

# Finite-Horizon Tracking Control for Repetitive Systems with Uncertain Initial Conditions

Sung Wook Yun, Yun Jong Choi, Kyong-min Lee, and Poogyeon Park

**Abstract**—Repetitive systems stand for a kind of systems that perform a simple task on a fixed pattern repetitively, which are widely spread in industrial fields. Hence, many researchers have been interested in those systems, especially in the field of iterative learning control (ILC). In this paper, we propose a finite-horizon tracking control scheme for linear time-varying repetitive systems with uncertain initial conditions. The scheme is derived both analytically and numerically for state-feedback systems and only numerically for output-feedback systems. Then, it is extended to stable systems with input constraints. All numerical schemes are developed in the forms of linear matrix inequalities (LMIs). A distinguished feature of the proposed scheme from the existing iterative learning control is that the scheme guarantees the tracking performance exactly even under uncertain initial conditions. The simulation results demonstrate the good performance of the proposed scheme.

**Keywords**—Finite time horizon, linear matrix inequality (LMI), repetitive system, uncertain initial condition.

## I. INTRODUCTION

THE repetitive system are widely spread over industrial fields such as robot manipulators, batch reactors, injection moldings and heating processes. For several decades, considerable efforts have been made on the development and analysis of tracking control for a repetitive system that performs a task on fixed pattern repetitively [7]. Among them, the Iterative Learning Control (ILC) has gained much attention as a useful method, which has an excellent tracking performance [1] – [6]. Until now, most researches regarding tracking control have been focused on the infinite horizon in the time axis and its analysis. Unfortunately, in the practical application one cannot but consider the finite horizon in which the tracking performance should be guaranteed. However, there are not much academic researches on guaranteed performance in finite time horizon. In this paper, we handle the finite-time horizon tracking control for repetitive systems with uncertain initial conditions

This paper is organized as follows. Section II describes the system model and formulates the problem. Section III finds the performance bound of uncertain initial conditions and calculates it by the proposed linear matrix inequality (LMI)

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conditions for three cases. Section IV presents simulation results and the performance of the proposed controller. Finally, section V concludes the paper with summarization and future works.

## II. PROBLEM FORMULATION

Consider the following linear discrete time-varying repetitive system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k, \end{aligned} \quad (1)$$

where  $k$  is discrete-time index. Reference trajectory  $y_0^T, y_1^T, \dots, y_n^T$  is previously given. Before proceeding to the next stage, we assume that

- (A1) Each successive operation ends after finite number of steps  $F(> 0)$
- (A2) Uncertain initial states are bounded by  $\varepsilon$

$$\|x_0 - x_c\|_P \leq \varepsilon,$$

where  $P$  is the dimensional weighting matrix and diagonal.

The output can be represented as follows:

$$Y = N x_0 + M u,$$

where

$$Y \triangleq \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad N \triangleq \begin{bmatrix} C_0 \\ C_1 A_0 \\ C_2 A_1 A_0 \\ \vdots \\ C_n A_{n-1} \cdots A_1 A_0 \end{bmatrix}, \quad u \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix},$$

$$M \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_1 B_0 & 0 & \cdots & 0 \\ C_2 A_1 B_0 & C_2 B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_n A_{n-1} \cdots A_1 B_0 & C_n A_{n-1} \cdots A_2 B_1 & \cdots & C_n B_{n-1} \end{bmatrix}.$$

Here, let us consider the tracking problem such that

$$\min_u \|Y_r - Y\|^2 \leq \gamma \text{ subject to } \|x_0 - x_c\|_P \leq \varepsilon. \quad (2)$$

Then, our goal of the paper is to find  $u$  minimizing upper bound of trajectory tracking error  $\gamma$  when initial condition is bounded by  $\varepsilon$ .

