Exact Solutions of Steady Plane Flows of an Incompressible Fluid of Variable Viscosity Using \((\xi, \psi)\)- Or \((\eta, \psi)\)- Coordinates

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Abstract—The exact solutions of the equations describing the steady plane motion of an incompressible fluid of variable viscosity for an arbitrary state equation are determined in the \((\xi, \psi)\) - or \((\eta, \psi)\) - coordinates where \(\psi(x,y)\) is the stream function, \(\xi\) and \(\eta\) are the parts of the analytic function, \(\psi = \xi(x,y) + i\eta(x,y)\). Most of the solutions involve arbitrary function/ functions indicating that the flow equations possess an infinite set of solutions.

Keywords—Exact solutions, Fluid of variable viscosity, Navier-Stokes equations, Steady plane flows

I. INTRODUCTION

NAEEM and Nadeem [1] extended Martin’s [2] approach to study the steady plane flows of an incompressible fluid of variable viscosity for an arbitrary state equation. Naeem and Nadeem determined some new exact solutions to the flow equations and also indicated applicability of some of the solutions to physically possible situations. In Martin’s approach a natural curvilinear coordinate system \((\phi, \psi)\) in the physical plane \((x,y)\) is introduced in which \(\psi = \text{constant}\) are the streamlines and \(\phi = \text{constant}\) is an arbitrary family of curves. In Martin’s approach, the transformed system of flow equations becomes undetermined and is due to arbitrariness of the coordinate lines \(\phi = \text{constant}\). The system can be made determinate in a number of ways. Naeem and Nadeem [1] made the system determinate by making system orthogonal, in which case coefficient \(F\) of the first fundamental element \(ds^2\) is zero. Naeem and Ali [3], following Martin’s approach made the system governing the motion of fluid in [1] determined by taking \(\phi = x\).

Recently Labropulu and Chandna [4] extended Martin’s approach to study the steady plane infinitely conducting MHD aligned flows and made their system of flow equations determinate by taking \(\phi(x,y) = \xi(x,y)\) or \(\phi(x,y) = \eta(x,y)\) where \(\xi(x,y)\) and \(\eta(x,y)\) are the real and imaginary parts of an analytic function \(\psi\). Labropulu and Chandna obtained exact solutions for the flows when the stream line pattern is of the form \(\frac{\eta - \xi(\xi)}{g(\xi)} = \text{Constant}\) or \(\xi - k(\eta) = \text{Constant}\).

In the present work, we extend Labropulu and Chandna approach to study the steady plane flows of an incompressible fluid of variable viscosity for arbitrary state equation and present some exact solutions. The most of the solutions contain arbitrary function(s) allowing us to construct an infinite set of solutions to flow equations. The plan of this is as follows:

In the next section description of basic flow equations are discussed. Section-III presents the flow equations in the physical plane and Martin’s system \((\phi, \psi)\). The coefficients \(E, F, G\) of first fundamental \(ds^2\) are also given in \((\xi, \psi)\) - and \((\eta, \psi)\) - coordinate system. In Section-IV, exact solutions to flow equations are determined.

II. BASIC FLOW EQUATIONS

The basic non-dimensional equations governing the steady plane motion of an incompressible fluid of variable viscosity in the presence of an unknown external force with no heat addition are:

\[
\begin{align*}
\mu_x + \nu_y &= 0 \\
\mu_u + \nu_v &= -p_x + \frac{1}{R_e} [(2\mu u_x) + (\mu(u_x + v_y))] + \lambda * f_1 \\
\mu v_x + \nu v_y &= -p_y + \frac{1}{R_e} [(2\mu v_y) + (\mu(u_x + v_y))] + \lambda * f_2 \\
u T_x + v T_y &= \frac{1}{R_e P_r} (T_{xx} + T_{yy}) + \frac{E}{R_e} [2\mu(u_x^2 + v_y^2)] + \mu (u_x + v_y)^2 \end{align*}
\]

where \(u, v\) are the velocity components, \(p\) the pressure, \(\mu\) the fluid viscosity, \(T\) the fluid temperature, \(R_e\) the Reynolds number, \(P_r\) the Prandtl number and \(E\), the Eckert number, \(\rho\) the density of the fluid and \(f_1, f_2\), are the components of the external force. In (2) and (3) \(\lambda^*\) is a non-dimensional
number, and in case of motion under the gravitational force, \( \lambda^* \), is called the Froude number \( (F_e) \).

