On Properties of Generalized Bi-\(\Gamma\)-Ideals of \(\Gamma\)-Semirings

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Abstract—The notion of \(\Gamma\)-semirings was introduced by Murali Krishna Rao as a generalization of the notion of \(\Gamma\)-rings as well as of semirings. We have known that the notion of \(\Gamma\)-semirings is a generalization of the notion of semirings. In this paper, extending Kaushik, Moin and Khan’s work, we generalize the notion of generalized bi-\(\Gamma\)-ideals of \(\Gamma\)-semirings and investigate some related properties of generalized bi-\(\Gamma\)-ideals.

Keywords—\(\Gamma\)-semiring, bi-\(\Gamma\)-ideal, generalized bi-\(\Gamma\)-ideal.

I. INTRODUCTION AND PRELIMINARIES

The notion of \(\Gamma\)-semirings was introduced and studied in 1995 by Murali Krishna Rao [10] as a generalization of the notion of \(\Gamma\)-rings as well as of semiring, and the notion of generalized bi-\(\Gamma\)-ideals of \(\Gamma\)-semirings. In 2009, Jagatap and Pawar [6] introduced the notion of bi-\(\Gamma\)-semirings and proved subsequently that these operations give rise to different structures such as commutative semigroups. Define the mapping \(a\alpha b\) for all \(a, b, c \in M\) and \(a, \beta \in \Gamma\) satisfying the following conditions:

1. \(a\alpha (b + c) = a\alpha b + a\alpha c\),
2. \((a + b)\alpha c = a\alpha c + b\alpha c\),
3. \((a + \beta)b = a\beta b + a\alpha b\),
4. \((a\alpha (b\beta) c) = (a\alpha b)c\beta\)

for all \(a, b, c \in M\) and \(\alpha, \beta \in \Gamma\).

Let \(M\) be a \(\Gamma\)-semiring, \(A\) and \(B\) nonempty subsets of \(M\), and \(\Lambda\) a nonempty subset of \(\Gamma\). Then we define

\[A + B := \{a + b \mid a \in A \text{ and } b \in B\}\]

and

\[A\Lambda B := \left\{\sum_{i=1}^{n} a_i \lambda_i b_i \mid n \in \mathbb{Z}^+, a_i \in A, b_i \in B \text{ and } \lambda_i \in \Lambda \text{ for all } i\right\}\]

If \(A = \{a\}\), then we also write \(\{a\} + B\) as \(a + B\), and \(\{a\} A\Lambda B\) as \(a\Lambda B\), and similarly if \(B = \{b\}\) or \(\Lambda = \{\lambda\}\).

Example I.2. [6] Let \(Q\) be set of rational numbers. Let \((S, +)\) be the commutative semigroup of all 2 \(\times\) 3 matrices over \(Q\) and \((\Gamma, +)\) commutative semigroup of all 3 \(\times\) 2 matrices over \(Q\). Define \(W\circ\alpha\) usual matrix product of \(W, \alpha\) and \(Y\) for all \(W, Y \in S\) and for all \(\alpha \in \Gamma\). Then \(S\) is a \(\Gamma\)-semiring but not a semiring.

Example I.3. [6] Let \(\mathbb{N}\) be the set of natural numbers and \(\Gamma = \{1, 2, 3\}\). Then \((\mathbb{N}, \max)\) and \((\Gamma, \max)\) are commutative semigroups. Define the mapping \(\mathbb{N} \times \Gamma \times \mathbb{N} \to \mathbb{N}\), by \(a\alpha b\) = \(\min\{a, \alpha, b\}\) for all \(a, \alpha, b \in \mathbb{N}\) and \(\alpha \in \Gamma\). Then \(\mathbb{N}\) is a \(\Gamma\)-semiring.

Example I.4. [6] Let \(Q\) be set of rational numbers and \(\Gamma = \mathbb{N}\) the set of natural numbers. Then \((Q, +)\) and \((\mathbb{N}, +)\) are commutative semigroups. Define the mapping \(Q \times \Gamma \times Q \to Q\), by \(a\alpha b\) usual product of \(a, \alpha, b\); for all \(a, b \in Q\) and \(\alpha \in \Gamma\). Then \(Q\) is a \(\Gamma\)-semiring.

Example I.5. [2] For consider the additively abelian groups \(Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}\) and \(\Gamma = \{2, 4, 6\}\). Let \(\cdot : Z_8 \times \Gamma \times Z_8 \to Z_8\), \((y, \alpha, s) = y\alpha s\). Then \(Z_8\) is a \(\Gamma\)-semiring.
Definition I.6. A nonempty subset A of a Γ-semiring M is called

1. a sub-Γ-semiring of M if (A, +) is a subsemigroup of 
   \((M, +)\) and \(a\gamma b \in A\) for all \(a, b \in A\) and \(\gamma \in \Gamma\).

