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Comparison of two types of preconditioners for Stokes and linearized Navier-Stokes equations

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Abstract—To solve saddle point systems efficiently, several preconditioners have been published. There are many methods for constructing preconditioners for linear systems from saddle point problems, for instance, the relaxed dimensional factorization (RDF) preconditioner and the augmented Lagrangian (AL) preconditioner are used for both steady and unsteady Navier-Stokes equations. In this paper we compare the RDF preconditioner with the modified AL (MAL) preconditioner to show which is more effective to solve Navier-Stokes equations. Numerical experiments indicate that the MAL preconditioner is more efficient and robust, especially, for moderate viscosities and stretched grids in steady problems. For unsteady cases, the convergence rate of the RDF preconditioner is slightly faster than the MAL perconditioner in some circumstances, but the parameter of the RDF preconditioner is more sensitive than the MAL preconditioner. Moreover the convergence rate of the MAL preconditioner is still quite acceptable. Therefore we conclude that the MAL preconditioner is more competitive than the RDF preconditioner. These experiments are implemented with IFISS package.

Keywords—Navier-Stokes equations, Krylov subspace method, preconditioner, dimensional splitting, augmented Lagrangian preconditioner.

I. Introduction

We consider the following incompressible Navier-Stokes equations describing the flow of viscous Newtonian fluids.

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{on} \quad \Omega\times(0,\mathrm{T}], \tag{1}$$

$$\mathrm{div}\mathbf{u} = 0 \quad \text{on} \quad \Omega\times[0,\mathrm{T}], \tag{2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial\Omega\times[0,\mathrm{T}], \tag{3}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{on} \quad \Omega, \tag{4}$$

where $\Omega \subset \mathbb{R}^d (d = 2,3)$ is an open bounded domain with boundary $\partial\Omega$, $[0,T]$ is the time interval, the unknown velocity fields $\mathbf{u}(\mathbf{x},t)$ and pressure fields $p(\mathbf{x},t)$, ν is the kinematic viscosity, Δ is the vector Laplacian operator, ∇ is the gradient operator, div is the divergence, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{u}_0$ are given functions. After implicit time discretization and linearization of the Navier-Stokes system of equations by the Picard fixed-point iteration, we get a sequence of the Oseen problems. Discretization of the Oseen problems using finite element methods results in a sequence of large sparse linear systems of equations. These equations are expressed in matrix structure as follows

$$H\mathbf{x} = b, \tag{5}$$

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where

$$H = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \text{ and } \quad b = \begin{pmatrix} f \\ g \end{pmatrix}$$

with $\mathbf{u}$ and p representing the discrete velocity and pressure, respectively. A denotes the discretization of the diffusion, convection and time-dependent terms. A is a diagonal block matrix , e.g. $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ in 2D. B^T is the discrete gradient, B denotes the (negative) discrete divergence, C is a stabilization matrix and depends on the discretization stability condition, f and g contain the forcing and boundary terms. If we use the LBB-stable finite elements to discretize the problems in order to $C = 0$, and use a simple transformation $\Pi = \begin{pmatrix} I & 0 \\ 0 & -J \end{pmatrix}$, where I and J are the identity matrix, whose ranks equal the rank of $\begin{pmatrix} A \\ B \end{pmatrix}$ and B^T, respectively; thus (5) can be rewritten in the mathematically equivalent system as

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{6}$$

where the spectrum of the coeffiecient matrix of (6) is entirely contained in the right half complex plane (e.g., see [5], [6]).

These systems can be solved by direct methods, but this requires extensive resources in terms of computational time and memory. For 3-dimensional and large 2-dimensional problems, iterative methods, in combination with suitable preconditioners, are the methods of choice. In recent years, much work is devoted to developing efficient preconditioner. For example, an important class of preconditioners is based on the block LU factorization of the coefficient matrix; see [2], [3], [16]-[18] etc.. In particular, there are a variety of block diagonal and block triangular preconditioners. Though this class of preconditioners is effective in many cases, they are not yet completely robust for smaller values of the viscosity. The AL preconditioner is based on the block triangular preconditioner for the augmented system in [3]. The MAL preconditioner is a variant of the AL preconditioner; see [10]-[12].

