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A Schur Method for Solving Projected Continuous-Time Sylvester Equations

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Abstract—In this paper, we propose a direct method based on the real Schur factorization for solving the projected Sylvester equation with relatively small size. The algebraic formula of the solution of the projected continuous-time Sylvester equation is presented. The computational cost of the direct method is estimated. Numerical experiments show that this direct method has high accuracy.

Keywords—Projected Sylvester equation, Schur factorization, Spectral projection, Direct method.

I. Introduction

In this paper we study the numerical solution of the projected continuous-time Sylvester equation of the form

$$\begin{cases} AX + XB + P_r C = 0, \\ X = P_r X, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times m}$ are given matrices, and $X \in \mathbb{R}^{n\times m}$ is the solution matrix sought. Here, A is a singular matrix and P_r is the spectral projector onto the right invariant subspace corresponding to the non-zero eigenvalues of the matrix A.

It is well known that the projected generalized continuous-time Lyapunov equations play an important role in stability analysis and control design problems for descriptor systems including the characterization of controllability and observability properties, balanced truncation model order reduction, determining the minimal and balanced realizations as well as computing H_2 and Hankel norms; see [1], [14], [22], [24], [29], [32] and the references therein. Recently, a new iterative method for solving the projected generalized continuous-time Lyapunov equations has been proposed in [3]. At each iterative step of this method, one needs to solve (1), see [3] for details.

During the past four decades, a number of numerical solution methods have been proposed for standard Lyapunov or Sylvester equations with $P_r = I$. Three classical direct methods are the Bartels-Stewart method [4], [28], the Hessenberg-Schur method [9], and the Hammarling method [13], [19]. These methods need to compute the real Schur forms of the coefficient matrices by means of the QR algorithm [10], and require $\mathcal{O}(n^3)$ operations and $\mathcal{O}(n^2)$ memory. Besides direct methods, we mention, among several iterative methods, the Smith method [27], the alternating direction implicit iteration (ADI) method [33], [21], the Smith(l) method [23], the modified low-rank Smith method [12], the Cholesky factor-alternating direction implicit (CF-ADI) method [20], and the matrix sign function method [5], [6], [7]. The matrix sign function method are appropriate for problems with the coefficient matrices dense and stable. There are also several other approaches to solve large-scale Lyapunov and Sylvester equations using Krylov subspace methods, see, for example, [2], [11], [15], [16], [17], [18], [25]. The ADI methods and Krylov subspace based methods are much suited for large-scale Lyapunov and Sylvester equations with sparse coefficient matrices.

In this paper, firstly, we make a detailed deduction for the algebraic expression of the solution of the projected continuous-time Sylvester equation (1) by exploiting the real Schur decomposition. Then, we propose a direct method for solving the projected equation. Since the real Schur decomposition is needed in the proposed method, it makes the direct method only applicable to problems with medium or relatively small size. We remark that to our knowledge, the method proposed in this paper is the unique direct solver for this projected equation.

Throughout this paper, we adopt the following notations. I denotes the identity matrix, 0 denotes the zero vector or zero matrix. The dimensions of these vectors and matrices are conformed with dimensions used in the context. The space of $m \times n$ real matrices are denoted by $\mathbb{R}^{m\times n}$. The Frobenius matrix norm is denoted by $\|\cdot\|_F$. The superscript "$\cdot^T$" denotes the transposition of a vector or a matrix and A^{-1} is the inverse of nonsingular matrix A. The spectrum of A is denoted by $\sigma(A)$. We shall also adopt MATLAB-like convention to access the entries of vectors and matrices. The set of integers from i to j inclusive is denoted by $i : j$. X's submatrices $X(k : l, i : j)$, $X(k : l, :)$ and $X(:, i : j)$ consist of intersections of row k to row l and column i to column j, row k to row l, and column i to column j , respectively.

The remainder of the paper is organized as follows. In Section 2, we first present the algebraic formula of the solution of the projected continuous-time Sylvester equation. Then, a direct method based on the real Schur factorization is proposed. The details of implementation are also included. Section 3 is devoted to some numerical tests. Some concluding remarks are given in the last section.

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II. The Schur method

Let the real Schur decompositions [10] of A and B be

$$A = U\begin{bmatrix} J_A & G_A \\ 0 & N_A \end{bmatrix}U^T, \quad B = V\begin{bmatrix} J_B & G_B \\ 0 & N_B \end{bmatrix}V^T, \quad (2)$$

where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are orthogonal matrices. $J_A \in \mathbb{R}^{n_1\times n_1}$ and $J_B \in \mathbb{R}^{m_1\times m_1}$ are upper quasi-triangular matrices and correspond to the non-zero eigenvalues of A and B, respectively, while N_A and N_B being upper triangular correspond to the zero eigenvalues of A and B, respectively. We shall always assume that $\sigma(J_A) \cap \sigma(-J_B) = \emptyset$.

