Fuzzy T-Neighborhood Groups Acting on Sets

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Abstract—In this paper, The T-G-action topology on a set acted on by a fuzzy T-neighborhood (T-neighborhood, for short) group is defined as a final T-neighborhood topology with respect to a set of maps. We mainly prove that this topology is a T-regular T-neighborhood topology.

Keywords—Fuzzy set, Fuzzy topology, Triangular norm, Separation axioms.

I. INTRODUCTION

A T-neighborhood topology on a set can be defined by several methods e.g., via closures, interiors, filters, etc. Sometimes a T-neighborhood topology constructed out of a given T-neighborhood topology may be useful. In the classical theory of topological groups, when a topological group acts on a set, it confers a topology on X, called the G-action topology on X. In this paper, we develop a fuzzy extension of that notion, in the case of classical theory of topological groups, when a topological group acts on a non-empty set X, if to each G acts on a non-empty set X, we define for each x ∈ X there exists v ∈ β(e) such that v · e ≤ μ₁⁻¹, i.e., r is continuous at e.

Definition 2.1 [8] A topological group G acts on a non-empty set X, if to each g ∈ G and each x ∈ X there exists a unique element gx such that gx(gx) = (g · g)x. When G acts on a set X, two families of functions can be defined as follows:

To each g ∈ G, we define

\[ g : X \rightarrow X, \]

\[ g(x) = gx. \]

To each x ∈ X, we define

\[ X : G \rightarrow X, \]

\[ X(g) = gx. \]

II. DEFINITION AND PRELIMINARIES

Theorem 2.1 [7] Let (G, .) be a group and β a T-neighborhood base on G. Then (G, .(β)) is a T-neighborhood group if and only if the following are fulfilled:

(a) For every a ∈ G we have

\[ β(a) = \{ \zeta(a) \mu / \mu \in β(e) \} \]

(res. \[ β(a) = \{ \zeta_0(a) \mu / \mu \in β(e) \} \] and \[ β(a) = \{ \zeta(e) \mu / \mu \in β(e) \} \] is a T-neighborhood base at a.

(b) For all μ ∈ β(e) and for all e ∈ I₀ there exists \( v \in β(e) \) such that \( v \cdot e \leq μ⁻¹ \), i.e., r is continuous at e.

(c) For all μ ∈ β(e) and for all e ∈ I₀ there exists \( v \in β(e) \) such that \( v \cdot e \leq μ⁻¹ \), i.e., m is continuous at \( e \).

(d) For all μ ∈ β(e), for all e ∈ I₀ and for all x ∈ G there exist \( v \in β(e) \) such that \( I_v \cdot I_{v^{-1}} \cdot e \leq μ \), i.e., int_x is continuous at e.

Where \( ζ : G \rightarrow G : z \mapsto xz \) (resp. \( R_0 : G \rightarrow G : z \mapsto xz \) is the left (resp. right) translation).

Theorem 2.2 [7] Let (G, .) be a group and \( τ \) a family of fuzzy subsets of G such that the following hold:

(a) \( τ \) is a filter basis, such that \( μ(e) = 1 \) for all \( μ \in τ \).

(b) For all \( μ \in τ \) and for all \( e \in I₀ \) there exists \( v \in τ \) such that \( v \cdot e \leq μ⁻¹ \).

(c) For all \( μ \in τ \) and for all \( e \in I₀ \) there exists \( v \in τ \) such that \( v \cdot e \leq μ \).

(d) For all \( μ \in τ \), for all \( e \in I₀ \) and for all \( x \in G \) there exists \( v \in τ \) such that \( I_v \cdot I_{v^{-1}} \cdot e \leq μ \).

Then there exists a unique T-neighborhood system \( β \) such that \( τ \) is a T-neighborhood basis for the T-neighborhood system at \( e \). Then \( β(e) \) is compatible with the group structure. This T-neighborhood system is given by

\[ β = \{ I_x μ / μ \in τ \}. \]

III. T-NEIGHBORHOOD TOPOLOGIES INDUCED BY T-NEIGHBORHOOD GROUP ACTIONS ON SET

Definition 3.1. Let \( (G, .) \) be a group acting on a set \( X \), then for all \( x ∈ X \), we define

\[ Γ(Γ(μ)(x)) = \sup \{ (g)(x)^T μ(x) : (g, x) \in G \times X and gx = y \}. \]