Another type of preconditioners is constraint preconditioner, but it is seldomly used for the Navier-Stokes system, where A is non-symmetric. Other types of preconditioenrs have been presented, such as the Hermitian and skew-Hermitian Splitting (HSS) preconditioner and Dimensional Splitting (DS) preconditioner. The HSS precnditioner is the high effective preconditioner and widely used as in [9]. The RDF preconditioner is introduced in [6], which is one kind of the DS preconditioner derived in [7], [8].

In [4], [6], [12], [19], results of numerical experiments including comparisons the RDF preconditioner or the (M)AL preconditioner with the the pressure convection diffusion preconditioner PCD, the modified version mPCD, and the least squares commutator preconditioner LSC are shown. In this paper, we shall compare the modified AL preconditioner with the RDF preconditioner, including aspects of the convergence rate of the preconditioned GMRES, the parameter stability, and computational time.

The rest of the paper is organized as follows. In the next section, we briefly introduce the RDF preconditioner and the (M)AL preconditioner. In section III, we present a series of numerical experiments illustrating the convergence behaviors of the RDF and MAL preconditioners. Finally, conclusions are drawn in section IV.

II. Preliminaries

In this section, we present briefly the RDF and (M)AL preconditioners. For the RDF preconditiner, according to the structure of H, these problems are discretized by stable finite elements, such as $Q2-Q1$ and $Q2-P1$ finite elements, so that we obtain the corresponding augmented Lagrangian system.

A. RDF preconditioner

For simplicity, we only consider the 2D case and use the LBB stable finite element to generate saddle point systems. Therefore the coefficient matrix H has the block structure therein (for further details, see [6], [8]):

$$H = \begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & 0 \end{pmatrix}, \tag{7}$$

where $H \in \mathbb{R}^{N\times N}$, $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{m\times n_i}$ and $\sum_{i=1}^{2} n_i + m = N$. For the discrete Stokes problem, A_i is symmetric and positive definite. For the discrete Oseen problem, $A_i \neq A_i^T$, but $A_i + A_i^T$ is positive definite.

Assume H is split as follows:

$$H = \begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & 0 & 0 \\ -B_1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & 0 \end{pmatrix} = H_1 + H_2. \tag{8}$$

Now let $\alpha > 0$ be a parameter, then $(\alpha I + H_1)$ and $(\alpha I + H_2)$ are both nonsymmetric, nonsingular, and positive definite. Obviously,

$$H = (\alpha I + H_1) - (\alpha I - H_2), \quad H = (\alpha I + H_2) - (\alpha I - H_1).$$

Applying the alternating iteration to the foregoing splitting can give

$$\begin{cases} (\alpha I + H_1)\mathbf{x}^{k+\frac{1}{2}} = (\alpha I - H_2)\mathbf{x}^k + b, \\ (\alpha I + H_2)\mathbf{x}^k = (\alpha I - H_1)\mathbf{x}^{k+\frac{1}{2}} + b. \end{cases} \tag{9}$$

As the stationary scheme, referred to [8], the following matrix can be used as a preconditioner:

$$P_{DS} = \begin{pmatrix} A_1 + \alpha I & -\frac{1}{\alpha}B_1^T B_2 & B_1^T \\ 0 & A_2 + \alpha I & B_2^T \\ -B_1 & -B_2 & \alpha I \end{pmatrix}. \tag{10}$$