To utilize the real Schur decomposition (2) to solve the projected Sylvester equation (1), it is necessary to block diagonalize the matrix A. It can be implemented by solving the following Sylvester equation

$$J_A Y - Y N_A - G_A = 0. \quad (3)$$

Note that J_A corresponds to the non-zero eigenvalues of A while N_A corresponds to the zero eigenvalues of A, the Sylvester equation (3) always has a unique solution Y for every G_A, see [8] for more theoretical results.

It follows from (2) that A can be written in factored form as follows:

$$\begin{aligned} A &= U\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}\begin{bmatrix} J_A & 0 \\ 0 & N_A \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}U^T \\ &= T\begin{bmatrix} J_A & 0 \\ 0 & N_A \end{bmatrix}T^{-1}, \end{aligned}$$

where the nonsingular matrix T is defined by

$$T = U\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}.$$

Then the spectral projector onto the right invariant subspace corresponding to the non-zero eigenvalues of A can be expressed by

$$P_r = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}T^{-1} = U\begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix}U^T. \quad (4)$$

By making use of the decompositions (2) and (4), we can reformulate $AX + XB + P_rC$ as

$$\begin{aligned} AX + XB + P_rC &= U\begin{bmatrix} J_A & G_A \\ 0 & N_A \end{bmatrix}U^T X \\ &\quad + XV\begin{bmatrix} J_B & G_B \\ 0 & N_B \end{bmatrix}V^T \\ &\quad + U\begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix}U^T C. \quad (5) \end{aligned}$$

Define $\widehat{X} = U^T XV$ and $\widehat{C} = U^T CV$, and partition $\widehat{X}$ and $\widehat{C}$ appropriately as

$$\widehat{X} = \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} \widehat{C}_{11} & \widehat{C}_{12} \\ \widehat{C}_{21} & \widehat{C}_{22} \end{bmatrix}.$$

By (5), the first equation of (1) is equivalent to

$$\begin{aligned} &\begin{bmatrix} J_A & G_A \\ 0 & N_A \end{bmatrix}\begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix} \\ &+ \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}\begin{bmatrix} J_B & G_B \\ 0 & N_B \end{bmatrix} \\ &+ \begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \widehat{C}_{11} & \widehat{C}_{12} \\ \widehat{C}_{21} & \widehat{C}_{22} \end{bmatrix} = 0. \end{aligned}$$

Then a simple calculation gives that

$$\begin{aligned} &\begin{bmatrix} J_A\widehat{X}_{11} + G_A\widehat{X}_{21} & J_A\widehat{X}_{12} + G_A\widehat{X}_{22} \\ N_A\widehat{X}_{21} & N_A\widehat{X}_{22} \end{bmatrix} \\ &+ \begin{bmatrix} \widehat{X}_{11}J_B & \widehat{X}_{11}G_B + \widehat{X}_{12}N_B \\ \widehat{X}_{21}J_B & \widehat{X}_{21}G_B + \widehat{X}_{22}N_B \end{bmatrix} \\ &+ \begin{bmatrix} \widehat{C}_{11} + Y\widehat{C}_{21} & \widehat{C}_{12} + Y\widehat{C}_{22} \\ 0 & 0 \end{bmatrix} = 0, \end{aligned}$$

which is equivalent to the following four equations:

$$J_A\widehat{X}_{11} + G_A\widehat{X}_{21} + \widehat{X}_{11}J_B + \widehat{C}_{11} + Y\widehat{C}_{21} = 0, \quad (6)$$

$$J_A\widehat{X}_{12} + G_A\widehat{X}_{22} + \widehat{X}_{11}G_B + \widehat{X}_{12}N_B + \widehat{C}_{12} + Y\widehat{C}_{22} = 0, \quad (7)$$

$$N_A\widehat{X}_{21} + \widehat{X}_{21}J_B = 0, \quad (8)$$

$$N_A\widehat{X}_{22} + \widehat{X}_{21}G_B + \widehat{X}_{22}N_B = 0. \quad (9)$$

Equation (8) is a standard continuous-time Sylvester equation. Since the spectrum of N_A and $-J_B$ are disjoint, (8) has a unique solution $\widehat{X}_{21} = 0$, see [8].