### III. MAIN RESULTS

#### A. Performance Bound of Uncertain Initial Condition

Let us define  $\bar{u}$  as

$$\bar{u} \triangleq Mu. \quad (3)$$

Then,  $\bar{u}$  can be uniquely specified if  $u$  is determined. From (2), we can establish two matrix inequalities

$$\begin{bmatrix} \gamma - (Y_r - Mu)^T(Y_r - Mu) & (*) \\ N^T(Y_r - Mu) & -N^TN \end{bmatrix} \geq 0, \quad (4)$$

$$\begin{bmatrix} \varepsilon^2 - x_c^T P x_c & (*) \\ P x_c & -P \end{bmatrix} \geq 0. \quad (5)$$

Combining (4) and (5) by the  $S$ -procedure and the Schur complement, we have

$$\begin{bmatrix} \gamma - \tau\varepsilon^2 + \tau x_c^T P x_c & (*) & (*) \\ (Y_r - Mu) & I & (*) \\ N^T(Y_r - Mu) - \tau P x_c & 0 & -N^TN + \tau P \end{bmatrix} \geq 0, \quad (6)$$

where  $\tau$  is a positive value. (6) is rewritten as

$$\gamma \geq \tau\varepsilon^2 - \tau x_c^T P x_c + J(\bar{u}), \quad (7)$$

where

$$J(\bar{u}) \triangleq [\Phi - \Psi\bar{u}]^T H [\Phi - \Psi\bar{u}], \quad \Phi \triangleq \begin{bmatrix} Y_r \\ N^T Y_r - \tau P x_c \end{bmatrix},$$

$$\Psi \triangleq \begin{bmatrix} I \\ N^T \end{bmatrix}, \quad H \triangleq \begin{bmatrix} I & 0 \\ 0 & (\tau P - N^T N)^{-1} \end{bmatrix}.$$

To minimize  $J(\bar{u})$ , we can have following result by weighted least squares method:

$$\bar{u}^* \triangleq Mu^* = (\Psi^T H \Psi)^{-1} \Psi^T H \Phi = Y_r - N x_c, \quad (8)$$

where  $\bar{u}^*$  and  $u^*$  are the optimum values of  $\bar{u}$  and  $u$ , respectively. Substituting (8) into (7) yields

$$\begin{aligned} \gamma &\geq \tau\varepsilon^2 - \tau x_c^T P x_c + J(\bar{u}^*) \\ &= \tau\varepsilon^2 - \tau x_c^T P x_c + x_c^T [N^T N + (\tau P - N^T N)] x_c \\ &= \tau\varepsilon. \end{aligned}$$

And, using the *Hessian* of  $J(\bar{u})$  ( $> 0$ ), we can obtain

$$\tau \geq \lambda_{max}(NP^{-1}N^T),$$

which leads to

$$\gamma_{min} = \lambda_{max}(NP^{-1}N^T)\varepsilon^2. \quad (9)$$

Therefore, we can summarize following theorem.

**Theorem 1:** (Performance bound of uncertain initial condition) For a given linear discrete time-varying system and initial states bounded by  $\varepsilon$  (i.e.  $\|x_0 - x_c\| \leq \varepsilon$ ), the maximum tracking error is

$$\|Y_r - Y\|^2 \leq \lambda_{max}(NP^{-1}N^T)\varepsilon^2, \quad (10)$$

where  $\lambda_{max}(A)$  means maximum eigenvalue of  $A$ .

*Proof:* It has been already provided. ■

#### B. Computation of $\lambda_{max}(NP^{-1}N^T)$

In this section, we will propose a way of calculating the performance bound in detail. The procedure is as follows.

