

# Variational Iteration Method for Solving Systems of Linear Delay Differential Equations

Sara Barati and Karim Ivaz

**Abstract**—In this paper, using a model transformation approach a system of linear delay differential equations (DDEs) with multiple delays is converted to a non-delayed initial value problem. The variational iteration method (VIM) is then applied to obtain the approximate analytical solutions. Numerical results are given for several examples involving scalar and second order systems. Comparisons with the classical fourth-order Runge-Kutta method (RK4) verify that this method is very effective and convenient.

**Keywords**—Variational iteration method; delay differential equations; multiple delays; Runge-Kutta method.

## I. INTRODUCTION

**T**IME delay systems arise from an inherent time delay in the components of the system or a deliberate introduction of time delay into the system for control purposes. Time delay systems can be represented by delay differential equations (DDEs) which belong to the class of functional differential equations and have been extensively studied [1]-[3]. Recent studies in diverse fields such as biology, economics, control and electromagnetic have shown that DDEs play an important role in explaining many physical phenomena. Indeed, many physical events do not occur instantaneously and can be modeled with delays.

The system of linear DDEs with multiple delays may be expressed as

$$\begin{cases} \dot{x}(t) = A_0x(t) + \sum_{d=1}^m A_d x(t - \tau_d) + p(t) & t \in [0, t_f], \\ x(t) = \phi(t) & t \in [-\tau, 0], \end{cases} \quad (1)$$

where  $x(t) = [x_1(t) \ x_2(t) \ \dots \ x_n(t)]^T \in \mathbb{R}^n$ ,  $\phi(t) = [\phi_1(t) \ \phi_2(t) \ \dots \ \phi_n(t)]^T \in \mathbb{R}^n$ ,  $A_0, A_1, \dots, A_m \in \mathbb{R}^{n \times n}$  and  $p(t) = [p_1(t) \ p_2(t) \ \dots \ p_n(t)]^T \in \mathbb{R}^n$  is a known continuous function representing the external excitation. The delays  $\tau_1, \tau_2, \dots, \tau_m$  are known, positive and constant numbers. The set  $[0, t_f]$  is a time interval and  $\tau = \max\{\tau_1, \tau_2, \dots, \tau_m\}$ .

Delay differential equations are often solved using numerical methods, asymptotic solutions and graphical tools. One of the approximation methods is the well-known Pade approximation which results in a shortened repeating fraction for the approximation of the characteristic equation of the delay [4]-[5]. However, the lack of an analytical solution form remains a major obstacle to the analysis and control of time delay systems. During recent decades, the Lambert W function has been used to develop an approach for the solution of linear time invariant systems of DDEs with a single delay (see [6]-[7] and the references therein).

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As we know, the variational iteration method (VIM) presented by He [8]-[13] is a powerful mathematical tool for finding solutions of linear and nonlinear problems and it can be implemented easily in practice. There are some papers concerning about the use of VIM for a particular class of DDEs [14]-[17]. In this paper, we apply the VIM to solve the problem (1) using a model transformation approach.

This paper is organized as follows. A brief description of the VIM is given in Section 2. In Section 3, first the problem (1) is converted to a system without time-delayed arguments and using the transition matrix method, a non-delayed initial value problem is obtained and then the VIM is applied to solve the attained problem. The accuracy of the method with several examples is demonstrated in Section 4. Section 5 contains a brief summary.

## II. A BRIEF DESCRIPTION OF THE VIM

The main idea in the variational iteration method is to construct an iterative sequence of the functions converging to an exact solution. Consider the following general problem:

$$\mathcal{L}u(t) + \mathcal{N}u(t) = g(t),$$

where  $\mathcal{L}$  is a linear operator,  $\mathcal{N}$  is a nonlinear operator and  $g(t)$  is a known analytical function. In the VIM a correction functional as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)(\mathcal{L}u_n(s) + \mathcal{N}\tilde{u}_n(s) - g(s))ds, \quad (2)$$

is made, where  $\lambda$  is a general Lagrange multiplier [12]-[13] which can be identified optimally via the variational theory. The index  $n$  denotes the  $n$ th iteration and  $\tilde{u}_n(s)$  is considered as the restricted variation, i. e.  $\delta\tilde{u}_n(s) = 0$  [12]-[13]. After determining the Lagrange multiplier  $\lambda$  and selecting an appropriate initial function  $u_0$ , the successive approximations  $u_n$  of the solution  $u$  can be readily obtained.