The matrix P_{DS} is called the *DS preconditioner*. Then we can get RDF preconditioner by removing αI in $(1,1)$ and $(2,2)$ blocks of P_{DS}. This is an improved variant of the DS preconditioner. Then the *RDF preconditioner* can be represented [6] as

$$P_{RDF} = \begin{pmatrix} A_1 & -\frac{1}{\alpha}B_1^T B_2 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & \alpha I \end{pmatrix}. \tag{11}$$

Numerical experiments show the preconditioner in (11) has better behavior than the DS preconditioner. For further details on the choice of parameter, readers can see [6]. In practice, the RDF preconditioner can be factorized into some factors which have simple structure. We pointed out that the RDF preconditioner is also scaled, that is a scaling is applied to the coefficient matrix before forming the preconditioner. The behavior of RDF preconditioning can be improved by diagonal scaling. Unless otherwise specified, we always perform a preliminary symmetric scaling of the system $Hx = b$ in the form $D^{-\frac{1}{2}}HD^{-\frac{1}{2}}y = D^{-\frac{1}{2}}b$, with $y = D^{\frac{1}{2}}x$ and $D = \mathrm{diag}(D_1, D_2, I_m)$, where $\mathrm{diag}(D_1, D_2)$ is the main diagonal of the velocity mass matrix for 2D problems. Obviously, this diagonal scaling is regarded as an easy preconditioner.

B. (M)AL preconditioner

For the steady-state problems (6), Benzi et al. have presented the equivalent AL formulation [3] as follows

$$\begin{pmatrix} A_\gamma & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f + \gamma B^T W^{-1} g \\ -g \end{pmatrix}, \tag{12}$$

where $A_\gamma := A + \gamma B^T W^{-1} B$, W is an arbitrary SPD matrix and $\gamma > 0$. Sometimes W may be selected as I in [13], [16]. In this case, it may get good results; see [13]. We can refer to [10], [16] on discussions of the choice of W. A good choice of W is the pressure mass matrix; in practice, we use the main diagonal of the pressure mass matrix. Since construction of the AL preconditioner needs to find a approximation for the pressure Schur complement $B^T A^{-1} B$ or its inverse, the AL preconditioner can be regarded as the Schur complement preconditioner or a block triangular preconditioner. In [10]-[12] the *AL preconditioner* is the block triangular matrix as follows

$$P_{AL} = \begin{pmatrix} A + \gamma B^T W^{-1} B & 0 \\ -B & \hat{S} \end{pmatrix},$$

where $\hat{S}$ is the approximate Schur complement and usually implicitly defined by

$$\hat{S}^{-1} = \nu Q_p^{-1} + \gamma W^{-1},$$

where Q_p denotes the approximate pressure mass matrix, ν is the viscosity. Usually, W is also replaced by Q_p. For decreasing calculation cost, in practice, Q_p is a diagonal matrix or is replaced by the spectrally equivalent diagonal matrix. We know the formulation as follows

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} = \begin{pmatrix} A_\gamma & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\gamma W^{-1} B & I \end{pmatrix}. \tag{13}$$

Then, $A_\gamma = A + \gamma B^T W^{-1} B$,

$$\begin{pmatrix} A_\gamma & B^T \\ -B & 0 \end{pmatrix} P_{AL}^{-1} = \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} A_\gamma^{-1} & 0 \\ (\hat{S}^{-1} + \gamma W^{-1}) B A_\gamma^{-1} \hat{S}^{-1} \end{pmatrix}.$$

If $\hat{S}^{-1} = \gamma W^{-1}$, we get

$$P_0(\gamma) = \begin{pmatrix} A + \gamma B^T W^{-1} B & 0 \\ -B & \gamma^{-1} W \end{pmatrix}. \tag{14}$$

Then applying (13) yields

$$\begin{pmatrix} A_\gamma & B^T \\ -B & 0 \end{pmatrix} P_0(\gamma)^{-1} = \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} A_\gamma & 0 \\ -2B\gamma^{-1} W \end{pmatrix}^{-1}. \tag{15}$$

$$\hat{P}_0(\gamma) = \begin{pmatrix} A_\gamma & 0 \\ -2B & \gamma^{-1} W \end{pmatrix}. \tag{16}$$

