Positive solutions of initial value problem for the systems of second order integro-differential equations in Banach space

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Abstract—In this paper, by establishing a new comparison result, we investigate the existence of positive solutions for initial value problems of nonlinear systems of second order integro-differential equations in Banach space. We improve and generalize some results (see [5, 6]), and the results is new even in finite dimensional spaces.

Keywords—Systems of integro-differential equations, monotone iterative method, comparison result, cone.

I. INTRODUCTION

In recent years, the research about initial value problem for second order integro-differential equation is more and more active (see [1-6]), paper [5] investigates the existence of solution for the following equation:

\[ \begin{align*}
  x''(t) &= f(t, x(t), x', Tx), \quad t \in [0, 1] \\
  x(0) &= x_0, x'(0) = x_1,
\end{align*} \]

paper [6] discusses the existence of solutions of initial value problems for the following systems of second order integro-differential equations:

\[ \begin{align*}
  x''(t) &= f(t, x(t), y, Tx), \quad x(0) = x_0, x'(0) = x_1, \\
  y''(t) &= g(t, y(t), x, Ty), \quad y(0) = y_0, y'(0) = y_1,
\end{align*} \]

Motivated by the paper [5, 6], we consider the existence of solutions of initial value problems for the following systems of second order integro-differential equations:

\[ \begin{align*}
  x''(t) &= f(t, x(t), y, Tx), \quad x(0) = x_0, x'(0) = x_1, \\
  y''(t) &= g(t, y(t), x, Ty), \quad y(0) = y_0, y'(0) = y_1,
\end{align*} \]

where \( t \in I, f,g \in C[I \times E \times E \times E \times E, E], Tx(t) = \int_0^t k(t, s)x(s)ds \in C[D, R^+] \), \( D = \{(t, s) \in R^2|0 \leq s \leq t \leq 1\} \).

\( \text{Suppose } (E, ||.||) \text{ is a real Banach space, } P \text{ is a normal cone in } E, \text{ and the normal constant is } 1, \text{ the partial order induced by } P \text{ is } \leq : x \leq y \Leftrightarrow y - x \in P. E^* = \{\varphi \in E^*: \varphi(x) \geq 0, \forall x \in P\} \text{ denote the dual cone of } P. \text{ Obviously, } x \in P \text{ if and only if } \varphi(x) \geq 0, \forall \varphi \in P^*. (C[I, E], ||.||_e) \text{ is also a Banach space, where } ||.||_e = \max \{||x(t)||\}. \text{ Let } I = [0, 1], P_c = \{x \in C[I, E]|x(t) \geq 0, \forall t \in I\}, \text{ then } P_c \text{ is a normal cone in } C[I, E], \text{ and normal constant is } 1, \text{ moreover, it defines the partial order of } C[I, E]. \forall u_0, v_0 \in C[I, E], \text{ and } u_0 < v_0, \text{ we define order interval } [u_0, v_0] = \{x \in C[I, E] \mid u_0 \leq x \leq v_0\}. \text{ For the sake of convenience, we first list some lemmas.}

Lemma 1 Let \( E \) be a real Banach space, \( P \) be a cone of \( E, \omega \in C^2[I, E] \) such that

\[ \omega''(t) \geq a \omega(t) - N \omega'(t) - LT \omega(t), \omega(0) \geq \theta, \omega'(0) \geq \theta, \]

where \( a \geq 0, N > 0, \ L \geq 0, \ k_0 = \max\{k(t, s) | (t, s) \in D\} \), which satisfy

\[ Lk_0 \leq a < N, \]

then \( \omega(t) \geq \theta, \omega'(t) \geq \theta. \)

Lemma 2 Let \( x, y \in C^2[I, E] \), and

\[ \begin{align*}
  x''(t) &= bx(t) + cy(t) - N x'(t) - L \int_0^t k(t, s)x(s)ds, \\
  x(0) &= \theta, x'(0) \geq \theta, \\
  y''(t) &= by(t) + cx(t) - N y'(t) - L \int_0^t k(t, s)y(s)ds, \\
  y(0) &= \theta, y'(0) \geq \theta,
\end{align*} \]

where \( b \geq c \geq 0 \) such that

\[ N \geq b + c - b \geq Lk_0, \]

then \( x(t) \geq \theta, y(t) \geq \theta, x'(t) \geq \theta, y'(t) \geq \theta. \)