We define the following functions:

\[
\begin{aligned}
\omega &= v_x - u_y \\
L &= p + \frac{(u^2 + v^2)}{2} - \frac{2\mu}{R_e}
\end{aligned}
\]

In term of these functions, the system (1-5) is replaced by the following system:

\[
\begin{aligned}
\nu &= -L_y - \frac{\mu(u_x + v_y)}{R_e} + F_1 \\
\omega &= -L_x - \frac{4\mu u_x}{R_e} + \frac{\mu(u_x + v_y)}{R_e} + F_2
\end{aligned}
\]

\[
\begin{aligned}
u_x + u_y &= 0
\end{aligned}
\]

\[
\begin{aligned}
\omega &= v_x - u_y \\
u T_x + v T_y &= \frac{1}{R_p} (T_{xx} + T_{yy}) \\
\mu &= \mu(T)
\end{aligned}
\]

Equation (8) implies the existence of a stream function \( \psi(x,y) \) such that

\[
\begin{aligned}
\mu = \psi_y, v = -\psi_x
\end{aligned}
\]

Let \( \psi(x,y) = \text{constant} \) defines the family of streamlines. Let us assume \( \phi(x,y) = \text{constant} \) to be some arbitrary family of curves such that it generates with \( \psi(x,y) = \text{constant} \) a curvilinear net \( (\phi, \psi) \) in the physical plane.

Let \( x = \phi(x,y), y = \psi(x,y) \)

\[
\begin{aligned}
F &= x\phi_y + y\phi_x \\
G &= x\psi_y + y\psi_x
\end{aligned}
\]

Equation (15) can be solved to obtain

\[
\begin{aligned}
\phi &= \phi(x,y), \psi = \psi(x,y)
\end{aligned}
\]

such that

\[
\begin{aligned}
x_\phi &= J\psi_y, x_\psi = -J\phi_y \\
y_\phi &= -J\psi_x, y_\psi = J\phi_x
\end{aligned}
\]

provided that \( 0 < |J| < \infty \), where \( J \) is the transformation Jacobian, and is defined as

\[
J = x_\phi y_\psi - x_\psi y_\phi
\]

If \( \alpha \) is the angle of inclination of the tangent to the coordinate line \( \psi = \text{constant} \), we have from differential geometry, the following results:

\[
\begin{aligned}
J &= \pm W \\
x_\phi &= \sqrt{E\cos\alpha}, x_\psi = \frac{F\cos\alpha - J\sin\alpha}{\sqrt{E}} \\
y_\phi &= \sqrt{E\sin\alpha}, y_\psi = \frac{F\sin\alpha + J\cos\alpha}{\sqrt{E}} \\
\alpha &= \frac{J}{E} \mu, \phi_\psi = \phi_\mu \mu \\
\end{aligned}
\]

where

\[
\begin{aligned}
J_1^2 &= \left[ -F \phi_\psi + 2EF_\phi - FE_\psi \right] \\
J_2^2 &= \frac{21W^2}{2}\mu^2 \\
W &= \sqrt{E - F^2}
\end{aligned}
\]

The three functions \( E, F, G \) of \( \phi, \psi \) satisfy the Gauss equation:

\[
\begin{aligned}
\frac{w_1^2}{E} - \phi_\psi = 0
\end{aligned}
\]

where \( K \) is the Gaussian curvature.

### III. TRANSFORMATION OF BASIC FLOW EQUATIONS IN THE STREAMLINED COORDINATE SYSTEM \((\phi, \psi)\)

If the arbitrary curve \( \phi(x,y) = \text{constant} \) and the streamlines \( \psi(x,y) = \text{constant} \) generate a curvilinear net in the physical plane, the system of equations (8-13) is transformed to the following system:

\[
\begin{aligned}
q &= \frac{\sqrt{E}}{W} \\
J_0 &= \frac{1}{J_0} \left[ 1 - \frac{J_0^2 \sin^{2}\alpha}{2E} \right] (25)
\end{aligned}
\]

\[
\begin{aligned}
+ \left( \frac{2F \sin^{2}\alpha - J^2 \cos^{2}\alpha}{E} \right) M_{\phi} + (F \sin\alpha + J \cos\alpha) M_{\psi}
\end{aligned}
\]

\[
\begin{aligned}
+ \frac{J}{\sqrt{E}} \left[ (F \sin\alpha + J \cos\alpha) + (J \sin\alpha + F \cos\alpha) \right] (26)
\end{aligned}
\]

\[
\begin{aligned}
+ (J \sin^{2}\alpha - F \cos^{2}\alpha) M_{\psi} + (E \cos^{2}\alpha + J \sin^{2}\alpha) + J \sqrt{E}(F \cos^{2}\alpha + J \sin^{2}\alpha) \\
\frac{G T_{\phi}}{J} + \frac{F T_{\psi}}{J} = \frac{E R_{\phi}}{J} \left( \alpha^2 + 4M^2 \right) + \frac{q T_{\psi}}{\sqrt{E}} (27)
\end{aligned}
\]

\[
\begin{aligned}
\omega &= \frac{W}{F} \left( \frac{E}{W} \right) \psi \\
K &= \frac{W_{11}}{E} - \frac{W_{12}^2}{W} = 0
\end{aligned}
\]

\[
\begin{aligned}
\mu &= \mu(T)
\end{aligned}
\]

which in \( \phi \) and \( \psi \) are considered as independent variables. This is a system of seven equations in ten unknown functions.
In (25-27), the functions $A$ and $M$ are given by

$$A = \frac{4\psi}{\sqrt{E\sin(\theta - \psi)}} \left[ F\sin \theta + Jcos \theta \right]$$

$$M = \frac{\mu}{\sqrt{E\sin(\theta - \psi)}} \left[ \psi \cos \theta + \sin \theta \right]$$