$\hat{P}_0(\gamma)$ is a preconditioner of (13). We can know that (16) and (14) act on (13) and (12), respectively; their effects are the same. In [13] the authors use splitting method and obtain the same preconditioner, i. e., assume $W = I$, we yield

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} = \begin{pmatrix} A + \gamma B^T B & 0 \\ -2B & \gamma^{-1} I \end{pmatrix} - \begin{pmatrix} \gamma B^T B & -B^T \\ -B & \gamma^{-1} I \end{pmatrix}. \tag{17}$$

Readers can see [13] for further details.

From [10]-[12], we know that the preconditioned matrix of (12) has eigenvalue 1 of multiplicity at least $n_1 + n_2$ and non-unity eigenvalue of multiplicity at most m. In the AL preconditioner, since computing the inverse of $A + \gamma B^T W^{-1} B$ costs too much, it should be simplified. Then we use an approximate solution method to compute its inverse in [11], [12]. Therefore it's necessary to deal with (1, 1) block of the preconditioner matrix. The (1, 1) block of the preconditioner matrix P_{AL} is handled by

$$\begin{aligned} A_\gamma &= \begin{pmatrix} A_1 + \gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & A_2 + \gamma B_2^T W^{-1} B_2 \end{pmatrix} \\ &\approx \begin{pmatrix} A_1 + \gamma B_1^T W^{-1} B_1 & 0 \\ \gamma B_2^T W^{-1} B_1 & A_2 + \gamma B_2^T W^{-1} B_2 \end{pmatrix}. \end{aligned} \tag{18}$$

The *MAL preconditioner* is given, then inverse of (18) is easier to be computed, and obviously there are less costs than before. Without special stated, we also consider the MAL preconditioner throughout this paper.

A good choice of parameter factors γ, W (or $\hat{S}^{-1}$) in the MAL preconditioner and α in RDF preconditioner is very important. Obviously, the MAL is more complicated since there are two parameter factors. Nearly optimal values of α, γ could be obtained values by resorting to Fourier Analysis tool; readers can refer to [6], [12] for more details.

III. Numerical experiments

In this section we compare the RDF preconditioner with the MAL preconditioner on a steady 2D problems. We consider both the steady and unsteady problems and use IFISS [15] to generate those problems. If necessary, the choice of the parameters is based on [6], [12]. These computations are performed on an Intel (R) Core 2 Duo CPU T6670 2.20 GHz and 2GB of memory.

In this section, The Stokes and the Navier-Stokes problems are solved as follow:

1. The channel domain problem. There are the Poisseuille channel flow in a square domain $(-1, 1) \times (-1, 1)$ with a parabolic inflow boundary condition and a natural outflow condition having the analytic solution: $u_x = 1 - y^2$, $u_y = 0$, and $p = 2\nu x$.

2. The leaky lid driven cavity problem. It is a model of the flow in a square cavity (the domain is $(-1, 1) \times (-1, 1)$). This problem again models flow in a cavity, with the lid moving from left to right. A Dirichlet no-flow condition is applied on the side and bottom boundaries. The leaky lid driven cavity computational models is $\{y = 1; 1 \le x \le 1 | u_x = 1\}$. In the case of Stokes $\nu = 1$.

3. The backward facing step problem. It's on the L-shaped domain $(-1, L) \times (-1, 1)$, a Poisseuille flow profile is imposed on the inflow $(x = -1; 0 \le y \le 1)$ and no slip conditions are imposed on the side walls. Neumann conditions are applied at the outflow that automatically sets the mean outflow pressure to zero.