We will use two important theorems which introduced in [7]. The first gives necessary and sufficient conditions for a group structure and T-neighborhood system to be compatible, and the second gives necessary and sufficient conditions for a filter to be the T-neighborhood filter of \( e \) in a T-neighborhood group.
Proposition 3.1. Let \( (G, .) \) be a group acting on a set \( X \) and \( \Psi, \Gamma, \in I^G, \mu \in I^X \). Then

(a) \( \Psi (\Gamma \mu) \leq (\Psi \Gamma) \mu \) \( \quad \) In particular \( \Psi (\Gamma \mu)(y) \leq (\Psi \Gamma)(\mu)(y) \)

(b) \( \Gamma_{\mu \lambda} = \vee_{x \in X} \Gamma_{\mu} \Gamma_{\lambda} \)

(c) \( \Gamma_{\mu}(y) = \sup \{ \Gamma_{\mu}(g) : g \in G \text{ and } \Gamma_{\mu}(y) \leq y \} \)

(d) \( \Gamma_{\mu}(y) = \sup \{ \mu(x) : x \in X \text{ and } \mu(x) = y \} \)

Proof: (b)-(e) follow immediately from Definition 3.1.

(a) For any \( y \in X \):
\[
\Psi (\Gamma \mu)(y) = \sup \{ \Psi (\Gamma \mu)(x) : (g, x) \in G \times X, \mu(x) = y \} \\
= \sup \{ \Psi (\mu)(y) T \Gamma(h) T \mu(z) : x = y \} \\
= \sup \{ \Psi (\mu)(y) T \Gamma(h) T \mu(z) : x = y \} \\
= \sup \{ \Psi (\mu)(y) T \Gamma(h) T \mu(z) : x = y \} \\
\leq \Psi (\Gamma)(\mu)(y)
\]

Hence \( \Psi(\Gamma \mu)(y) = (\Psi \Gamma)(\mu)(y) \). If both \( \Gamma \), \( \mu \) are crisp, then \( \Gamma \mu \) is also crisp and is given by \( \mu \Gamma \mu(x) : g \in G \text{ and } x \in x \).

Note that \( \Gamma_{\mu \lambda} = \vee_{x \in X} \Gamma_{\mu \lambda} \) and \( \Gamma_{\mu}(y) \) is 0 if \( y \) \( \notin \) orbit of \( x \).

Theorem 3.1. Let \( G \) be a \( T \)-neighborhood group acting on a set \( X \), and let \( \Re \) be a fundamental system of \( G \) at \( e \). For each \( x \in X \), let \( \beta \in \{ \Gamma \in G : \Gamma e = e \} \). Then \( \beta_{\Gamma_{\mu}(x)} \) is a \( T \)-neighborhood basis on \( X \). The resulting \( T \)-neighborhood space is denoted by \( \tau \). Its fuzzy closure operator \( \cdot : I^X \rightarrow I^X \) is given by: For all \( \eta \in I^X \), \( x \in X \):

\[
\eta(x) = \inf \sup \Gamma(g) \eta(gx)
\]

Proof. First, we verify that \( \beta_{\Gamma_{\mu}(x)} \) is a \( T \)-neighborhood basis in \( X \). Let \( x \in X \), \( \Psi \in \Re \), \( \mu = \Gamma_{\mu}(y) \in \beta_{\mu} \). (i) \( \mu(x) = \Gamma_{\mu}(x) = \sup \{ \Gamma_{\mu}(g) : g \in G \text{ and } \mu(x) = x \} \)

(ii) There exists \( A \in \Re : \Psi \cap A \geq A \). Hence \( \mu \wedge \lambda = \mu \wedge \lambda \wedge \Psi \geq A \).

(iii) T-kernel condition:
Recall that \( \{ \Re \} \) is a \( T \)-neighborhood basis of the \( T \)-neighborhood group \( G \) Theorem 2.2. Let, as before, \( \mu = \Gamma_{\mu}(y) \in \beta_{\mu} \). By the T-kernel condition for \( \Gamma_{\mu}(y) \leq \Gamma \), for all \( x \in I_0 \) there exists a family \( \{ \mu \in \Re \} \) such that for each \( k \in G \), \( \Gamma_{\mu}(y) \leq \Gamma \).

We take \( \eta = \Gamma_{\mu}(y) \). For each \( y \in X \), if \( y \notin \) orbit of \( x \), take for \( \eta \) any element of \( \beta_{\eta} = \Re_{\eta} \).

