)) d\tau), 0 \leq t \leq \frac{1}{2}, \\
-\phi_0(\int_{t}^{1} h_2(\tau) g(u(\tau)) d\tau), \frac{1}{2} \leq t \leq 1,
\end{cases}
\]

correspondingly \((\phi_0(v'))' = -h_2(t)g(u) \leq 0, 0 < t < 1\), so \( v \) is concave in \([0, 1]\).

Next we show \( v \) is symmetric in \([0, 1]\).

When \( t \in [0, \frac{1}{2}], 1 - t \in [\frac{1}{2}, 1]\), so

\[
v(1-t) = \begin{cases} 
\phi_0(\int_{0}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \phi_0(\int_{0}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds,
\end{cases}
\]

Similarly, we have \((Tu)'(t) = (Tu)(t), t \in [\frac{1}{2}, 1]. \) So \( v \) is a symmetric concave function in \([0, 1]\).

Next we show \( Tu \) is symmetric in \([0, 1]\), when \( t \in [0, \frac{1}{2}], 1 - t \in [\frac{1}{2}, 1]\), so

\[
(Tu)(1-t) = \begin{cases} 
\phi_0(\int_{0}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \phi_0(\int_{0}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds,
\end{cases}
\]

Similarly, we have \((Tu)'(t) = (Tu)(t), t \in [\frac{1}{2}, 1]. \) So \( Tu \) is concave in \([0, 1]\), \( TK \subset K \). On the other hand, let \( D \) is a arbitrary bounded set of \( K \), then there exist constant \( c > 0 \), such that \( D \subset \{u \in K \mid ||u|| \leq c\} \). Let \( b = \max_{u \in [a, b]} g(u) \), so \( \forall u \in D \), we have

\[
||v|| = \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds \leq \frac{c}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) d\tau) ds = a.
\]

Let \( L = \max_{u \in [a, b]} f(u, v) \), so \( \forall u \in D \), we have

\[
||Tu|| = \int_{0}^{t} \phi_0(\int_{s}^{t} h_1(\tau) f(u(\tau), v(\tau)) d\tau) ds + \\
\frac{1}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) g(u(\tau), v(\tau)) d\tau) ds \leq \frac{c}{2} \int_{0}^{t} \phi_0(\int_{s}^{t} h_2(\tau) d\tau) ds.
\]
\[
\| (Tu) \| = \max \{ | \phi_q(\int_0^{t} h_1(\tau)f(u(\tau), v(\tau))d\tau) |, \quad \phi_q(\int_0^{t} h_1(\tau)f(u(\tau), v(\tau))d\tau) \} \leq L^{r-1}\}\phi_q(\int_0^{t} h_1(\tau)d\tau).
\]

By Arzela-Ascoli theorem, we know TD is compact set. By Lebesgue dominated convergence theorem, it is easy to show T is continuous in K, so \( T : K \rightarrow K \) is completely continuous.

**Lemma 1.2** For any \( 0 < \varepsilon < \frac{1}{4} \), \( u \in K \), we have

1. \( u(t) \geq \| u \|(1 - t), \forall t \in \{0, 1\} \);
2. \( u(t) \geq \varepsilon^2\| u \|, \quad t \in [\varepsilon, 1 - \varepsilon] \). (the proof is elementary, we omit it.)

**Lemma 1.3** (see [8]) Let \( K \) is a cone of \( E \) in Banach space, \( \Omega_1 \) and \( \Omega_2 \) are open subsets in \( E, \theta \in \Omega_1, \Omega_2 \subseteq \Omega_2 \), and \( T : K \bigcap (\Omega_2 \setminus \Omega_1) \rightarrow K \) is a completely continuous operator, and satisfy one of the following conditions:

1. \( ||Tx|| \leq ||x||, \forall x \in K \bigcap \partial \Omega_1, ||Tx|| \geq \varepsilon , \forall x \in K \bigcap \partial \Omega_2 \).
2. \( ||Tx|| \geq ||x||, \forall x \in K \bigcap \partial \Omega_1, ||Tx|| \leq \varepsilon , \forall x \in K \bigcap \partial \Omega_2 \), then A has at least one fixed point in \( K \bigcap (\Omega_2 \setminus \Omega_1) \).

**Lemma 1.4** (see [9]) Let \( K \) is a cone of \( E \) in Banach space, \( K_\rho = \{ x \in K \mid || x || \leq \rho \} \), suppose \( A : K_\rho \rightarrow K \) is a completely continuous, and satisfy \( Tx \neq x, \forall x \in \partial K_\rho \),

1. \( ||Tx|| \leq \varepsilon, \forall x \in \partial K_\rho \), then \( i(T, K_\rho, K) = 1 \).
2. \( ||Tx|| \geq \varepsilon, \forall x \in \partial K_\rho \), then \( i(T, K_\rho, K) = 0 \).