In this paper, we focus on problem 2 discretized by $Q2 - P1$ finite element on stretch grids, also consider problem 1 discretized by $Q2 - Q1$ element and problem 3. Since the RDF preconditioner is based on the problems discretized by the stable finite elements, hence we only consider the case that the MAL preconditioner is based on (12). The basic Krylov solver used in all our experiments is restarted GMRES [1], where the maximum subspace dimension is 20, the number of maximum iteration is 120, and we always use a zero initial guess. The iteration is stopped when the relative residual norm is reduced below 10^{-6}. Before forming the preconditioner, Benzi et al. have pointed out that the performance of preconditioning can be remarkably improved by diagonal scaling in [6], [8]. This method is fit for the DS and the RDF preconditioners, we find that the scaling is beneficial for MAL preconditioner as well.

Here we perform exact solves by applying a column approximate AMD permutation [14] (using the MATLAB function colamd) to $A_i + \gamma B_i^T W^{-1} B_i$ $(i = 1, 2)$ in the MAL preconditioner and $A_i + \frac{1}{\alpha} B_i^T B_i$ $(i = 1, 2)$ in the RDF preconditioner, then using sparse LU factorization in MATLAB. Thus they can easily computed. In all experimental results, we set $W = \hat{M}_p = \mathrm{diag}(M_p)$, here M_p denotes the pressure mass matrix, $M_p = Q$ in the IFISS package. Unless otherwise specified, $\hat{S}^{-1} = \gamma W^{-1}$.

3.1 Steady problem 1 and 2. At first, we present the Stokes problem with $\nu = 1$. Experimental results show that the RDF and MAL preconditioners work well in Table I for steady case. The parameter of the MAL preconditioner equals one, it is apparent that the MAL preconditioner is better than the RDF preconditioner. From Table II, it is shown that effects of the MAL preconditioner is much better than the RDF preconditioner. The Schur complement of the MAL preconditioner is $\frac{1}{\gamma} W$, we also deal with the MAL preconditioner by the diagonal scaling technique that is the same to the RDF preconditioner. Those optimal values of the parameters are found by experiments as so to result in those iteration counts. We find out the parameter of the MAL is quite insensitive, the convergence rate of GMRES is almost independent on mesh size and mild dependent on the viscosity (see Figure 1), but the parameter of the RDF preconditioner is grid-dependent and

TABLE I
GMRES(20) ITERATIONS FOR STEADY STOKES PROBLEMS (LID CAVITY, Q2-Q1, UNIFORM GRIDS). THE OPTIMAL PARAMETER VALUES OF THE RDF PRECONDITIONER ARE FOUND EXPERIMENTALLY, THE PARAMETER VALUES OF THE MAL PRECONDITIONER EQUAL ONE.

Grid	RDF	MAL
16×16	11(0.006)	9(1)
32×32	13(0.002)	9(1)
64×64	12(0.0004)	9(1)
128×128	12(0.0001)	8(1)

TABLE II
GMRES(20) ITERATIONS FOR STEADY OSEEN PROBLEMS (CHANNEL, $Q2-Q1$, UNIFORM GRIDS). THOSE ITERATION COUNTS ARE BASED ON OPTIMAL VALUES CHOSEN BY EXPERIMENTS.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	13	10	15	11	24	22
32×32	14	10	14	11	27	19
64×64	14	10	14	10	27	15
128×128	15	9	15	9	24	12

becomes smaller with the refining of the grid. This behavior of the parameter is always held, we can find from the latter numerical experiments. Next, these problems are discretized by Q2-Q1 and Q2-P1 elements on uniform grids and stretched grids in order to illustrate behaviors of the RDF and MAL preconditioners.