2. a Γ-ideal of M if \((A, +)\) is a subsemigroup of 
   \((M, +)\), and \(x\gamma a \in A\) and \(a\gamma x \in A\) for all \(a \in A, x \in M\) and 
   \(\gamma \in \Gamma\).

3. a quasi-Γ-ideal of M if A is a sub-Γ-semiring of M and 
   \(\Gamma M \cap MG \subseteq A\).

4. a bi-Γ-ideal of M if A is a sub-Γ-semiring of M and 
   \(\Gamma M \Gamma A \subseteq A\).

5. a generalized bi-Γ-ideal of M if \(\Gamma M \Gamma A \subseteq A\).

Remark I.7. Let M be a Γ-semiring. We have the following:

1. Every quasi-Γ-ideal of M is a bi-Γ-ideal.
2. Every bi-Γ-ideal of M is a generalized bi-Γ-ideal.

Definition I.8. A Γ-semiring M is called a GB-simple Γ-
semiring if M is the unique generalized bi-Γ-ideal of M.

II. MAIN RESULTS

Before the characterizations of generalized bi-Γ-ideals of 
Γ-semirings for the main results, we give some auxiliary results 
which are necessary in what follows. By Lemma I.7 (2) and 
[7], we have the following lemma.

Lemma II.1. Let M be a Γ-semiring and \(a \in M\). Then \(aGM\) 
and \(MGa\) are generalized bi-Γ-ideals of M.

Lemma II.2. Let M be a Γ-semiring and \(\{B_i \mid i \in I\}\) 
a nonempty family of generalized bi-Γ-ideals of M with 
\(\bigcap_{i \in I} B_i \neq \emptyset\). Then \(\bigcap_{i \in I} B_i\) is a generalized bi-Γ-ideal of M.

Proof: For all \(i \in I\), we have 
\[
\left(\bigcap_{i \in I} B_i\right) \Gamma M \Gamma \left(\bigcap_{i \in I} B_i\right) \subseteq B_i \Gamma M \Gamma B_i \subseteq B_i.
\]
Thus
\[
\left(\bigcap_{i \in I} B_i\right) \Gamma M \Gamma \left(\bigcap_{i \in I} B_i\right) \subseteq \bigcap_{i \in I} B_i.
\]
Hence \(\bigcap_{i \in I} B_i\) is a generalized bi-Γ-ideal of M.

Lemma II.3. Let M be a Γ-semiring and \(\emptyset \neq A \subseteq M\). Then
\[
A \cup \Gamma M \Gamma A
\]
is the smallest generalized bi-Γ-ideal of M containing A.

Proof: Let \(B = A \cup \Gamma M \Gamma A\). Then \(A \subseteq B\). Therefore
\[
B \Gamma M \Gamma B = (A \cup \Gamma M \Gamma A) \Gamma M \Gamma (A \cup \Gamma M \Gamma A)
\]
\[
\subseteq [A \Gamma (\Gamma M \Gamma A) \cup \Gamma (A \Gamma M \Gamma A) \cup \Gamma (A \Gamma M \Gamma A)]
\]
\[
\subseteq [A \Gamma (\Gamma M \Gamma A) \cup \Gamma (A \Gamma M \Gamma A) \cup \Gamma (A \Gamma M \Gamma A)]
\]
\[
\subseteq A \cup \Gamma M \Gamma A \subseteq \Gamma M \Gamma A
\]
Thus \(B = A \cup \Gamma M \Gamma A \subseteq C\).

Hence \(B\) is the smallest generalized bi-Γ-ideal of \(M\) containing A.

By Lemma II.3, let \(A\) be the smallest generalized bi-Γ-
ideal of \(M\) containing \(\{a\}\) as \(\{a\}\).

Lemma II.4. Let \(T\) be a sub-Γ-semiring of a Γ-semiring M, 
\(a \in M\) and \((a \Gamma T \Gamma a) \cap T \neq \emptyset\). Then \((a \Gamma T \Gamma a) \cap T\) is a 
generalized bi-Γ-ideal of T.

Proof: Consider
\[
(a \Gamma T \Gamma a) \cap T \Gamma T \Gamma (a \Gamma T \Gamma a) \cap T
\]
\[
\subseteq [(a \Gamma T \Gamma a) \Gamma T \cap TT \Gamma (a \Gamma T \Gamma a) \cap T]
\]
\[
\subseteq [(a \Gamma T \Gamma a) \Gamma T \cap TT \Gamma (a \Gamma T \Gamma a) \cap T]
\]
\[
\subseteq [(a \Gamma T \Gamma a) \cap TT \Gamma (a \Gamma T \Gamma a) \cap T] \cap T
\]
\[
\subseteq [TT \Gamma (a \Gamma T \Gamma a) \cap T] \cap T.
\]
Hence \((a \Gamma T \Gamma a) \cap T\) is a generalized bi-Γ-ideal of T.