By $\widehat{X}_{21} = 0$, (6), (7) and (9) are simplified as

$$J_A\widehat{X}_{11} + \widehat{X}_{11}J_B + \widehat{C}_{11} + Y\widehat{C}_{21} = 0, \quad (10)$$

$$J_A\widehat{X}_{12} + G_A\widehat{X}_{22} + \widehat{X}_{11}G_B + \widehat{X}_{12}N_B + \widehat{C}_{12} + Y\widehat{C}_{22} = 0, \quad (11)$$

$$N_A\widehat{X}_{22} + \widehat{X}_{22}N_B = 0. \quad (12)$$

Equation (10) is also a standard continuous-time Sylvester equation and, has a unique solution $\widehat{X}_{11}$ since $\sigma(J_A) \cap \sigma(-J_B) = \emptyset$. Clearly, Equation (12) have many solutions including $\widehat{X}_{22} = 0$.

By utilizing the expression of P_r defined by (4), the second equation of (1) can be rewritten as

$$X = P_rX = U\begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix}U^T X. \quad (13)$$

Multiply (13) from the left by U^T and multiply it from the right by V to obtain

$$\begin{aligned} \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix} &= \begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \widehat{X}_{11} + Y\widehat{X}_{21} & \widehat{X}_{12} + Y\widehat{X}_{22} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which shows that

$$\widehat{X}_{21} = 0 \quad \text{and} \quad \widehat{X}_{22} = 0.$$

By $\widehat{X}_{22} = 0$, Equation (11) is simplified as

$$J_A\widehat{X}_{12} + \widehat{X}_{12}N_B + (\widehat{C}_{12} + Y\widehat{C}_{22} + \widehat{X}_{11}G_B) = 0, \quad (14)$$

Equation (14) is also a standard continuous-time Sylvester equation, and has a unique solution $\widehat{X}_{12}$ since $\sigma(J_A) \cap \sigma(-N_B) = \emptyset$.

We may summarize these results in the following theorem on the unique solution of the projected continuous-time Sylvester equation (1).

Theorem 2.1: Assume that $\sigma(J_A) \cap \sigma(-J_B) = \emptyset$. Then the unique solution of the projected continuous-time Sylvester equation (1) can be formulated as

$$X = U\widehat{X}V^T = U \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ 0 & 0 \end{bmatrix} V^T,$$

where $\widehat{X}_{11}$ and $\widehat{X}_{12}$ are the unique solutions of the Sylvester equations (10) and (14), respectively.

The previous discussion also provides an approach for solving the projected continuous-time Sylvester equation (1). The corresponding algorithm is described as follows.

Algorithm 2.1: **The Schur method**
Input: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times m}$.
Output: $X \in \mathbb{R}^{n\times m}$ is the unique solution of (1).

1) Compute the real Schur factorizations (2) of the matrices A and B, respectively.
2) Solve the Sylvester equation (3) to obtain Y.
3) Compute
$$\widehat{C} = U^TCV = \begin{bmatrix} \widehat{C}_{11} & \widehat{C}_{12} \\ \widehat{C}_{21} & \widehat{C}_{22} \end{bmatrix}.$$
4) Solve the Sylvester equation (10) for $\widehat{X}_{11}$.
5) Solve the Sylvester equation (14) for $\widehat{X}_{12}$.
6) Form the solution
$$X = U \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ 0 & 0 \end{bmatrix} V^T.$$

About Algorithm 2.1, some remarks of implementation details are in order:

- At Step 1, we take use of the MATLAB function *"schur.m"* to compute the real Schur factorizations and adopt the MATLAB function *"ordschur.m"* to reorders the Schur factorization. It costs about $10n^3$ and $10m^3$ flops for computing the real Schur factorizations (2) of the matrices A and B, respectively.
- At Step 2, the Sylvester equation (3) can be solved by the direct method, such as the Bartels-Stewart method [4]. This method is based on the Schur decomposition, by which the original equation is transformed into a form that is easy to be solved by a forward substitution. Since the coefficient matrices in (3) are already in the real Schur form. It requires only $2.5(n_1(n-n_1)^2 + (n-n_1)n_1^2) = 2.5n(n-n_1)n_1$ flops to solve (3).
- At Step 3, one need about $2nm(n+m-1)$ flops to compute $\widehat{C}$.
- At Step 4, computing $\widehat{C}_{11}+Y\widehat{C}_{21}$ costs about $2n_1m_1(n-n_1)$ flops and solving the Sylvester equations (10) by the Bartels-Stewart method costs about $2.5(n_1m_1^2 + m_1n_1^2)$ flops.
- At Step 5, it costs about $2n_1(m-m_1)(n+m_1-n_1)$ flops for computing $\widehat{C}_{12} + Y\widehat{C}_{22} + \widehat{X}_{11}G_B$ and it costs about $2.5(n_1(m-m_1)^2 + (m-m_1)n_1^2)$ flops for solving the Sylvester equations (14) by the Bartels-Stewart method.
- At Step 6, we only need to compute $U(1 : n_1, :)[\widehat{X}_{11} \quad \widehat{X}_{12}]V^T$ for forming the solution X. It requires $(2n_1-1)nm + (2m-1)n^2$ flops.
- When $n \approx m$ and $n_1 \approx m_1$, the total cost of Algorithm 2.1 for solving the projected continuous-time Sylvester equation (1) is about $24n^3 + 7n_1n^2 - 5n_1^2(n-n_1)$. Therefore, Algorithm 2.1 is applicable to problems of medium or relatively small size.