If \( y \) \( \in \) orbit of \( x \), choose some \( h \in G \) such that \( y = hx \), and \( \delta = \Gamma(h) \geq \sup \{ \Gamma_{\mu}(y) : k = y \} \)

where \( \delta \in I_0 \) is a real number that satisfies \( \delta \in I_0 \) for all \( h, c \in I_0 \). Such \( \delta \) exists by the uniform continuity of \( T \). Take \( \eta = \Gamma_{\mu}(y) \in \beta_{\mu} \). Then, if \( y \notin \) orbit of \( x \), we find for all \( z \in X \)

\[
2c + \mu(z) \geq \sup \{ \Gamma_{\mu}(y) \cap T \} = \nu_{\mu}(z)
\]
because then \( \nu_{\mu}(y) = (\Gamma_{\mu}(y) \cap T) = 0 \). And when \( y \) \( \in \) orbit of \( x \), we find for all \( z \in X \):

\[
2c + \mu(z) = 2c + \mu(z) \geq \sup \{ \Gamma_{\mu}(y) \cap T \} = \nu_{\mu}(z)
\]

Thus, the kernel condition holds for \( \mu \in \beta_{\mu} \) in both cases of \( y \). Finally, for all \( \eta \in I^X \)

\[
\eta(x) = \inf \sup \Gamma(g) \eta(gx)
\]

Because \( y \notin \) orbit of \( x \), then \( \eta_{\mu}(y) = 0 \).

\[
\eta(x) = \sup \eta(y) \cap T \sup \{ \Gamma(g) : g \in G \text{ and } gx = y \}.
\]

Thus, fuzzy rendering (2).

Proposition 3.2. Let \( \Gamma \in I^G \), \( \varnothing \subseteq I^G \), \( g \in G \), \( x \in X \)

then \( (\Gamma_{\mu}(y) \cap T) = (\Gamma_{\mu}(y) \cap T) \in I^X \). and hence

\[
(\varnothing, 1) \subseteq \varnothing \subseteq I^X.
\]

Proof:

\[
(\Gamma(\mu) \cap T) = \sup \{ \Gamma(k, y) : k \in G \text{ and } ky = y \} = \sup \{ \Gamma(k, y) : k \in G \text{ and } ky = y \} = \sup \{ \Gamma(k, y) : k \in G \text{ and } ky = y \}
\]

This completes the proof.

Proposition 3.3. For each filterbasis \( F \) in \( I^G \) and for \( x \in X \):

\[
\{ \Gamma_{\mu} : \Gamma \in F^+ \} \subseteq \{ \Psi_{\mu} : \Psi \in F^{-} \} \subseteq I^X
\]

Proof: Let \( \Gamma \in F^+ \). Then for all \( \varepsilon > 0 \), there exists \( \Gamma_\varepsilon \in \Gamma \) such that \( \Gamma + \varepsilon \geq \Gamma_\varepsilon \). Then for all \( y \in X \), we have

\[
(\Gamma_{\mu}(y) \cap T) = \sup \{ \Gamma(k, y) : k \in G \text{ and } ky = y \} = \sup \{ \Gamma(k, y) : k \in G \text{ and } ky = y \}
\]

Rendering (2).

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\[ \geq \sup \{ \Gamma(g): g x = y \} = \{ \Gamma(x), y \} \]

Thus, \( e + \Gamma(x) \geq \Gamma(x) = \{ \forall \in \forall \in \forall \in \forall \} \). Hence \( \Gamma(x) = \{ \forall \in \forall \in \forall \in \forall \} \). This proves (5).

**Proposition 3.4.** The fuzzy closure operator on \( X \) defined in (2) does not depend on the particular choice of a fundamental system \( \mathcal{R} \) of \( e \).

**Proof:** All fundamental systems \( \mathcal{R} \) of \( G \) at \( e \) have the same saturation \( \mathcal{R} \). Also, for each \( x \in X \)
\[ \beta_x = \{ \Gamma(x): \Gamma \in \mathcal{R} \} \]
\[ \subseteq \{ \Gamma(x): \Gamma \in \mathcal{R} \} \]
\[ \subseteq \{ \Gamma(x): \Gamma \in \mathcal{R} \} \]

As \( \{ \beta_x \} \), \( \{ \beta_x \} \) induce the same fuzzy closure operator on \( X \), then the fuzzy closure operator defined in (2) is also given by
\[ \eta(x) = \inf \sup \{ \Gamma(g) \mid \eta \in \mathcal{R} \} \eta(gx) \quad (6) \]

Which is independent of the particular choice of a fundamental system \( \mathcal{R} \) of \( e \).