Lemma II.5. Let M be a Γ-semiring and \(a \in M\). Then 
\(a \Gamma M \Gamma a\) is a generalized bi-Γ-ideal of M.

Proof: Consider
\[
(a \Gamma M \Gamma a) \Gamma M \Gamma (a \Gamma M \Gamma a) = a \Gamma (M \Gamma a \Gamma M \Gamma M a \Gamma M) \Gamma a \subseteq a \Gamma M \Gamma a
\]
Hence \(a \Gamma M \Gamma a\) is a generalized bi-Γ-ideal of M.

Proposition II.6. Let M be a Γ-semiring and \(T\) a sub-Γ-
semiring of M. Then every subset of \(T\) containing \(M T T\) is 
a sub-Γ-semiring of M.
Proof: Let $A$ be a subset of $T$ such that $\Gamma T \subseteq A$. Then

$$A \Gamma A \subseteq M \Gamma T \subseteq A.$$  

Hence $A$ is a sub-$\Gamma$-semiring of $M$.


Proposition II.7. Let $M$ be a $\Gamma$-semiring and $T$ a $\Gamma$-ideal of $M$. Then every subset of $T$ containing $M \Gamma T \cup T T M$ is a $\Gamma$-ideal of $M$.

Proof: Let $B$ be a subset of $T$ such that $M \Gamma T \cup T T M \subseteq B$. Then

$$M \Gamma B \subseteq M \Gamma T \subseteq M \Gamma T \cup T T M \subseteq B$$

and

$$B \Gamma M \subseteq T T M \subseteq T T M \cup M \Gamma T \subseteq B.$$ 

Hence $B$ is a $\Gamma$-ideal of $M$.


Proposition II.8. Let $M$ be a $\Gamma$-semiring and $T$ a quasi-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $T T M \cap M \Gamma T$ is a quasi-$\Gamma$-ideal of $M$.

Proof: Let $C$ be a subset of $T$ such that $T T M \cap M \Gamma T \subseteq C$. Then

$$C \cap T M \cap M \Gamma T \subseteq C$$

and

$$C \cap M \Gamma C \subseteq T T M \cap M \Gamma T \subseteq C.$$ 

Hence $C$ is a quasi-$\Gamma$-ideal of $M$.


Proposition II.9. Let $M$ be a $\Gamma$-semiring and $T$ a $\alpha$-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $T T M \Gamma T$ and all of its images is a $\alpha$-$\Gamma$-ideal of $M$.

Proof: Let $D$ be a subset of $T$ such that $T T M \Gamma T \subseteq D$ and $D \Gamma D \subseteq D$. Then

$$D \Gamma M \Gamma D \subseteq T T M \Gamma T \subseteq D.$$ 

Hence $D$ is a $\alpha$-$\Gamma$-ideal of $M$.


Proposition II.10. Let $M$ be a $\Gamma$-semiring and $T$ a generalized $\beta$-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $T T M \Gamma T$ is a generalized $\beta$-$\Gamma$-ideal of $M$.

Proof: Let $E$ be a subset of $T$ such that $T T M \Gamma T \subseteq E$. Then

$$E \Gamma M \Gamma E \subseteq T T M \Gamma T \subseteq E.$$ 

Hence $E$ is a generalized $\beta$-$\Gamma$-ideal of $M$.


Theorem II.11. Let $M$ be a $\Gamma$-semiring. Then the following statements are equivalent.

1. $M$ is a GB-simple $\Gamma$-semiring.
2. $\alpha \Gamma M \Gamma a = M$ for all $a \in M$.
3. $(a) = M$ for all $a \in M$.

Proof: (1) $\Rightarrow$ (2) Assume that $M$ is a GB-simple $\Gamma$-semiring and $a \in M$. By Lemma II.5, we have $\alpha \Gamma M \Gamma a$ is a generalized $\alpha$-$\Gamma$-ideal of $M$. Since $M$ is a GB-simple $\Gamma$-semiring, we have $\alpha \Gamma M \Gamma a = M$.

(2) $\Rightarrow$ (3) Assume that $\alpha \Gamma M \Gamma a = M$ for all $a \in M$ and let $a \in M$. Then, by (2), we have

$$\{a\} \cup a \Gamma M \Gamma a = \{a\} \cup M = M.$$ 

(3) $\Rightarrow$ (1) Assume that $(a) = M$ for all $a \in M$, and let $A$ be a generalized $\beta$-$\Gamma$-ideal of $M$ and $a \in A$. Then $(a) \subseteq A$. By assumption, we have

$$M = \{a\} \subseteq A \subseteq M.$$ 

Thus $M = A$. Therefore $M$ is a GB-simple $\Gamma$-semiring.