In Tables III-VI, we present preconditioned GMRES iteration counts for the lid driven cavity problem. In Tables III and V, we compare the RDF preconditioner and the MAL prconditioner based on the nearly optimal values of the parameter obtained by Fourier analysis for the Q2-Q1 discretization of the cavity problem on uniform and stretched grids. With respect to the case of the Q2-Q1 discretization of this problem, readers can refer to [6], [11], [12]. In Table IV, we show that times for LU factorization (including reordering), iteration times for preconditioned GMRES. It's shown that the

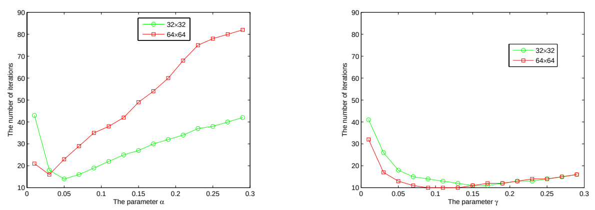

Fig. 1. Number of iterations vs. parameter(steady channel Oseen problem, Q2-Q1, uniform grids, $\nu = 0.01$ in Table 3). Left: the RDF preconditioner. Right: the MAL preconditioner

TABLE III
GMRES(20) ITERATIONS FOR STEADY OSEEN PROBLEMS (CAVITY, $Q2-Q1$, UNIFORM GRIDS). THE NEARLY OPTIMAL PARAMETER VALUES ARE OBTAINED BY FOURIER ANALYSIS.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	11	9	15	10	52	47
32×32	12	9	15	10	56	33
64×64	12	9	12	9	38	26
128×128	12	9	11	10	24	18

TABLE IV
TIMINGS(S) OF GMRES(20) ITERATIONS AND LU FACTORIZATION (INCLUDING REORDERING) WITH THE RDF AND MAL PRECONDITIONERS, THE ITERATION COUNTS ARE 5, AND THE PARAMETER IS OBTAINED BY FOURIER ANALYSIS.

Grid	Iter time-R	LU time-R	Iter time-M	LU time-M
16×16	0.0629	0.0312	0.0625	0.0315
32×32	0.0782	0.1563	0.2813	0.1563
64×64	0.3438	1.6251	1.3282	1.6406
128×128	1.7656	18.2344	5.9219	18.2813

TABLE V
GMRES(20) ITERATIONS FOR STEADY OSEEN PROBLEMS (CAVITY, $Q2-Q1$, STRETCHED GRIDS). THE NEARLY OPTIMAL PARAMETER VALUES ARE OBTAINED BY FOURIER ANALYSIS.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	16	8	17	9	44	30
32×32	32	9	27	9	67	23
64×64	23	9	25	9	74	20
128×128	16	9	18	9	58	14

iteration costs of the RDF preconditioner is much less than the MAL preconditioner if their iteration counts are equivalent, the timings of LU factorization of both preconditioners cost too much. Thus we can see that how to solve the "velocity" Schur complement $A_i + \gamma B_i^T W^{-1} B_i$ and $A + \frac{1}{\alpha} B_i^T B_i$ for $i = 1, 2$ efficiently in both preconditioners is important, Benzi et al. have discussed this issue. In Table VI, we display the number of iterations for the lid driven cavity problem discretized with Q2-P1 elements. In this case, the experimentally optimal values of the parameters are very good. Streamline and pressure plots of the approximate solutions of this problem are shown in Figure 2.

3.2 Unsteady problem 2. Similarly, we present the unsteady

TABLE VI
GMRES(20) ITERATIONS FOR STEADY OSEEN PROBLEMS (CAVITY, $Q2-P1$, STRETCHED GRIDS). THOSE ITERATION COUNTS ARE BASED ON OPTIMAL VALUES CHOSEN BY EXPERIMENTS.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	14(0.1)	7(0.3)	15(0.5)	10(0.08)	33(1)	20(0.03)
32×32	18(0.02)	7(0.3)	19(0.2)	10(0.08)	38(0.4)	20(0.03)
64×64	22(0.01)	7(0.3)	22(0.05)	10(0.08)	51(0.2)	20(0.03)
128×128	24(0.003)	7(0.3)	27(0.01)	8(0.08)	55(0.04)	15(0.03)