Recently Labropulu and Chandna [4] presented a new approach for the determination of exact solutions of steady plane infinitely conducting MHD aligned flows. In their approach, the coordinate net or $(\eta,\psi)$-coordinate is used to obtain exact solution of these flows where coordinates $\psi(x,y)$ is the stream function and $(\eta(x,y))_{\psi}$ are the real and imaginary parts of an analytic function $\varphi = \xi(x,y) + i\eta(x,y)$, Labropulu and Chandna following Martin's transform their

$$\psi_0 = \psi_1 = \psi_2 \quad \text{and} \quad \psi_3 = \psi_4$$

by taking $(\varphi,\psi)$-system where $\psi$ is constant.

Equations:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

The equations

$$\xi = \xi(x,y), \eta = \eta(x,y)$$

can be solved to get

$$x = x(\xi,\eta), y = y(\xi,\eta)$$

such that

$$\frac{\partial x}{\partial \xi} = J, \frac{\partial \eta}{\partial \xi} = J, \frac{\partial y}{\partial \eta} = J, \frac{\partial \xi}{\partial \eta} = J, \frac{\partial \psi}{\partial \xi} = J, \frac{\partial \psi}{\partial \eta} = J$$

provided that $0 < J < \infty$ where $J$ is given by

$$J = \frac{\partial x}{\partial \xi}, \frac{\partial \eta}{\partial \xi}, \frac{\partial y}{\partial \eta}, \frac{\partial \xi}{\partial \eta}, \frac{\partial \psi}{\partial \xi}, \frac{\partial \psi}{\partial \eta}$$

Employing (35), (36) and (39) in $ds^2 = dx^2 + dy^2$, we get

$$J = \frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \eta}, \frac{\partial \psi}{\partial \xi}, \frac{\partial \psi}{\partial \eta}$$

Equation (39), employing (35) and (37), yields

$$ds^2 = J^2 [d\xi^2 + d\eta^2]$$

To analyze whether a family of curves $\frac{\psi_0}{\psi_1} = \psi_2$ is constant, can or can’t be streamlines in $(\xi,\eta)$-coordinate net, they assumed affirmative so that their exist some function $\gamma(\psi)$ such that

$$\gamma - f(\psi) = \gamma(\psi_0), \gamma(\psi_1)$$

$$d^2 = J^2 [\gamma(\psi_0) + \gamma(\psi_1)]$$

Similarly to analyze whether a family of curves $\frac{\xi - k(\eta)}{m(\eta)}$ is constant, can or can’t be streamlines in $(\eta,\psi)$-coordinate net, they assumed affirmative so that their exist some function $\gamma(\psi)$ such that

$$\gamma - f(\psi) = \gamma(\psi_0), \gamma(\psi_1)$$

$$d^2 = J^2 [\gamma(\psi_0) + \gamma(\psi_1)]$$

For both family of streamlines there exists some function $\gamma(\psi)$ such that

$$\xi - k(\eta) = \text{Constant}, \gamma(\psi) = \text{Constant}$$

In the absence of external force.

(i) Assume

$$\varphi = \xi + i\eta = \ln z$$

where $z = x + iy$. The (56) yields.