Proof Let \( \omega(t) = x(t) + y(t), t \in I \), by (4), we have

\[ \omega''(t) \geq (b + c) \omega(t) - N \omega'(t) - L \int_0^t k(t, s)\omega(s)ds, \]

\( \omega(0) \geq \theta, \omega'(0) \geq \theta, \) by (5) and lemma 1, we can get

\[ \omega(t) \geq \theta, \omega'(t) \geq \theta, \]

i.e.

\[ x(t) + y(t) \geq \theta, x'(t) + y'(t) \geq \theta. \]

Next, we prove

\[ x(t) \geq \theta, y(t) \geq \theta, x'(t) \geq \theta, y'(t) \geq \theta, \forall t \in I. \]

Actually, by (4) and (6), we can get

\[ x'' \geq (b - c)x - N x' - L \int_0^t k(t, s)x(s)ds, \]

\[ x(0) \geq \theta, x'(0) \geq \theta. \]
\[ y'' \geq (b - c)y - N'y - L \int_0^t k(t, s)y(s)ds, \quad y(0) \geq \theta, \quad y'(0) \geq \theta. \]

In the same way, by (7) and (8), we can get
\[ x(t) \geq \theta, \quad y(t) \geq \theta, \quad x'(t) \geq \theta, \quad y'(t) \geq \theta. \]

Let \( \alpha(.) \) be Kuratowski noncompact measure, we have the following lemmas.

**Lemma 3** \(^{[2]}\) If \( B \subset C[I, E] \) is a countable bounded set, then
\[ \alpha(B(t)) \in L[I, R^+], \quad \alpha(\int I x(t)dt | x \in B) \leq 2 \int I \alpha(B(t))dt. \]

**Lemma 4** \(^{[3]}\) If \( B \subset C[I, E] \) is bounded and equicontinuous, let \( m(t) = \alpha(B(t)), t \in I, \) then \( m(t) \) is continuous on \( I, \) and
\[ \alpha(\int I B(t)dt) \leq \int I \alpha(B(t))dt. \]

## II. CONCLUSIONS

In this paper, we suppose that the following conditions hold:

**H1** There exist \( x_0, y_0 \in C^2[I, E], \) such that \( x_0(t) \leq y_0(t), t \in I, \) and
\[ \begin{align*}
x_0'' &\leq f(t, x_0(t), y_0(t), \tau x_0(t)), \forall t \in I, \\
x_0(0) &\leq x_0(0) \leq x_1, \\
y_0(0) &\leq f(t, y_0(t), 0, \tau y_0(t)), \forall t \in I, \\
y_0(0) &\geq y_0(0) \geq y_1.
\end{align*} \]

**H2** There exist non-negative constants \( b, c, N, L \) satisfying inequality (5), such that
\[ \begin{align*}
f(t, x_{n+1}, y_{n+1}, x'_{n+1} + T x_{n+1}) - f(t, x_n, y_n, x'_n, x_n) &\geq b(x_{n+1} - x_n) + c(y_{n+1} - y_n) - N(x'_{n+1} - x'_n) - LT(x_{n+1} - x_n), \\
g(t, y_{n+1}, x_{n+1}, y'_{n+1} + T y_{n+1}) - g(t, y_n, x'_n, y_n) &\leq b(y_{n+1} - y_n) + c(x_{n+1} - x_n) - N(y'_{n+1} - y'_n) - LT(y_{n+1} - y_n),
\end{align*} \]
\[ \begin{align*}
&g(t, y_{n+1}, x_{n+1}, y'_{n+1} + T y_{n+1}) - f(t, x_n, y_n, x'_n, x_n) \geq b(y_{n+1} - y_n) + c(x_{n+1} - x_n) - N(y'_{n+1} - y'_n) - LT(y_{n+1} - y_n),
\end{align*} \]
where \( x_n, y_n, x_{n+1}, y_{n+1} \in [x_0, y_0], \) and \( x_{n+1} \geq x_n \geq y_n, \)
\[ x_n, y_n, x_{n+1}, y_{n+1} \in [x_1, y_1], \quad x_{n+1} \geq x_n \geq y_n \geq y_{n+1}, n = 1, 2, 3, \ldots. \]