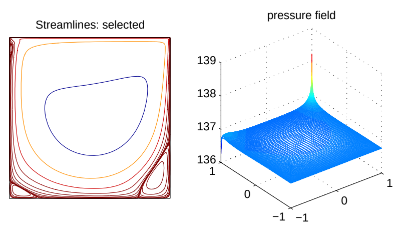

Fig. 2. Navier - Stokes flow solution plot(steady cavity problem, Q2-P1, 128×128 stretched grids, $\nu = 0.001$). Left: streamline plot. Right: pressure plot.

TABLE VII
GMRES(20) ITERATIONS FOR GENERALIZED OSEEN PROBLEMS (CAVITY, $Q2-Q1$, STRETCHED GRIDS). THE PARAMETER OF THE MAL PRECONDITIONER IS TAKEN TO BE ONE, $\hat{S}^{-1} = \gamma W^{-1}$.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	13(0.02)	10(1)	12(0.02)	10(1)	12(0.02)	11(1)
32×32	18(0.005)	12(1)	15(0.008)	14(1)	16(0.01)	17(1)
64×64	21(0.001)	14(1)	18(0.02)	20(1)	19(0.003)	35(1)
128×128	23(0.0002)	13(1)	21(0.0005)	24(1)	25(0.0008)	52(1)

TABLE VIII
GMRES(20) ITERATIONS FOR GENERALIZED OSEEN PROBLEMS (CAVITY, $Q2-P1$, UNIFORM GRIDS). THE OPTIMAL PARAMETER VALUES ARE FOUND EXPERIMENTALLY, $\hat{S}^{-1} = \gamma W^{-1}$.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	11(0.03)	12(1.5)	13(0.06)	13(1)	13(0.08)	15(0.2)
32×32	13(0.01)	14(1.5)	15(0.02)	15(0.9)	14(0.03)	15(0.2)
64×64	13(0.003)	16(1.5)	12(0.008)	16(0.8)	12(0.01)	16(0.2)
128×128	12(0.005)	18(1.5)	9(0.01)	15(0.8)	8(0.02)	15(0.2)

problems discretized with the LBB-stable Q2-P1 finite element. Linear systems of this type tend to be easier to solve than the ones arising from the steady case, since the presence of the additional positive definite term σM matrix makes the A block more diagonally dominant, where σ is the reciprocal of the time step Δt, $\sigma = 1/h$ (h is the size of mesh) and M is the velocity mass matrix. At first we show the number of iterations with Q2-Q1 element on the stretched grids, then display Q2-P1 discretization of case. From Tables VII-IX, we can see that the MAL preconditioner also works well for unsteady problem. In Table VII, the parameter values are '1' in the MAL preconditioner, but the parameter values are experimentally optimal in the RDF preconditioner, in order to show that the MAL preconditioner is efficient for unsteady problem as well though the parameter is chosen poorly, it is shown that the rates of convergence of GMRES with respect to the MAL preconditioner are mildly dependent on the mesh size in Table VII, [11], [12] have pointed out that the Schur complement chosen suitably can reform this phenomenon.

In Tables VIII and IX, we present preconditioned GMRES iteration counts with the optimal parameter values found experimentally. We observe that the convergence rate of the MAL

TABLE IX
GMRES(20) ITERATIONS FOR UNSTEADY OSEEN PROBLEMS (CAVITY, $Q2-P1$,STRETCHED GRIDS). THOSE PARAMETER VALUES OF BOTH PRECONDITIONER ARE OPTIMAL FOUND EXPERIMENTLY, $\hat{S}^{-1} = (\nu+\gamma)W^{-1}$.