## III. MODEL TRANSFORMATION AND SOLUTION METHOD

Now we study the model (1) with multiple delays  $\tau_1, \tau_2, \dots, \tau_m$ . We shall use a transformation technique that requires the technical assumption that the interval length  $\Delta t = t_f$  is a rational multiple of all the delays  $\tau_1, \tau_2, \dots, \tau_m$ :

$$\Delta t = q_1\tau_1, \quad \Delta t = q_2\tau_2, \quad \dots, \quad \Delta t = q_m\tau_m,$$

where  $q_1, q_2, \dots, q_m \in \mathbb{Q}$ . With this assumption, we divide the time interval  $[0, t_f]$  into  $l$  equidistant subintervals such that  $l$  is the minimum number of divisions of  $[0, t_f]$  which satisfies all of the following conditions:

$$\frac{l\tau_1}{\Delta t} = z_1 \in \mathbb{N}, \quad \frac{l\tau_2}{\Delta t} = z_2 \in \mathbb{N}, \quad \dots, \quad \frac{l\tau_m}{\Delta t} = z_m \in \mathbb{N}.$$

Let  $T = \frac{\Delta t}{l}$  and  $N = nl$ . For each  $t \in [0, T]$ , we define

$$y_k(t) = x_j(t + (k-1 - (j-1)l)T),$$

$$k = (j-1)l + 1, \dots, jl, \quad j = 1, 2, \dots, n. \quad (3)$$

Then the problem (1) is converted to the following  $N$ -dimensional non-delayed system:

$$\begin{cases} \dot{y}(t) = Cy(t) + f(t) & t \in [0, T], \\ y(0) = y_0 \end{cases} \quad (4)$$

where  $C \in \mathbb{R}^{N \times N}$ ,  $y(t) = [y_1(t) \ y_2(t) \ \dots \ y_N(t)]^T \in \mathbb{R}^N$ ,  $f(t) = [f_1(t) \ f_2(t) \ \dots \ f_N(t)]^T \in \mathbb{R}^N$  and the initial value  $y_0 = [y_{10} \ y_{20} \ \dots \ y_{N0}]^T \in \mathbb{R}^N$  is determined as follows:

$$y_{k0} = \phi_j(0), \quad k = (j-1)l + 1, \quad j = 1, 2, \dots, n,$$

$$y_{k0} = y_{k-1}(T), \quad k = (j-1)l + 2, \dots, jl, \quad j = 1, 2, \dots, n.$$

Let  $y(T) = [r_1 \ r_2 \ \dots \ r_N]^T \in \mathbb{R}^N$  denote the final value. The transition matrix method assumes the existence of two transition matrices  $Y(t) \in \mathbb{R}^{N \times N}$  and  $F(t) \in \mathbb{R}^{N \times N}$  such that

$$y(t) = Y(t)y(T), \quad (5)$$

$$f(t) = F(t)y(T). \quad (6)$$

According to (5) and (6), it is easy to see that the transition matrix  $Y(t)$  must satisfy the following non-delayed initial value problem:

$$\begin{cases} \dot{Y}(t) = CY(t) + F(t) & t \in [0, T], \\ Y(T) = I \end{cases} \quad (7)$$

Now we construct the correction functional (2) for problem (7) as follows:

$$Y_{n+1}(t) = Y_n(t) + \int_T^t \lambda(s)(\dot{Y}_n(s) - CY_n(s) - F(s))ds.$$

By taking variation with respect to independent variable  $Y_n$  and noticing that  $\delta Y_n(T) = 0$  and  $\delta \tilde{Y}_n = 0$ , we get

$$\delta Y_{n+1}(t) = \delta Y_n(t) + \lambda(s)\delta Y_n(s)|_{s=t} - \int_T^t \dot{\lambda}(s)\delta Y_n(s)ds = 0,$$

which imply the following stationary conditions

$$1 + \lambda(s)|_{s=t} = 0, \quad \dot{\lambda}(s) = 0.$$

The general Lagrange multiplier, therefore, can be readily identified as  $\lambda = -1$ . As a result, we obtain the following iteration formula:

$$Y_{n+1}(t) = Y_n(t) - \int_T^t (\dot{Y}_n(s) - CY_n(s) - F(s))ds, \quad (8)$$

with initial approximation  $Y_0(t) = I$ . So we achieve the approximate value for  $y(t)$  as

$$y(t) \simeq Y_n(t)y(T). \quad (9)$$

Incorporating the initial condition of the system (4) into (9), we can determine the unknown constants  $r_1, r_2, \dots, r_N$ . Finally, from (3), the solution of (1) can be easily obtained.