**H3** There exists constant \( d > 0, \) for any bounded equicontinuous set \( B_i(i = 1, 2, 3, 4) \) in \([x_0, y_0]\) and \([x_1, y_1]\), we have
\[ \begin{align*}
\alpha(f(t, B_1(t), B_2(t), B_3(t), B_4(t))) &\leq d\alpha(B_1(t)) + d\alpha(B_2(t)) + d\alpha(B_3(t)) + d\alpha(B_4(t)), \\
\alpha(g(t, B_1(t), B_2(t), B_3(t), B_4(t))) &\leq d\alpha(B_1(t)) + d\alpha(B_2(t)) + d\alpha(B_3(t)) + d\alpha(B_4(t)).
\end{align*} \]

**Theorem 1** Suppose \( P \subset E \) is a normal cone, and conditions \( (H1) - (H3) \) hold, then initial value problem (1) has solutions \( x^*, y^* \in [x_0, y_0]. \) Moreover, there exist monotone iterative sequences \( \{x_n(t)\}, \{y_n(t)\} \subset [x_0, y_0] \) and \( \{x'_n(t)\}, \{y'_n(t)\} \subset [x_1, y_1], \) which converge uniformly to \( x^*, y^* \) and \( (x^*)', (y^*)' \) on \( I, \) where \( x_n(t), y_n(t) \) and \( x'_n(t), y'_n(t) \) satisfy
\[ \begin{align*}
x_n(t) &= x_0 + tx_1 + \int_0^t (t - s)[f(s, x_{n-1}(s), y_{n-1}(s), x'_{n-1}(s), T x_{n-1}(s)) + b(x_n(s) - x_{n-1}(s)) + c(y_n(s) - y_{n-1}(s))]ds, \\
y_n(t) &= y_0 + ty_1 + \int_0^t (t - s)[g(s, y_{n-1}(s), x'_{n-1}(s), T y_{n-1}(s)) + b(y_n(s) - y_{n-1}(s)) + c(x_n(s) - x_{n-1}(s))]ds,
\end{align*} \]
\[ \begin{align*}
x'_n(t) &= x_1 + \int_0^t [f(s, x_{n-1}(s), y_{n-1}(s), x'_{n-1}(s), T x_{n-1}(s)) + b(x_n(s) - x_{n-1}(s)) + c(y_n(s) - y_{n-1}(s))]ds, \\
y'_n(t) &= y_1 + \int_0^t [g(s, y_{n-1}(s), x'_{n-1}(s), T y_{n-1}(s)) + b(y_n(s) - y_{n-1}(s)) + c(x_n(s) - x_{n-1}(s))]ds.
\end{align*} \]

**Proof** Firstly, by mathematical induction, we can prove \( \{x_n(t)\}, \{y_n(t)\} \) satisfy
\[ \begin{align*}
x_{n-1} &\leq x_n \leq y_n \leq y_{n-1}, n = 1, 2, 3, \ldots, \\
x'_{n-1} &\leq x'_n \leq y'_n \leq y'_{n-1}, n = 1, 2, 3, \ldots.
\end{align*} \]

Obviously, \( \forall x_{n-1}, y_{n-1} \in C[I, E] \) \( n = 1, 2, 3, \ldots \), it is easy to see equations (9) and (10) have only a couple of solutions \( x_n, y_n \in C[I, E] \), by (9)(10), we have
\[ \begin{align*}
x''_n(t) &= f(t, x_{n-1}(t), y_{n-1}(t), x'_{n-1}(t), T x_{n-1}(t)) + b(x_n(t) - x_{n-1}(t)) + c(y_n(t) - y_{n-1}(t)) + N(x_n(t) - x_{n-1}(t)) - LT(x_n(t) - x_{n-1}(t)), \\
x(0) &= x_0, x'(0) = x_1, n = 1, 2, 3, \ldots.
\end{align*} \]
\[ \begin{align*}
y''_n(t) &= f(t, y_{n-1}(t), x'_{n-1}(t), y'_{n-1}(t), T y_{n-1}(t)) + b(y_n(t) - y_{n-1}(t)) + c(x_n(t) - x_{n-1}(t)) + N(y_n(t) - y_{n-1}(t)) - LT(y_n(t) - y_{n-1}(t)), \\
y(0) &= y_0, y'(0) = y_1, n = 1, 2, 3, \ldots.
\end{align*} \]