$$\xi = \frac{1}{2} \ln \left( x^2 + y^2 \right)$$

$$\eta = \tan^{-1} \left( \frac{y}{x} \right)$$

OR

$$x = e^\xi \cos \eta$$

$$y = e^\xi \sin \eta$$

A Example1 (Flows with $\xi$ = constant as streamlines)
We let \( [4] \)
\[
\xi = \gamma(\psi), \quad \gamma'(\psi) \neq 0
\]
where \( \gamma(\psi) \) is an unknown function and \( \xi \) is given by (57).
Comparing (59) with (55), we get
\[
k(\eta) = 0, \quad m(\eta) = 1
\]
Utilizing (60) in (49-53), we get
\[
\begin{align*}
E & = J', \\
F & = 0, \\
G & = J'\gamma'(\psi) \\
J & = J'\gamma'(\psi) \\
W & = J'\gamma'(\psi)
\end{align*}
\]
where
\[
J' = e^{2\gamma(\psi)}
\]
Equations (24-29), utilizing (61) and (62), become
\[
\begin{align*}
\frac{dA}{d\psi} & = 1, \\
\frac{d\psi}{d\psi} & = \frac{1}{\gamma(\psi)}
\end{align*}
\]
where in (64-66), the functions \( A \) and \( M \) are given by
\[
\begin{align*}
A & = \frac{2a}{R_e e^{2\gamma(\psi)}(\psi)\gamma'(\psi)} \left[ 2 - \frac{\gamma''(\psi)}{2\gamma'(\psi)} \right] \sin 2\eta \\
M & = \frac{\mu}{R_e e^{2\gamma(\psi)}(\psi)\gamma'(\psi)} \left[ 2 - \frac{\gamma''(\psi)}{2\gamma'(\psi)} \right] \cos 2\eta
\end{align*}
\]
In order to determine the solutions of (64-66), we make use of the compatibility condition \( L_{\theta\psi} = L_{\psi\theta} \) and this yield
\[
\gamma' \gamma'' Z_{\theta\eta} - \gamma' Z_{\psi\psi} = 0
\]
where
\[
Z = \frac{\mu}{R_e e^{2\gamma(\psi)}} \left[ 2 - \frac{\gamma''(\psi)}{\gamma'(\psi)} \right]
\]
Equation (70) is the equation which the viscosity \( \mu \) and \( \gamma(\psi) \) must satisfy.
Equation (70) possesses many solutions and we consider only those solutions for which the exact solution of (66) can be determined. These solutions are for the following cases:

**Case I** \( \gamma'' = 0 \)

**Case II** \( \gamma'' \neq 0 \)

We study these cases separately

**Case I**

When \( \gamma'' = 0 \) we get
\[
\gamma = a\psi + b
\]
The (70) provides
\[
a^2 \gamma_{\theta\eta} - 2a\gamma_{\theta\psi} - \gamma_{\psi\psi} = 0
\]

Two solutions of (73) are determined which employing (74) give us
\[
\mu = \frac{aR_e e^{2\gamma(\psi)}}{2a} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) + \frac{1}{2} \frac{1}{e^{2\gamma(\psi)}} + \frac{1}{a}
\]
where \( t_1(\neq 0), t_2(\neq 0), t_3, t_4, t_5, t_6, t_7 \) are arbitrary constant, and \( t_r = t_e + \frac{t_1}{2} - \frac{t_2}{2a} \).

The temperature distribution \( T \) given by (75) satisfies the equation
\[
a^2 T_{\theta\psi} + T_{\psi\psi} - aR_P T_\eta = -2aR_P E_1 \psi(\eta, \psi)
\]
whose solution is
\[
T = a(RP, E_1, \eta, \psi) \left( \frac{t_1}{2} + \frac{t_2}{2a} \right) + \frac{t_3}{4a^2}
\]
For \( \mu \) given by (76), the temperature distribution \( T \) satisfies the equation
\[
a^2 T_{\theta\psi} + T_{\psi\psi} - aR_P T_\eta = -2aR_P E_1 \psi(\eta, \psi)
\]

The solution of (79) is
\[
T = e^{2\gamma(\psi)} \left[ \frac{R_P E_1}{a} \left( \frac{S_2}{a} \right) \right] \left( \frac{t_1}{2} + \frac{t_2}{2a} \right) + \frac{t_3}{4a^2}
\]

Equation (83) yields
\[
\gamma'' = -\frac{1}{2} \ln(2a + c_1) + c_2
\]
where \( c_1, c_2 \) are arbitrary constants. Equations (68) and (69), employing (83), yield
\[
A = 0 \\
M = 0
\]

Equations (64) and (65), utilizing (85) yields
\[
L = \text{constant} = c_3 \quad \text{(say)}
\]
We note that in this case the viscosity function \( \mu \) is arbitrary.
Now the (66), on using (84) and (85), becomes
\[
T_{\eta\psi} - 2(2\psi + c_1) R_P T_{\psi\theta} + (2\psi + c_1) T_{\psi\psi} + 2(2\psi + c_1) H_{\psi} = 0
\]

The solution of (87) is
\[ T = c_4 \eta + \frac{c_5 R \rho L}{4}(2 \psi + c_1) + c_6 \ln(2 \psi + c_1) + c_6 \]

where \( c_4(\neq 0), c_5(\neq 0), c_6 \) are arbitrary constants. For a non-trivial solution of (70) we let

\[ Z = \text{constant} = C_7 \quad \text{(say)}. \]

Then (71) yields

\[
\mu = \frac{c_5 R \rho L e^{2 \psi}}{2} + \frac{\gamma'''}{\gamma'} \neq 0
\]

Equations (64) and (65) on using \( Z = C_7 \), yield

\[
L_{\psi} = -e^{\psi}
\]

\[
L_{\psi} = -2c_7(1 + \cos 2\eta)
\]

Equations (90) and (91) give

\[
L = \int \frac{\gamma'''}{\gamma'} d\eta + 2C_8
\]

where \( c_7(\neq 0) \) and \( c_8 \) are arbitrary constants.