Let the initial condition  $Q_F \triangleq C_F^T C_F$ . Recursively finding  $Q_{n-1}$  such that

$$Q_{n-1} = A_{n-1}^T Q_n A_{n-1} + C_{n-1}^T C_{n-1}, \quad (11)$$

we can obtain  $Q_0 = N^T N$ . Because  $P$  is diagonal,  $\gamma_{min} = \lambda_{max}(Q_0 P^{-1})\varepsilon^2$ . If the system matrix  $A$  is not Hurwitz,  $\lambda_{max}(NP^{-1}N^T)$  become very large as the time index increasing, and thus it is difficult to obtain a minimum  $\gamma$  with an appropriate level. In this case,  $N$  must be stabilized through feedback mechanism. Modifying the recursive formula (11) as

$$Q_{n-1} \geq A_{n-1}^T Q_n A_{n-1} + C_{n-1}^T C_{n-1}, \quad (12)$$

we can minimize  $\lambda_{max}(Q_0)$  by finding

$$\begin{aligned} &\min \operatorname{tr}(Q_{n-1}) \text{ s.t.} \\ &\begin{bmatrix} Q_{n-1} & (*) & (*) \\ Q_n^{\frac{1}{2}} A_{n-1} & I & (*) \\ C_{n-1} & 0 & I \end{bmatrix} \geq 0. \end{aligned}$$

Now, let us apply this algorithm to feedback systems.

1) *State feedback:* Using the feedback control law

$$u_k = K_k x_k + u_k^*,$$

where  $K_k x_k$  is a feedback component and  $u_k^*$  is a feedforward component, analytic solution of state feedback parameter is

$$Q_{n-1} = (A_{n-1} + B_{n-1} K_{n-1})^T Q_n (A_{n-1} + B_{n-1} K_{n-1}) + C_{n-1}^T C_{n-1},$$

$$K_{n-1} = -2(B_{n-1}^T Q_n B_{n-1})^{-1} B_{n-1}^T Q_n A_{n-1},$$

which can be formulated with the following LMI:

$$\begin{aligned} &\min \operatorname{tr}(Q_{n-1}) \text{ s.t.} \\ &\begin{bmatrix} Q_n & (*) & (*) \\ Q_{n+1}^{\frac{1}{2}} (A_n + B_n K_n) & I & (*) \\ C_n & 0 & I \end{bmatrix} \geq 0. \end{aligned}$$

2) *Output feedback:* Using the dynamic output feedback control

$$\begin{aligned} x_{k+1}^c &= A_k^c x_k^c + B_k^c y_k, \\ u_k &= C_k^c x_k^c + D_k^c y_k + u_f, \end{aligned}$$

where  $u_f$  is the feedforward input, then we have the following LMI condition

$$\begin{aligned} &\min_{\Sigma_{n-1}} \operatorname{tr}(Q_{n-1}) \text{ s.t.} \\ &\begin{bmatrix} Q_n & (*) & (*) \\ Q_{n+1}^{\frac{1}{2}} (\tilde{A}_n + \tilde{B}_n \Sigma_n \tilde{C}_n) & I & 0 \\ C_n & 0 & I \end{bmatrix} \geq 0, \end{aligned}$$

where

$$\begin{aligned} \tilde{A}_n &\triangleq \begin{bmatrix} A_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_n \triangleq \begin{bmatrix} 0 & I \\ B_k & 0 \end{bmatrix}, \\ \Sigma_n &\triangleq \begin{bmatrix} A_k^c & B_k^c \\ C_k^c & D_k^c \end{bmatrix}, \quad \tilde{C}_n \triangleq \begin{bmatrix} I & 0 \\ 0 & C_k \end{bmatrix}. \end{aligned}$$

### C. Input Constrained System

In practical systems, since the input is often restricted in a level, we might consider the tracking performance under the input saturation. In this section, we deal with guaranteed performance when the system input has a limited boundary. If the system is unstable, feedback input is required. Due to the input constraint, it is difficult to find a feasible solution of (6). Therefore, let us consider that the system is stable, which leads to the following LMI conditions

$$\begin{aligned} & \min_u \gamma \text{ s.t.} \\ & \begin{bmatrix} \gamma - \tau\epsilon^2 + \tau x_c^T P x_c & (*) & (*) \\ (Y_r - M u) & I & (*) \\ N^T (Y_r - M u) - \tau P x_c & 0 & -N^T N + \tau P \end{bmatrix} \geq 0, \\ & \tau > 0, |u_k| < \delta, k \in [0 \ F]. \end{aligned}$$

It is very hard job to find analytic solution of the input constrained system or it may be impossible. But numerical solution using LMI method is still available. This means that we can still guarantee performance when input condition is limited, and it may be applied to design the reference trajectory under uncertain initial conditions.