By (17)(18), \( (H_1), (H_2) \), we have
\[ \begin{align*}
(x_1 - x_0)''(t) &\geq b(x_1 - x_0)(t) + c(y_1(t) - y_0(t)) - N(x_1 - x_0)(t) - LT(x_1 - x_0)(t),
\end{align*} \]
Now, suppose that for $x \leq k$, (15) and (16) hold. Consequently $0 \leq x \leq x_1 \geq \theta,$

$$m_1(t) \leq 2 \int_{t}^{*} \alpha((t-s)g(s, B_1(s))B_2(s), B_3(s), TB_1(s)) \\left[ (2b + d) \alpha(s) + (2c + d) \right] ds$$

$$m_2(t) \leq 2 \int_{t}^{*} \alpha((t-s)[g(s, B_2(s))B_1(s), T B_2(s)]) \\left[ (2b + d) \alpha(s) + (2c + d) \right] ds$$

$$m_3(t) \leq 2 \int_{t}^{*} \alpha((t-s)[g(s, B_3(s))B_1(s), T B_2(s)]) \\left[ (2b + d) \alpha(s) + (2c + d) \right] ds$$

By lemma 1 and lemma 2, we have

$$\begin{align*}
(x_1 - x_0)(0) &= x_0 - x_0 = \theta, \\
(x_1 - x_0)(0) &= x_1 - x_1 = \theta, \\
(y_1 - y_0)(t) &\leq b(y_1 - y_0(t)) + c(x_1(t) - x_0(t)) \\
&\quad - N(y_1 - y_0)(t) - LT(y_1 - y_0)(t), \\
(y_1 - y_0)(0) &= y_0 - y_0 = \theta, \\
(y_1' - y_0')(0) &= y_1 - y_1 = \theta.
\end{align*}$$

Next, we will show it also hold for $k+1$. Actually, by (15), (16) and $(H_2)$, we have

$$\begin{align*}
(x_{k+1} - x_k)(t) &\geq b(x_{k+1} - x_k(t)) + c(x_{k+1}(t) - x_k(t)) \\
&\quad - N(x_{k+1} - x_k)(t) - LT(x_{k+1} - x_k)(t), \\
(x_{k+1} - x_k)(0) &= x_0 - x_0 = \theta, \\
(x_{k+1} - x_k)(0) &= x_1 - x_1 = \theta.
\end{align*}$$

Therefore, by lemma 1 and lemma 2, we have

$$\begin{align*}
(x_k - x_{k+1}) &\leq y_k \leq x_k \leq x_{k+1} \leq y_k' \leq y_k, \\
so, \forall n \in N, (15) and (16) hold. Consequently
\end{align*}$$

$$\begin{align*}
x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \leq y_n \leq \ldots \leq y_0, \quad (19)
\end{align*}$$

and

$$\begin{align*}
x' \leq x' \leq \ldots \leq x' \leq \ldots \leq y' \leq \ldots \leq y' \leq y'. \quad (20)
\end{align*}$$

Let $B_1(t) = \{x_n(t)\}$, $B_2(t) = \{y_n(t)\}$, $B_3(t) = \{x_n'(t)\}$, $B_4(t) = \{y_n'(t)\}$, where $n \in N$, $m_i(t) = \alpha(B_i(t))(i = 1, 2, 3, 4)$. $P_e$ is normal cone, by the normality of cone $P_e$, $B_i(i = 1, 2, 3, 4)$ is the bounded set in $C[I, E]$, obviously, it is equicontinuous on $I$, by (9), (10), $(H_3)$ and lemma 3, we have

$$\begin{align*}
m_1(t) &\leq 2 \int_{t}^{*} \alpha((t-s)g(s, B_1(s))B_2(s), B_3(s), TB_1(s)) \\left[ (2b + d) \alpha(s) + (2c + d) \right] ds, \\
m_2(t) &\leq 2 \int_{t}^{*} \alpha((t-s)[g(s, B_2(s))B_1(s), T B_2(s)]) \\left[ (2b + d) \alpha(s) + (2c + d) \right] ds, \\
m_3(t) &\leq 2 \int_{t}^{*} \alpha((t-s)[g(s, B_3(s))B_1(s), T B_2(s)]) \\left[ (2b + d) \alpha(s) + (2c + d) \right] ds.
\end{align*}$$

By lemma 4, we have

$$\begin{align*}
\alpha(T B_1(s)) &= \alpha\int_{0}^{s} k(s, \tau) B_1(\tau) d\tau, \\
&\leq k_0 \int_{0}^{s} \alpha(B_1(\tau)) d\tau, \\
&= k_0 \int_{0}^{s} m_1(\tau) d\tau, \\
&\leq k_0 \int_{0}^{s} \alpha(B_2(\tau)) d\tau, \\
&= k_0 \int_{0}^{s} m_2(\tau) d\tau.
\end{align*}$$