The temperature distribution in this case, satisfies

\[
\gamma^3 T_{\eta \eta} - \gamma^2 R_{\eta} P_{\eta} \gamma - \gamma R_{\psi \psi} - \gamma'''' = -\gamma^2 c_8 R_{\psi} P_{\psi} \gamma(2 + \gamma'')
\]

whose solution is

\[
T = a_{\eta \eta} \int \gamma^{2 \psi} \left( \left| \gamma^{2 \psi} R_{\psi} P_{\psi} - c_8 \gamma^{2 \psi} E_{\psi} P_{\psi} \gamma \right| + \gamma^{2 \psi} \right) d\eta + \gamma^2
\]

where \( c_9(\neq 0), c_{10}, c_{11} \) are arbitrary constants. We mention that the function \( \gamma(\psi) \) is arbitrary in this case, and therefore we can construct an infinite set of solutions to the flow equations.

**Example-2** (Flows with \( \eta = \text{constant as streamlines})

Assume \[ \eta = \gamma(\psi) \]

where \( \gamma(\psi) \) is unknown function and \( \eta \) is given by (57).

Equations (96) and (54), give

\[
\gamma = 0, \gamma_{\psi} = 1
\]

Equations (43-47), employing (96) yield

\[
E = 2^\psi \gamma
\]

\[
G = 2^\psi \gamma^2 \psi
\]

\[
J = 2^\psi \gamma'^2
\]

\[
W = 2^\psi \gamma'^3
\]

Equations (24-29), employing (97), give

\[
\omega = -L_{\psi} - \cos \gamma(\psi) A_{\psi} - \sin \gamma(\psi) M_{\psi} - \frac{\gamma''(\psi)\sin 2\gamma(\psi)}{2} A_{\psi}
\]

\[
+ \gamma(\psi) \cos 2\gamma(\psi) M_{\psi}
\]

\[
0 = -L_{\psi} - \cos \gamma(\psi) \sin 2\gamma(\psi) A_{\psi} - \cos \gamma(\psi) 2\gamma' M_{\psi} - \sin \gamma(\psi) 2\gamma' A_{\psi}
\]

\[
+ \sin 2\gamma(\psi) M_{\psi}
\]

\[
\gamma'(\psi) T_{\xi \xi} + \frac{R_{\psi}}{\gamma'(\psi)} = -\frac{2^\psi \gamma'(\psi) E_{\psi} P_{\psi} \gamma^2}{\mu} + \mu R_{\psi} T_{\xi \xi}
\]

\[
\omega = \frac{\gamma''(\psi)}{\gamma'(\psi)^3}
\]

where the functions \( A \) and \( M \) are given by

\[
A = \frac{4\mu}{\gamma'(\psi)^3} \left( -C_2 \gamma(\psi) - \frac{\gamma''(\psi) \sin 2\gamma(\psi)}{2} \right)
\]

\[
M = \frac{2\mu}{\gamma'(\psi)^3} \left( \sin 2\gamma(\psi) \frac{\gamma''(\psi)}{2} \right)
\]

Equations (99) and (100), employing (102-104), can be rewritten as

\[
-\gamma''(\psi)\gamma'(\psi) - 2\gamma' \gamma''(\psi) + 4\gamma''^{2}(\psi)
\]

\[
-4\gamma' \cos 2\gamma(\psi) - 2\gamma' \gamma''(\psi) + 4\gamma''^{2}(\psi)
\]

\[
0 = -\gamma' L_{\psi} - 4\gamma' \sin 2\gamma(\psi) - 4\gamma' X - \gamma' \sin 2\gamma(\psi) Y
\]

where

\[
X = \frac{\mu}{R_{\psi} \gamma'(\psi)^3}
\]

\[
Y = \frac{\gamma''(\psi)}{\gamma'(\psi)^3}
\]

Proceeding in the same manner as in example–1, a solution of (105) and (106) is

\[
\mu = a R_{\psi} e^{2\psi} \int \left( \int Z_{\xi}(\xi) d\xi + Z_{\zeta}(\psi) \right)
\]

\[
L = e^{-2\psi} \int \left( \int Z_{\xi}(\xi) d\xi + Z_{\zeta}(\psi) \right) - 2Z_{\zeta}(\xi) - 4Z_{\zeta}(\xi) d\xi + g_1
\]

provided \( \gamma = av + b \). In (109) and (110), \( Z_{\xi}(\xi) \) and \( Z_{\zeta}(\psi) \) are arbitrary functions and \( g_1, g_2 \) are arbitrary constants.