Lemma II.12. Let $B$ be a generalized bi-$\Gamma$-ideal of a $\Gamma$-semiring $M$ and $T$ a sub-$\Gamma$-semiring of $M$. If $T$ is a GB-simple $\Gamma$-semiring such that $T \cap B \neq \emptyset$, then $T \subseteq B$.

Proof: Assume that $T$ is a GB-simple $\Gamma$-semiring such that $T \cap B \neq \emptyset$ and let $a \in T \cap B$. By Lemma II.3, we have $(a) \cup a T T a$ is a generalized bi-$\Gamma$-ideal of $T$. Since $T$ is a GB-simple $\Gamma$-semiring, we have $(a) \cup a T T a = T$. Thus

$$T = (a) \cup a T T a \subseteq B \cup B \Gamma M \Gamma B \subseteq B \cup B \subseteq B.$$ 

Hence $T \subseteq B$.


Theorem II.13. Let $M$ be a $\Gamma$-semiring, $B$ a generalized bi-$\Gamma$-ideal of $M$ and $\emptyset \neq A \subseteq M$. Then $B \Gamma A$ and $A \Gamma B$ are generalized bi-$\Gamma$-ideals of $M$.

Proof: Since $B$ is a generalized bi-$\Gamma$-ideal of $M$, we have

$$(B \Gamma A) \Gamma M \Gamma (B \Gamma A) = (B \Gamma (A \Gamma M \Gamma A)) \Gamma B \subseteq (B \Gamma M \Gamma B) \Gamma A \subseteq B \Gamma A$$

and

$$(A \Gamma B) \Gamma M \Gamma (A \Gamma B) = A \Gamma (B \Gamma M \Gamma B) \Gamma B \subseteq A \Gamma (B \Gamma M \Gamma B) \subseteq A \Gamma B.$$ 

Therefore $B \Gamma A$ and $A \Gamma B$ are generalized bi-$\Gamma$-ideals of $M$.


Theorem II.14. Let $M$ be a $\Gamma$-semiring and $B$ a bi-$\Gamma$-ideal of $M$. Then $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$ if and only if $B$ is a GB-simple $\Gamma$-semiring.

Proof: Assume that $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$. By assumption, $B$ is a $\Gamma$-semiring. Let $C$ be a generalized bi-$\Gamma$-ideal of $B$. Then

$$C \cap B \Gamma C \subseteq C \subseteq B.$$ 

Since $B$ is a generalized bi-$\Gamma$-ideal of $M$ and by Theorem II.13, we have $C \cap B \Gamma C$ is a generalized bi-$\Gamma$-ideal of $M$. Since $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$, we get $C \cap B \Gamma C = B$. Thus, by (3), we have $B = C$. Hence $B$ is a GB-simple $\Gamma$-semiring.

Conversely, assume that $B$ is a GB-simple $\Gamma$-semiring. Let $C$ be a generalized bi-$\Gamma$-ideal of $M$ such that $C \subseteq B$. Then

$$C \cap B \Gamma C \subseteq C \subseteq C \cap B \Gamma C \subseteq C \subseteq B.$$ 

Thus $C$ is a generalized bi-$\Gamma$-ideal of $B$. Since $B$ is a GB-simple $\Gamma$-semiring, we have $B = C$. Hence $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$.


Theorem II.15. Let $M$ be a $\Gamma$-semiring having a proper generalized bi-$\Gamma$-ideal. Then every proper generalized bi-$\Gamma$-ideal of $M$ is minimal if and only if the intersection of any two distinct proper generalized bi-$\Gamma$-ideals is empty.
Proof: Assume that every proper generalized bi-$\Gamma$-ideal of $M$ is minimal and let $B_1$ and $B_2$ be two distinct proper generalized bi-$\Gamma$-ideals of $M$. By assumption, we have $B_1$ and $B_2$ are minimal. We shall show that $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 \cap B_2 \neq \emptyset$. By Lemma II.2, we have $B_1 \cap B_2$ is a proper generalized bi-$\Gamma$-ideal of $M$. Since $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$, we get $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$. Thus $B_1 = B_2$ which is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

Conversely, assume that the intersection of any two distinct proper generalized bi-$\Gamma$-ideals is empty. Let $B$ be a proper generalized bi-$\Gamma$-ideal of $M$ and $C$ a generalized bi-$\Gamma$-ideals of $M$ such that $C \subseteq B$. Suppose that $C \neq B$. Then $C$ is a proper generalized bi-$\Gamma$-ideal of $M$. Since $C \subset B$ and by assumption, we have $C = C \cap B = \emptyset$ which is a contradiction. Therefore $C = B$, so $B$ is minimal.

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REFERENCES


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