	Viscosity					
	0.1		0.01		0.001	
Grid	RDF	MAL	RDF	MAL	RDF	MAL
16×16	14(0.04)	12(2)	16(0.08)	14(0.8)	15(0.05)	14(0.5)
32×32	18(0.008)	14(2)	20(0.02) 30(0.08)	16(0.6)	19(0.02) 24(0.05)	18(0.2)
64×64	23(0.002)	16(2)	25(0.005) >120(0.08)	15(0.4)	23(0.005) 66(0.05)	15(0.04)
128×128	33(0.0001)	15(2)	>120(0.08) 30(0.0004)	12(0.1)	>120(0.05) 28(0.0004)	10(0.02)

TABLE X
GMRES(20) ITERATIONS WITH THE RDF, MAL PRECONDITIONERS FOR STEADY BACKWARD FACING STEP PROBLEM (UNIFORM GRIDS, VISCOSITY $\nu = 0.005$). THE ITERATION COUNTS ARE BASED ON THE EXPERIMENTALLY OPTIMAL PARAMETERS.

	Q2-Q1		Q2-P1	
Grid	RDF	MAL	RDF	MAL
16×48	22	18	23	19
32×96	23	18	23	18
64×192	24	16	25	18

preconditioner is independent of mesh size if the parameter is correctly chosen in Table IX. In this circumstance, we suitably change the choice of the implicit Schur complement in order to test the convergence rates of the MAL preconditioner.

3.3 Problem 3. We briefly show some experimental results for a steady backward facing step Oseen problem with $\nu = 0.005$ to try to ascertain which is a better preconditioner, since this problem would be not stable solution for the smaller viscosity; see [2, pp. 316]. The flow already becomes unstable for the value of fairly moderate viscosity and computing a steady solution for smaller viscosity would not be meaningful. In Table X, it's simple to see that the MAL preconditioner is better than the RDF preconditioner from the number of iterations.

IV. CONCLUSION

Now, we can get some conclusions from numerical results on which preconditioner can quickly and effectively solve problems. Obviously, the parameter in the MAL preconditioner is more robust and insensitive than that in the RDF preconditioner; the RDF preconditioner has difficulty in dealing with problems on stretched grids (worse than on uniform grids). The convergence rate of the MAL preconditioner may be much faster than the RDF preconditioner in many circumstances if the parameter and the Schur complement (for the MAL preconditioner) is correctly chosen.

For steady problems, the convergence rate of the MAL preconditioner is much faster than that of the RDF preconditioner; it is much evident on the stretched grids. For unsteady problems, the number of iterations of the RDF preconditioner is slight less than the MAL perconditioner in some cases. If the parameter is optimal, the convergence rate of the MAL preconditioner may be faster than the RDF preconditioner. Even if the parameter value equals one (Table VII), those iteration counts still are acceptable.

The MAL preconditioner is more flexible in application than the RDF preconditioner. The RDF preconditioner formulation is relatively fixed. We may use different approximately Schur complements to construct the preconditioner for the different problems, e.g., we modify $\hat{S}^{-1}$ by using the pressure and velocity mass matrix in [11], it can greatly improve the degree of independence of mesh size for unsteady problems. In addition, the MAL preconditioner can be applied into those cases where problems are discretized by stabilized finite elements.

During computation, they need to do inverse matrix-vector multiplication. Since directly solving inverse matrix is prohibitively expensive cost, reordering, high effective solves, and iteration scheme also are used. In addition, the coefficient matrix of the augmented system may lead more computational costs in the MAL preconditioner compared with the RDF preconditioner.

According to the above numerical results and [6], [11], [12], we can see the MAL preconditioner can work better. The nearly optimal values of the parameter can be respectively obtained by Fourier Analysis. The experimentally optimal parameter of the MAL preconditioner is almost independent of the mesh size in many cases, but the experimental optimal parameter of the RDF greatly depends on the mesh size. Moreover, the viscosity ν may influence the choice of the parameter in order to affect the convergence rates of the preconditioners. Overall, the MAL preconditioner appears to be quite competitive with the RDF preconditioner.

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