## IV. SIMULATION RESULTS

### (CASE I) The case of the unconstrained input

The case of the unconstrained input : The simulation parameters are as followings:

- Simulation condition.

$$\begin{aligned} A_k &= \begin{bmatrix} 1.5 & 0.1 \sin(k) \\ 0.2 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \epsilon = 0.2, \\ y_r(n) &= \frac{1}{2} e^{\frac{n}{50}} \sin \frac{6n}{50}. \end{aligned}$$

- Output feedback control is applied.
- $P$  is the identity matrix  $I$ .

The experiment is repeated by 20 times. In each trial, initial conditions are randomly chosen within a given boundary  $\epsilon$ . Fig.1 shows that output follows the reference well under the uncertain initial conditions. The upper bound of the tracking performance of this system is  $\gamma_{min} = 0.04$ .

### (CASE II) The case of the constrained input

The simulation parameters are as followings:

- Simulation condition.

$$\begin{aligned} A_k &= \begin{bmatrix} 0.7 & 0.1 \sin(k) \\ 0.2 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \epsilon = 0.2, |u| < 0.4 \\ y_r(n) &= \frac{1}{2} e^{\frac{n}{50}} \sin \frac{6n}{50}. \end{aligned}$$

- Output feedback control is applied.
- $P$  is the identity matrix  $I$ .

In this case,  $\gamma_{min} = 1.08$  which is much larger than (CASE I). But we can still guarantee the performance bound of the system with uncertain initial condition. The performance of (CASE II) is depicted in the Fig. 2.

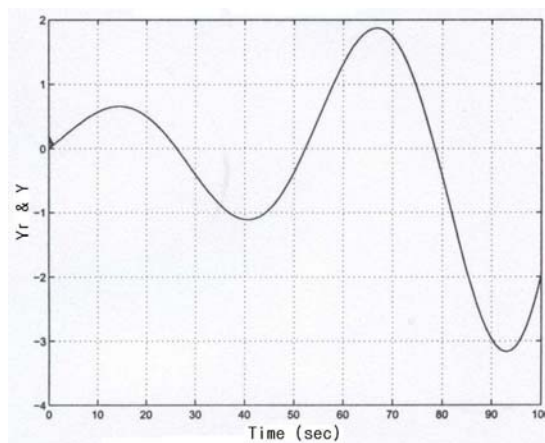


Fig. 1 The trajectories of the reference and the output

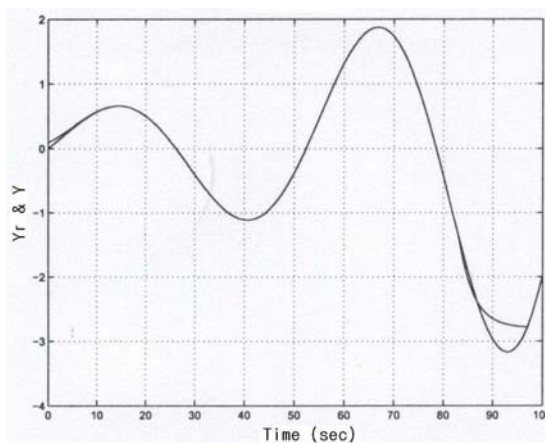


Fig. 2 A reference trajectory and its tracking curves of saturated input

## V. CONCLUSION

In this paper, we handled the finite time horizon system with uncertain initial condition by a repetitive manner. As a main results, we first found the performance bound of the uncertain initial conditions and then calculated it by the proposed LMI conditions in three cases: the state feedback, the output feedback and the constrained input. The simulation results demonstrated the performance of the proposed controller. However, there were some limitations to apply input saturation. Therefore, the future work is to overcome these limitations for the case of input saturation.

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