III. NUMERICAL EXPERIMENTS

In this section, we present two numerical examples to illustrate the performance of the Schur method for the projected Sylvester equation (1). To our knowledge, the Schur method proposed in this paper is the unique direct solver for this projected equation. So, we do not compare its performance with other methods. We only present the accuracy of the Schur method. The accuracy is depicted by the relative residual

$$\frac{\|AX + XB + P_rC\|_F}{\|P_rC\|_F},$$

where X is the computed solution by the Schur method.

All numerical experiments are performed on a PC with the usual double precision, where the floating point relative accuracy is 2.22×10^{-16}.

Example 1. For the first experiment, we consider the 2D instationary Stokes equation that describes the flow of an incompressible fluid in a domain. The spatial discretization of this equation by the finite difference method on a uniform staggered grid leads to the descriptor system

$$\begin{aligned} E\dot{x}(t) &= Fx(t) + M_1u(t), \\ y(t) &= K_1x(t). \end{aligned} \tag{15}$$

This example for the projected generalized Lyapunov equations was presented by Stykel, see [30], [31], [32] and the references therein. The matrix coefficients in (15) are given by

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Here, F is nonsingular and the matrix A in (1) is given by $A = F^{-1}E$. Obviously, the matrix A is singular. In this example, the state space dimensions of the descriptor system (15) is $n = 442$.

We discretize the 2D instationary Stokes equation with $m = 308$ to generate another descriptor system

$$\begin{aligned} G\dot{x}(t) &= Hx(t) + M_2u(t), \\ y(t) &= K_2x(t). \end{aligned}$$

Analogously, we can get the singular matrix $B = H^{-1}G$. Let C be 442×308 random matrices. Elements of C are chosen from a normal distribution with mean zero, variance one and standard deviation one.

For this example, the relative residual of the solution computed by the Schur method is 6.16×10^{-15}.

Example 2. For the second experiment, we consider a holonomically constrained damped mass-spring system with g masses as in [31]. The ith mass is connected to the $(i+1)$th mass by a spring and a damper and also to the ground by another spring and damper. Moreover, the first mass is connected to the last one by a rigid bar and it can be influenced by a control. The vibration of this system is described by the descriptor system (15) with the matrices

$$D_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & I & 0 \\ K_1 & H_1 & -N_1^T \\ N_1 & 0 & 0 \end{bmatrix},$$

where $M_1 \in \mathbb{R}^{g\times g}$ is the symmetric positive definite mass matrix, $K_1 \in \mathbb{R}^{g\times g}$ is the stiffness matrix, $H_1 \in \mathbb{R}^{g\times g}$ is the damping matrix and N_1 is the matrix of constraints, see [26] for details. Here, F_1 is nonsingular and the matrix A in (1) is given by $A = F_1^{-1} D_1$. It is obvious that A is a singular matrix. In this example, the state space dimensions of the problem is $n = 501$.

We consider another holonomically constrained damped mass-spring system

$$D_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & I & 0 \\ K_2 & H_2 & -N_2^T \\ N_2 & 0 & 0 \end{bmatrix},$$

to get the singular matrix $B = F_2^{-1} D_2$ with the state space dimensions $m = 401$. Let C be 501×401 random matrices. Elements of C are chosen from a normal distribution with mean zero, variance one and standard deviation one.

For this example, the relative residual of the solution computed by the Schur method is 7.58×10^{-15}.

IV. Conclusions and Future Work

In this paper, we give out an algebraic formula of the solution of a projected continuous-time Sylvester equation, and propose a Schur method to compute its solution. Numerical experiments presented in this paper show the Schur method has high accuracy. Future work should include perturbation analysis of the projected continuous-time Sylvester equation, including the condition number and perturbation bounds.

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