The temperature distribution \( T \) satisfies the equation

\[
a^2 T_{\xi \xi} + T_{\psi \psi} - a R_{\psi} T_{\xi \xi} = \frac{4\mu}{a^2}
\]

where \( \mu \) is given by (109). The solution of (111) is

\[
T = \frac{4E, P, R, c_1 \gamma(\psi)}{a} \int \left( \int Z_{\xi}(\xi) d\xi \right) - \frac{a a}{P_{\psi}} - \frac{a a}{R_{\psi}}
\]

\[
-4a E, P, R_{\psi} \int \left( \int Z_{\xi}(\xi) d\xi \right) - a a R_{\psi}
\]

\[
+ a \cos \psi a^2 + a P, R, P, R_{\psi}, a \sin \psi a + a P, R, P, R_{\psi}
\]

\[
+ \frac{1}{a^2} \left( \int Z_{\xi}(\xi) d\xi \right) - a a R_{\psi}
\]

\[
E, P, R_{\psi}
\]

for \( a = R_{\psi} \),

where \( a_1, a_2, a_3, a_4, a_5, a_6 \) are all arbitrary constants.

Note that the expressions for \( \mu L \) and \( T \) involve arbitrary functions, and this allows us to construct a large number of solution to the flow equations.

(2) Assume

\[
\omega = \xi + i \eta = a^2 z + b
\]

where \( a = a_1 + ia_2, b = b_1 + ib_2 \)
For this example, in (24-29), employing (114), become

\[ q = \frac{1}{\gamma'(\psi)} \left[ \frac{1 + \lambda^2}{\gamma'(\psi)} \right] \]

\[ = \frac{1}{4\mu} \left( \frac{1 + \lambda^2}{\gamma'(\psi)} \right) \]

where

\[ A = -\frac{4\mu(1 + \lambda^2)\gamma''(\psi)}{\gamma'(\psi)} \]

\[ M = \frac{1 + \lambda^2}{\gamma'(\psi)} \]

and \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \) are given in appendix B.

Equations (116-118), employing (119) and (120), become

\[ L_v = -\beta_{14} \frac{\gamma''(\psi)}{\gamma'(\psi)} + \beta_{15} \gamma'(\psi) X_1 + \beta_{16} X_v \]

\[ L_z = -\beta_{13} \gamma'(\psi) + \frac{X_z}{\gamma'} \]

\[ \gamma''(\psi) T_v - 2\lambda T_v + \frac{1 + \lambda^2}{\gamma'} T_v + \gamma''(\psi) X_v \]

\[ = -\frac{E_P R_P(\gamma' \psi^3)}{\mu} \]

where

\[ X(\xi, \psi) = \frac{\gamma''(\psi)}{R_P(\gamma' \psi^3)} \psi_0(\xi, \psi) \]

and \( \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14} \) are given in appendix B.

On eliminating the generalized energy function \( L \) from (121) and (122), we obtain

\[ \frac{\partial}{\partial \psi} X_{\psi \psi} + \gamma'(\psi) \beta_{12} X_{\psi} - \beta_{11} \gamma'(\psi) X_v + (\beta_{10} + \beta_{13}) X_{\psi \psi} = 0 \]  

Equation (125) is the compatibility equations for example (3).

We found that this compatibility equation possesses solutions for the following possible cases for which equation (123) is exactly solvable.

Case I \( \gamma''(\psi) = 0 \)

Case II \( \gamma''(\psi) \neq 0 \)

We study these two cases separately as follows.

Case-I

When \( \gamma''(\psi) = 0, \gamma'(\psi) = a \psi + b \). Equation (125) is identically satisfied. The viscosity function \( \mu \) is arbitrary and the generalized energy function \( L \) turns out to be constant.

The solution of (123), for any value of \( \lambda \), is given by

\[ T = m_1 \psi^2 \left( \frac{2 \lambda \gamma'(\psi) - m_2}{2(1 + \lambda^2)} \right) \]

\[ + m_3 \psi + m_4 \]

where \( m_1, m_2, m_3, m_4, m_5, m_6 \) are all non-zero arbitrary constants.

Case-II

When \( \gamma''(\psi) \neq 0 \), the function \( \chi \) in (124) can either be considered constant or non-constant. When \( \chi = \) constant = \( \omega \)

(say), the equation is identically satisfied and therefore

\[ \mu = \frac{m_7 \gamma(\psi)}{\gamma''(\psi)} \]

Equation (123), employing (127), yields

\[ \gamma''(\psi) T_v - 2\lambda T_v + \frac{1 + \lambda^2}{\gamma'} T_v + \gamma''(\psi) X_v \]

\[ = -\frac{E_P R_P(\gamma' \psi^3)}{\mu} \]

whose solution is

\[ T = \int \left[ \frac{\gamma''(\psi)}{\gamma'} \right] \left[ \frac{1}{\gamma'} \right] \left( \int \frac{\gamma''(\psi)}{\gamma'} \right) d\psi + m_5 \]

\[ + m_6 \xi + m_7 \]

where \( m_7, m_8, m_9, m_{10} \) are all non-zero arbitrary constants. We note that in this case the function \( \gamma(\psi) \) is arbitrary. The generalized energy function \( L \) can easily be determined from (121) and (122).