Let $p(t) = \max\{m_1(t), m_2(t), m_3(t), m_4(t)\}$, by (21)-(26), we have

$$\begin{align*}
p(t) &\leq 2 \int_{0}^{t} [(2b + 2c + 2N + 3d) p(s) + (2L + d) k_0 \int_{0}^{s} p(\tau) d\tau] ds \\
&= 2 \int_{0}^{t} [(2b + 2c + 2N + 3d) p(s) + (2L + b) k_0 (t-s)] p(s) ds \\
&\leq 2 [2b + 2c + 2N + 3d] p(t) \\
&= (2L + b) k_0 \int_{0}^{t} p(s) ds.
\end{align*}$$
similarly to the proof of paper [7], we can get p(t) = 0, \forall t \in I, therefore \( m_n(t) = 0 \) (i=1,2,3,4), so \( \{x_n(t)\}, \{y_n(t)\}, \{y_n'(t)\} \) is relatively compact set in \( C[I,E] \). By the Arzela-Ascoli theorem there exist subsequences \( \{x_{n_k}(t)\}, \{y_{n_k}(t)\}, \{y_{n_k}'(t)\} \), which converge uniformly to \( x^*, y^*, (x')^* \) and \( (y')^* \) on \( I \). By the normality of \( P \) and the monotonicity of \( \{x_n(t)\}, \{y_n(t)\}, \{x_n'(t)\}, \{y_n'(t)\} \) we know \( \{x_n(t)\}, \{y_n(t)\}, \{x_n'(t)\}, \{y_n'(t)\} \) converge to \( x^*, y^*, (x')^* \) and \( (y')^* \) on \( I \). Taking limits in (9),(10), we have

\[
x^*(t) = x_0 + tx_1 + \int_0^t (t-s)f(s,x(s),y(s),(x')^*(s),Tx^*(s))ds,
\]

\[
y^*(t) = y_0 + ty_1 + \int_0^t (t-s)g(s,y(s),x(s),(y')^*(s),Ty^*(s))ds,
\]

it is easy to see that \( x^*(t), y^*(t) \) are solutions of systems of integro-differential equations (1), and (13)(14) hold evidently.

**Theorem 2** Suppose \( P \subset E \) is a regular cone, and conditions \( (H_1) - (H_2) \) hold, then there exist the same result as theorem 1.

**Proof** Similarly to the proof of theorem 1, the only difference is that we get the result \( m_n(t) = a(B_i(t)) = 0 (i = 1, 2, 3, 4) \) from \( (H_3) \) in theorem 1, but we can get it by (19) (20) and the regularity of cone in theorem 2.

### III. Example

As an application of theorem 1 and theorem 2, we give an example:

**Example** using the result of this paper, we study the initial problem for the following integro-differential equation in Banach space \( E \):

\[
\omega'' = H(t,\omega,\omega',T\omega), \omega(0) = \omega_0, \omega'(0) = \omega_1,
\]

(29)

Suppose \( H, \omega_0, \omega_1 \) have the following decomposition:

\[
H(t,\omega,\omega') = f(t,x,y,y'), T\omega = g(t,y,\omega,\omega'),
\]

\[
\omega = x + y, \omega_0 = x_0 + y_0, \omega_1 = x_1 + y_1, \text{ where } f, g \in C[I \times E \times E \times E, E],
\]

\[
T_2(x) = \int_0^t k(t,s)x(s)ds, x(0) = x_0 \leq y_0 = y(0), x(t) = x_1 \leq y_1 = y_1(0).
\]

As the direct result of theorem 1 and theorem 2, we get the following conclusions:

**Conclusion 1** Let \( P \subset E \) be a normal cone, \( f, g \) satisfy conditions \( (H_1) - (H_3) \), then there exists solutions \( \omega^* \in [2x_0, 2y_0] \), and iterative sequence \( \{\omega_n\} = \{x_n + y_n\} \) converges uniformly to \( \omega^* \) on \( I \), where \( x_n, y_n \) are defined by (9) and (10).

**Conclusion 2** Let \( P \subset E \) be a regular cone, \( f, g \) satisfy conditions \( (H_1) - (H_2) \), then the same result as conclusion 1 holds.

### References