II. Conclusion

**Theorem 2.1** Suppose \((H_1)-(H_4)\) hold, then problem (1) has at least one positive solution.

**Proof** By \((H_3)\), there exist \( \nu \) and a sufficient large number \( M > 0 \), such that

\[
f(u, v) \geq \nu^{p-1} \nu^{(p-1)\alpha}, \forall u \in R^+, \quad v > M, \quad (5)
\]

\[
g(u) \geq C_0 \nu^{p-1} u^{\frac{1}{\nu}}, \forall u > M, \quad (6)
\]

where \( C_0 = \max \{ \frac{1}{p} \int_0^1 \phi_q(\frac{k_2(s)}{s})ds \}^{-1}, \)

\[
(\frac{2}{\nu^{p-1}} \int_0^1 \phi_q(\frac{k_1(s)}{s})ds)^{-1}.
\]

Let \( N = (M + 1)\varepsilon^{-2} \), if \( u \in K \bigcap \partial K\), by Lemma 2, \( \min_{\varepsilon \leq t \leq 1 - \varepsilon} u(t) \geq \varepsilon^2\| u \| = \varepsilon^2N = M + 1 \), by (3)-(6) and the symmetric property, for any \( t \in [\varepsilon, 1 - \varepsilon] \)

\[
v(t) = \int_0^t \phi_q(\int_r^s h_2(\tau)g(u(\tau))d\tau)ds + \int_0^r \phi_q(\int_s^r h_2(\tau)g(u(\tau))d\tau)ds,
\]

\[
\geq \frac{\varepsilon^2}{(1 - \varepsilon)^2} \int_0^r \phi_q(\int_s^r h_2(\tau)g(u(\tau))d\tau)ds,
\]

\[
\geq \frac{1}{(1 - \varepsilon)^2} \int_0^r \phi_q(\int_s^r h_2(\tau)g(u(\tau))d\tau)ds,
\]

\[
\geq \frac{1}{(1 - \varepsilon)^2} \int_0^r \phi_q(\int_s^r h_2(\tau)g(u(\tau))d\tau)ds
\]

\[
\geq \frac{1}{(1 - \varepsilon)^2} \int_0^r \phi_q(\int_s^r h_2(\tau)d\tau)ds(\varepsilon^2\| u \|)^{\frac{1}{\nu}}
\]

\[
\geq 2\| u \|.
\]

so \( ||Tu|| > ||u||, \forall u \in K \bigcap K_\rho \), by lemma 1.4, we can get

\[
i(T, K \bigcap K_\rho, K) = 0. \quad (7)
\]

On the other hand, by the second limit of \( H_4 \), there exists a sufficient small number \( r_1 \in (0, 1) \) such that

\[
C_1^{p-1} = \sup \{ \frac{f(u, v)}{\nu^{(p-1)s}} | u \in R^+, v \in (0, r_1) \} < +\infty. \quad (8)
\]

Let \( \varepsilon = \min \{ \frac{r_1}{1 - \gamma}, \frac{1}{(1 - \varepsilon)^2} \int_0^1 \phi_q(\int_0^1 h_2(\tau)g(u(\tau))d\tau)ds \} \), by the first limit of \( H_4 \), there exist a sufficient small number \( r_2 \in (0, 1) \) such that

\[
g(u) \leq \varepsilon^{p-1} u^{\frac{1}{\nu}}, \forall u \in [0, r_2]. \quad (9)
\]
Take $r = \min\{r_1, r_2\}$, by (9), we can get

$$v(t) = \int_0^t \phi_i(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds +$$

$$\int_0^t \int_s^t \phi_i(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds,$$

$$\leq \frac{\epsilon}{T^\gamma} \int_0^t \phi_i(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds$$

$$\leq \frac{\epsilon}{T^\gamma} \int_0^t \phi_i(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds$$

$$\leq \frac{\epsilon}{T^\gamma} \int_0^t \phi_i(\int_s^t h_2(\tau)g(u(\tau))d\tau)ds\|u\|^\beta$$

$$\leq r_1^{1+\delta} < r_1, \forall u \in K \cap \partial K_\sigma, s \in [0, 1].$$