When \( \chi \) is not a constant, the solution of (125) is

\[ X = m_{11} + m_{12} \int e^{\gamma''(\psi)} d\psi + m_3 \]

where \( m_{11}, m_{12}, m_{13} \) are non-zero arbitrary constants. The viscosity \( \mu \) is given by

\[ \mu = \frac{\gamma(\psi)}{m_{14} + \chi(\psi)} \]

Equation (123), in this case becomes.

\[ T_v - 2\lambda T_v + \frac{1 + \lambda^2}{\gamma'} T_v + \gamma''(\psi) X_v \]

\[ = -\frac{E_P R_P(\gamma' \psi^3)}{\mu} \]

\[ (m_{14} + \chi(\psi)) \]

The two solutions of (132) are obtained and these are
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\[ T = \int \left[ \frac{E_{\psi\psi}}{y^2} + \left( \frac{P_{\psi\psi} + 2\lambda m_{21}}{y^2} \right) d\psi \right] y \]

(i) and

\[ \mu, m_{21} = \mu + m_{22} + m_{23} \psi^2 + m_{24} \psi + m_{25} \psi^4 \]

\[ \text{for } m_{21} = \frac{E_{\psi\psi} P_{\psi\psi}}{1 + \frac{2}{2}m_{21}} \]

(ii) and

\[ T = e^{(1+\lambda)} \left( \frac{m_{21} + 2m_{22}}{1 + \frac{2}{2}m_{21}} \right) \]

\[ \left( \frac{m_{21} + 2m_{22}}{1 + \frac{2}{2}m_{21}} \right) \left( \frac{m_{21} + 2m_{22}}{1 + \frac{2}{2}m_{21}} \right) \]

\[ \int \left( 1 + \frac{2}{2}m_{21} \right) \left( \frac{m_{21} + 2m_{22}}{1 + \frac{2}{2}m_{21}} \right) \left( \frac{m_{21} + 2m_{22}}{1 + \frac{2}{2}m_{21}} \right) \]

\[ \phi = \text{constant} \]

\[ \text{for solution given by (133), the function } \psi \text{ is arbitrary, and for (134) it is given by} \]

\[ \psi = \frac{1}{1 + \frac{2}{2}m_{21} - E_{\psi\psi} P_{\psi\psi}} \]

\[ \ln \left( \frac{m_{21} + 2m_{22}}{1 + \frac{2}{2}m_{21} - E_{\psi\psi} P_{\psi\psi}} \right) \]

The constants \( m_{16}, m_{17}, m_{18}, m_{19}, m_{20}, m_{21} \) are all non zero arbitrary constants. In (131) and (132) \( X_{\phi}(\psi) \) is given by

\[ X_{\phi}(\psi) = m_{12} \left[ \frac{P_{\psi\psi}}{m_{14}} \right] \]

\[ \text{The generalized energy function } L \text{ can easily be determined in this case in the same manner as in examples 1 and 2.} \]

\[ x = x(\phi, \psi), y = (\phi, \psi) \]

\[ \text{define a system of curvilinear coordinates in } (x, y) \text{ plane such that the Jacobian, } J = \frac{\partial(x, y)}{\partial(\phi, \psi)} \text{ of the transformation (A.1) is non-zero and finite. The first fundamental form of differential element is defined by} \]

\[ ds^2 = E(x, y) dx^2 + 2F(x, y) dx dy + G(x, y) dy^2 \]

in which E, F, G are given by

\[ E(\phi, \psi) = x_\phi^2 + y_\phi^2 \]

\[ F(\phi, \psi) = x_\phi y_\psi + y_\phi x_\psi \]

\[ G(\phi, \psi) = y_\psi^2 \]

Differentiating (A.1) with respect to \( x \) and \( y \) and solving the resulting equations for \( \psi_x, \psi_y, \phi_x, \phi_y \) yields

\[ x_\phi = J \psi_x, y_\phi = -J \psi_y \]

\[ x_\psi = -J \psi_y, y_\psi = J \psi_x \]

wherein

\[ J = \sqrt{E G - F^2} \]

\[ = \pm (x_\phi y_\psi - y_\phi x_\psi) = \pm W \text{(say)} \]

Let \( \alpha \) be the angle between the tangent vector at the point \( P(x, y) \) [see Fig.1] to the coordinate line \( \psi = \text{constant} \) then

\[ \phi = \text{constant} \]

\[ \psi = \text{constant} \]

\[ x \text{ - axis} \]

\[ \text{Fig. 1 (} \phi, \psi \text{) coordinate system} \]

\[ \text{Equation (A.4), on utilizing (A.3, A.6), gives} \]

\[ \begin{align*}
  x_\phi &= \sqrt{E \cos \alpha} x_\psi = \frac{FC_{\cos \alpha} - J \sin \alpha}{\sqrt{E}} \\
  y_\phi &= \sqrt{E \sin \alpha} y_\psi = \frac{F \sin \alpha + J \cos \alpha}{\sqrt{E}}
\end{align*} \]