The following definition is well phrased by virtue of Theorem 3.1, and Proposition 3.4;

**Definition 3.2.** Let \( G \) be a T-neighborhood group acting on a set \( X \). A T-G-action-topology on \( X \) denoted by \( \mathcal{T} \) is introduced through its closure operator \( \cdot \), defined in (2).

**Proposition 3.5.** Let \( \mathcal{R} \) be a fundamental system at \( e \) of \( G, \mu \in \mathcal{R} \). Then
\[ \mathcal{R} \cdot \mathcal{R}, \mathcal{R} : \mathcal{R} \subseteq \mathcal{R} \]

**Proof:** From condition (d) in Theorem 2.1, for all \( \mu > 0 \) there exists \( v \in \mathcal{R} \) such that
\[ v \cdot v \leq \mathcal{R} \cdot \mathcal{R} \]

This proves that \( \mathcal{R} \cdot \mathcal{R}, \mathcal{R} : \mathcal{R} \subseteq \mathcal{R} \).

**Notion:** In T-G-action topology
(1) We denote the T-neighborhood system at \( x \in X \) by \( \mathcal{N}(x) \).

(2) Let \( \mathcal{R} \) be the T-neighborhood system of \( G \) at \( e, x \in X \). We denote \( \mathcal{R} \mathcal{L}(x) \) by \( \mathcal{C}(x) \). Recall that \( \mathcal{C} = \mathcal{N}, \text{i.e} \mathcal{C}(x) \) is a T-neighborhood basis at \( x \) for this space.

**Definition 3.3.** Let \( (X, \cdot, (\beta)) \) be a T-neighborhood space, \( M \) be a non-empty set in \( X \). Then \( \mu \in \mathcal{F}^X \) is said to be a T-neighborhood of \( M \) if \( \mu \) is a T-neighborhood of all points \( x \) in \( M \). It follows that the set of all T-neighborhoods of \( M \) (called the T-neighborhood system of \( M \)) is the set \( \mathcal{N}(x) \).

**Proposition 3.6.** Let \( \Gamma \in \mathcal{F}, g \in G, z \in X \) then
\[ \Gamma(x)(y) = \{ \forall \in \forall \in \forall \in \forall \} \]

**Proof:**
\[ \mathcal{R} \cdot \mathcal{R}(y) = \{ \forall \in \forall \in \forall \in \forall \} \]
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Next, let $\Gamma$ be a T-neighborhood of $e$. Then $\Gamma$ contains a symmetric T-neighborhood $\Delta$ of $e$. For any $x \in X$

$$(1_M)(x) = \inf_{x \in X} \sup_{y \in X} 1_M(y) T \Delta I_1(y)$$

$$\leq \sup_{y \in X} \Delta I_1(y) = \sup_{y \in M} \Delta I_1(y) \text{ by Proposition 3.7}$$

$$= \sup_{y \in M} \Delta I_1(y) = (\Delta I_1)(y) \leq (\Gamma I_1)(y).$$

This proves (8).

**Proposition 3.9.** Let $\mathcal{R}$ be a fundamental system of T-neighborhoods of $e$. For any subset $M$ of $X$

$$\{1_M\} = \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$

**Proof:** From Proposition 3.8, $(1_M) \leq \Gamma I_M$ for every $\Gamma \in \mathcal{R}$. Then

$$(1_M) \leq \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

Next we prove that

$$\bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M = \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

Since $\mathcal{R} \subset \mathcal{R}^-$, then

$$\bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M \geq \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

Also, let $\Gamma \in \mathcal{R}^-$, for all $\varepsilon > 0$ there exists $\Gamma \in \mathcal{R}$ such that $\varepsilon + \Gamma \geq \Gamma$. Then

$$\varepsilon + \Gamma I_M \geq (\varepsilon + \Gamma) I_M \geq \Gamma I_M \geq \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

Since this holds for all $\varepsilon > 0$, then

$$\bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M \geq \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

This inequality holds for all $\Gamma \in \mathcal{R}^-$. Consequently,

$$\bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M \geq \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

Hence, equality holds.