By (8), we can get

$$|u| \leq \int_0^t \phi_i(\int_s^t h_1(\tau)f(u(\tau), v(\tau))d\tau)ds$$

$$\leq \frac{\epsilon}{T^\gamma} \int_0^t \phi_i(\int_s^t h_1(\tau)d\tau)ds$$

$$\leq \frac{\epsilon}{T^\gamma} \int_0^t \phi_i(\int_s^t h_1(\tau)d\tau)ds\|u\|^\alpha$$

$$= C_1 \epsilon \beta(\frac{1}{T^\gamma} \int_0^t \phi_i(\int_s^t h_1(\tau)d\tau)ds)^{\beta+1}\|u\|^\alpha$$

$$\leq \|u\|, \forall u \in K \cap \partial K_\sigma, t \in [0, 1].$$

So $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial K_\sigma$, by lemma 1.4, we get

$$i(T, K \cap K_\sigma, K) = 1. \quad (10)$$

By lemma 1.5, $T$ has at least one fixed point in $K \cap (\overline{K_N} \setminus K_\sigma)$, so problem (1) has at least a system positive solution.

**Theorem 2.2** Suppose $(H_1), (H_2), (H_3), (H_4), (H_5)$ hold, then problem (1) has at least two systems of positive solutions.

**Proof** By $(H_2)$, there exists $\mu > 0$ and a sufficiently small number $\xi \in (0, 1)$, such that

$$f(u, v) \geq \mu v^{p-1} v^{n-1}, \forall u \in R^+, 0 \leq v \leq \xi; \quad (11)$$

$$g(u) \geq (C_2u)^{\psi+1}, \forall 0 \leq u \leq \xi, \quad (12)$$

where

$$C_2 = 2(\frac{\mu^2}{\xi^{1+\delta}})^\gamma \int_0^\gamma \phi_i(\int_s^\gamma h_2(\tau)d\tau)ds$$

$$= 2(\frac{\mu^2}{\xi^{1+\delta}})^\gamma \int_0^\gamma \phi_i(\int_s^\gamma h_2(\tau)d\tau)ds \int_\gamma^\gamma (\phi_i(\int_s^\gamma h_2(\tau)d\tau)ds)^{-1}$$

since $g \in C(R^+, R^+), g(0) \equiv 0$, so there exists $\sigma \in (0, \xi)$ such that $\forall u \in [0, \sigma]$, we have

$$g(u) \leq \frac{1}{1 - \gamma} \int_0^u \phi_i(\int_s^u h_1(\tau)d\tau)ds^{-1},$$

this imply

$$v(t) \leq \frac{1}{1 - \gamma} \int_0^u \phi_i(\int_s^u h_2(\tau)g(u(\tau))d\tau)ds \leq \xi, \forall u \in K \cap \partial K_\sigma.$$

By using Jensen inequality, $0 < q \leq 1$, and (11)-(13), we can get

$$(Tu)(\frac{1}{u}) \geq \frac{1}{\gamma} \int_0^\gamma \phi_i(\int_s^\gamma h_1(\tau)d\tau)ds$$

$$(\frac{1}{\gamma} \int_0^\gamma \phi_i(\int_s^\gamma h_2(\tau)g(u(\tau))d\tau)ds)\|u\|^\alpha$$

$$\geq \frac{1}{\gamma} \int_0^\gamma \phi_i(\int_s^\gamma h_1(\tau)d\tau)ds$$

$$\int_\gamma^\gamma (\frac{1}{\gamma} \int_\gamma^\gamma \phi_i(\int_s^\gamma h_2(\tau)g(u(\tau))d\tau)ds)^n\|u\|$$

$$\int_\gamma^\gamma (\phi_i(\int_s^\gamma h_2(\tau)d\tau)ds\|u\|$$

$$= 2\|u\|, \forall u \in K \cap \partial K_\sigma.$$

So $\|Tu\| > \|u\|, \forall u \in K \cap \partial K_\sigma$, by lemma 1.4, we can get

$$i(T, K \cap K_\sigma, K) = 0. \quad (14)$$

We can choose $N > R > \sigma$, such that (7),(14) hold together.

On the other hand by (3),(4) and $H_6$ we can get

$$(Tu)(t) < \frac{1}{1 - \gamma} \int_0^1 \phi_i(\int_s^1 h_1(\tau)f(u(\tau), v(\tau))d\tau)ds$$

$$\leq \frac{1}{\gamma} \int_0^\gamma \phi_i(\int_s^\gamma h_1(\tau)d\tau)ds$$

$$f(R, 0) \leq \phi_i(\int_s^\gamma h_2(\tau)d\tau)ds g(R)$$

$$< R, \forall u \in K \cap K_\sigma, \forall t \in [0, 1].$$

So for any $u \in K \cap K_\sigma$, by lemma 1.4, we can get

$$i(T, K \cap K_\sigma, K) = 1. \quad (15)$$

By (7),(14),(15), we have

$$i(T, K \cap (K_N \setminus K_\sigma), K) = i(T, K \cap K_N, K) - i(T, K \cap K_\sigma, K) = 1.$$
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