The integrability conditions

\[ x_\psi y_\phi - x_\phi y_\psi = y_\psi \psi \]

for variables \( x \) and \( y \), yield

\[ \phi \text{ is constant} \]

\[ \psi \text{ is constant} \]

\[ x \text{ - axis} \]

\[ \text{Fig. 1 (} \phi, \psi \text{) coordinate system} \]

\[ \text{Equation (A.4), on utilizing (A.3, A.6), gives} \]

\[ \begin{align*}
  x_\phi &= \sqrt{E \cos \alpha} x_\psi = \frac{FC_{\cos \alpha} - J \sin \alpha}{\sqrt{E}} \\
  y_\phi &= \sqrt{E \sin \alpha} y_\psi = \frac{F \sin \alpha + J \cos \alpha}{\sqrt{E}}
\end{align*} \]

The integrability conditions

\[ x_\psi y_\phi - x_\phi y_\psi = y_\psi \psi \]

for variables \( x \) and \( y \), yield

\[ \phi \text{ is constant} \]

\[ \psi \text{ is constant} \]

\[ x \text{ - axis} \]

\[ \text{Fig. 1 (} \phi, \psi \text{) coordinate system} \]
\[
\alpha_x = \frac{\mathcal{R}_1^2}{E}
\]
\[
\alpha_y = \frac{\mathcal{R}_2^2}{E}
\]
\[
\text{wherein}
\]
\[
\begin{align*}
\mathcal{R}_1^2 &= \frac{[-FE_0 + 2E_0 - EF_0]}{2W^2} \\
\mathcal{R}_2^2 &= \frac{[EG_0 - FE_0]}{2W^2}
\end{align*}
\]
Equation (A.9), on employing integrability condition for \(\alpha(\phi, \psi)\), \(\alpha_{\psi\phi} = \alpha_{\phi\psi}\) yields
\[
\kappa = \frac{\psi_1^2}{w_1} - \frac{\psi_2^2}{w_2} = 0
\]
wherein \(K\) is called the Gaussian Curvature and (A.11) is called the Gauss equations. This equation represents a necessary and sufficient condition that \(E, F, G\), are coefficient of the first fundamental form in (A.2).

**APPENDIX B**

\[
\beta_1 = \frac{a_1(2a_1 + a_2)}{\left(1 + \lambda_1^2\right)\left(a_1^2 + a_2^2\right)}
\]
(B-1)

\[
\beta_2 = \frac{\left(2a_2 + 2a_2 a_1 - 2a_1\right)}{\left(1 + \lambda_1^2\right)\left(a_1^2 + a_2^2\right)}
\]
(B-2)

\[
\beta_3 = \frac{a_1}{\left(a_1^2 + a_2^2\right)}
\]
(B-3)

\[
\beta_4 = \frac{a_2}{\left(a_1^2 + a_2^2\right)}
\]
(B-4)

\[
\beta_5 = \frac{1}{\left(a_1^2 + a_2^2\right)}
\]
(B-5)

\[
\beta_6 = \frac{a_1}{\left(a_1^2 + a_2^2\right)}
\]
(B-6)

\[
\beta_7 = \frac{\left(2a_2 + 2a_2 a_1 - 2a_1\right)}{\left(1 + \lambda_1^2\right)\left(a_1^2 + a_2^2\right)}
\]
(B-7)

\[
\beta_8 = \frac{\left(2a_2 + 2a_2 a_1 - 2a_1\right)}{\left(1 + \lambda_1^2\right)\left(a_1^2 + a_2^2\right)}
\]
(B-8)

\[
\beta_9 = \frac{2a_1 a_2}{\left(a_1^2 + a_2^2\right)}
\]
(B-9)

\[
\beta_{10} = \frac{4\beta_4 a_1}{\beta_1}
\]
(B-10)

\[
\beta_{11} = \frac{4\beta_4 a_1}{\beta_1} + \frac{4\beta_4 a_1}{\beta_1}
\]
(B-11)

\[
\beta_{12} = \frac{4\beta_4 a_1}{\beta_1} - \frac{4\beta_4 a_1}{\beta_1}
\]
(B-12)

\[
\beta_{13} = \frac{4\beta_4 a_1}{\beta_1} + \frac{4\beta_4 a_1}{\beta_1}
\]
(B-13)

\[
\beta_{14} = \frac{\left(1 + \lambda_1^2\right)\left(a_1^2 + a_2^2\right)}{\beta_1}
\]
(B-14)

**REFERENCES**