It is clear that if $O$ is the set of symmetric elements in $\mathcal{R}^-$ then

$$\bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M = \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M \leq \bigwedge_{\Gamma \in \mathcal{R}} \Gamma I_M.$$ 

Conversely, let $O$ be the set of symmetric elements in $\mathcal{R}^-$. Then $O$ is a fundamental system at $e:

$$(\bigwedge_{\Delta \in O} \Delta I_1)(x) = \inf_{\Delta \in O} (\Delta I_1)(x) = \inf_{\Delta \in O} \sup_{x \in X} 1_M(y) T \Delta I_1(y) \text{ by Proposition 3.7} = (1_M)(x)$$

because the set $\{\Delta I_1 : \Delta \in O\}$ is a T-neighborhood basis at $x$.

**Theorem 3.3.** A T-G-action topology on $X$ is a T-regular T-neighborhood topology.

**Proof:** Let $M \subset X$ and $x \in X$. We establish condition (N²-T-regularity) of Theorem 3.2 in [6], which is equivalent to the T-regularity of $X$. For all $M \subset X$, $x \in X$ such that

$$\inf_{\Delta \in O} \sup_{x \in M} 1_M(y) \geq \sup_{x \in M} \sup_{y \in \Delta} (\Gamma I_1)(y): y \in \Delta \geq (\Gamma I_1)(y) \geq (\Delta I_1)(y) \geq \sup_{y \in \Delta} \inf_{x \in M} 1_M(y) T \Delta I_1(y) \geq \Delta I_1(y) \geq \Delta I_1(y).$$

So, (call $y \in kx$)

$$\inf_{\Delta \in O} \sup_{x \in M} 1_M(y) \geq \sup_{x \in M} \sup_{y \in \Delta} (\Gamma I_1)(y): y \in \Delta \geq (\Gamma I_1)(y) \geq (\Delta I_1)(y) \geq \sup_{y \in \Delta} \inf_{x \in M} 1_M(y) T \Delta I_1(y).$$

But by Theorem 2.2 in [7], for every $\Delta \in O, \varepsilon \geq 0$ there exists $\Delta \in \mathcal{R}$ such that $\Delta \leq \Delta + \varepsilon$. Hence,

$$\inf_{\Delta \in O, \varepsilon > 0} \sup_{x \in M} 1_M(y) \geq \sup_{x \in M} \sup_{y \in \Delta} (\Gamma I_1)(y): y \in \Delta \geq (\Gamma I_1)(y) \geq (\Delta I_1)(y) \geq \sup_{y \in \Delta} \inf_{x \in M} 1_M(y) T \Delta I_1(y).$$

The opposite inequality is always valid.

**Theorem 3.4.** A T-G-action topology $\tau_{X}^{T-G}$ coincides with the final T-neighborhood topology $\tau_f$ on $X$ defined by the set of functions

$$\{ \hat{x} : G \to X : x \in X, \hat{x}(g) = gx \}$$

**Proof:** For any $x \in X$, the function $\hat{x} : G \to (X, \tau_{X}^{T-G})$ is continuous, because for all $g \in G$ and for each neighborhood $\Gamma I_1(x)$ in the fundamental system $\mathcal{R} I_1(x)$ of $\hat{x}(g) = gx$, where $\Gamma \in \mathcal{R}$, we have

$$\hat{x} : (\Gamma I_1)(y) = \sup_{h \in G} (\hat{x} I_1)(h) = \sup_{h \in G} (\hat{x} I_1)(h) = \sup_{h \in G} (\hat{x} I_1)(h) = (\Gamma I_1)(y).$$

then $\hat{x} : (\Gamma I_1)(y) = (\hat{x} I_1)(y) = \Gamma I_1(x)$ and $\Gamma I_1$ is a T-neighborhood of $g$ by Theorem 2.3 in [7]. Therefore
$\tau^T \subseteq \tau_f$ since $\tau_f$ is the finest T-neighborhood topology making all $x$ continuous.

Next, let $x \in X$, $\mu$ a T-neighborhood of $x$ in $\tau_f$. Then $\hat{x}^{-1}(\mu)$ is a T-neighborhood of $e$ in $G$; i.e. $(\hat{x}^{-1}(\mu))_{1x}$ is a T-neighborhood of $x$ in $\tau^T_{X}$. But $(\hat{x}^{-1}(\mu))_{1x} = \mu \wedge \text{range} \hat{1} \leq \mu$.

This proves that $\mu$ is a T-neighborhood of $x$ in $\tau^T_{X}$. Then $\tau_f \subseteq \tau^T_{X}$. Hence, equality holds.

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