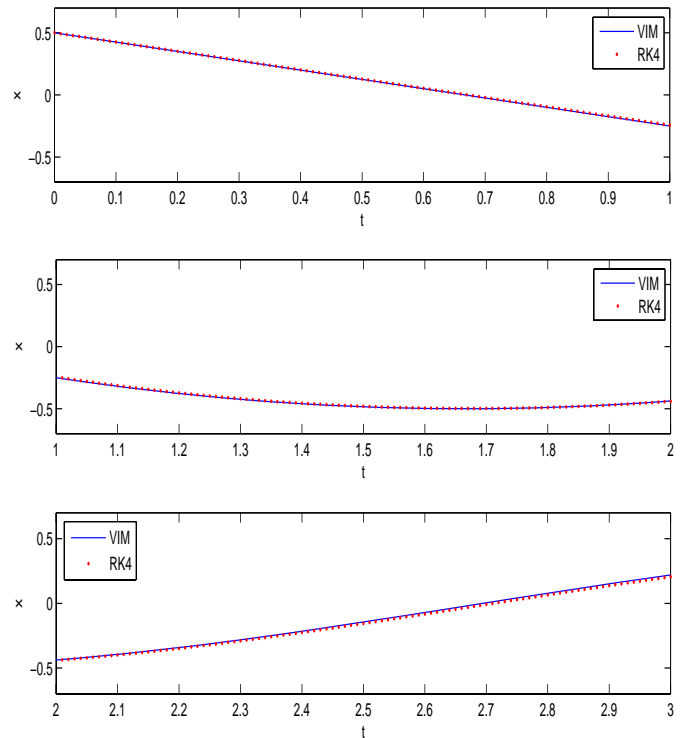


Fig. 1. Comparison of the VIM with the numerical RK4 for Example IV.1.

#### IV. ILLUSTRATIVE EXAMPLES

In this section, we present several examples to show the efficiency of the method described in the previous section. The package MATLAB 9 is used for computation.

**Example IV.1.** Consider a single-delay scalar system as follows [18]:

$$\begin{cases} \dot{x}(t) = -\frac{3}{2}x(t-1), & t \in [0, 3] \\ x(t) = 0.5, & t \in [-1, 0] \end{cases}$$

Following transformations reformulate Example IV.1 into a non-delayed system on the interval  $[0, 1]$ :

$$\begin{cases} y_1(t) = x(t), & t \in [0, 1], \\ y_2(t) = x(t+1), & t \in [0, 1], \\ y_3(t) = x(t+2), & t \in [0, 1]. \end{cases}$$

The resulting non-delayed reformulation of Example IV.1 is as follows:

$$\dot{y}(t) = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \end{bmatrix} y(t) + \begin{bmatrix} -\frac{3}{4} \\ 0 \\ 0 \end{bmatrix},$$

with initial value

$$y(0) = \begin{bmatrix} 0.5 \\ r_1 \\ r_2 \end{bmatrix},$$

where  $y(t) = [y_1(t) \ y_2(t) \ y_3(t)]^T$  and  $y(1) = [r_1 \ r_2 \ r_3]^T$ . Now the non-delayed initial value problem (7) can be solved

on the interval  $[0, 1]$  by the iteration formula (8) with

$$C = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \end{bmatrix},$$

$$F(t) = \begin{bmatrix} -\frac{3}{4r_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad r_1 \neq 0.$$

Figure 1 depicts the approximate analytical solution of Example IV.1 after using 3 iterations by the VIM. Comparison with the numerical RK4 ( $h = 0.01$ ) shows good agreement between two methods.

**Example IV.2.** Consider the following multiple-delays scalar system [19]:

$$\begin{cases} \dot{x}(t) = -x(t) - 5x(t-1) - 2x(t-2), & t \in [0, 3] \\ x(t) = \sin(t), & t \in [-2, 0] \end{cases}$$

Following transformations reformulate Example IV.2 into a non-delayed system on the interval  $[0, 1]$ :

$$\begin{cases} y_1(t) = x(t), & t \in [0, 1], \\ y_2(t) = x(t+1), & t \in [0, 1], \\ y_3(t) = x(t+2), & t \in [0, 1]. \end{cases}$$

The resulting non-delayed reformulation of Example IV.2 is as follows:

$$\dot{y}(t) = \begin{bmatrix} -1 & 0 & 0 \\ -5 & -1 & 0 \\ -2 & -5 & -1 \end{bmatrix} y(t) + f(t),$$

$$f(t) = \begin{bmatrix} -5 \sin(t-1) - 2 \sin(t-2) \\ -2 \sin(t-1) \\ 0 \end{bmatrix}$$

with initial value

$$y(0) = \begin{bmatrix} 0 \\ r_1 \\ r_2 \end{bmatrix},$$

where  $y(t) = [y_1(t) \ y_2(t) \ y_3(t)]^T$  and  $y(1) = [r_1 \ r_2 \ r_3]^T$ . Now the non-delayed initial value problem (7) can be solved on the interval  $[0, 1]$  by the iteration formula (8) with

$$C = \begin{bmatrix} -1 & 0 & 0 \\ -5 & -1 & 0 \\ -2 & -5 & -1 \end{bmatrix},$$

$$F(t) = \begin{bmatrix} \frac{-5 \sin(t-1) - 2 \sin(t-2)}{r_1} & 0 & 0 \\ 0 & \frac{-2 \sin(t-1)}{r_2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where  $r_1, r_2 \neq 0$ .

Figure 2 depicts the approximate analytical solution of Example IV.2 after using 5 iterations by the VIM. Comparison with

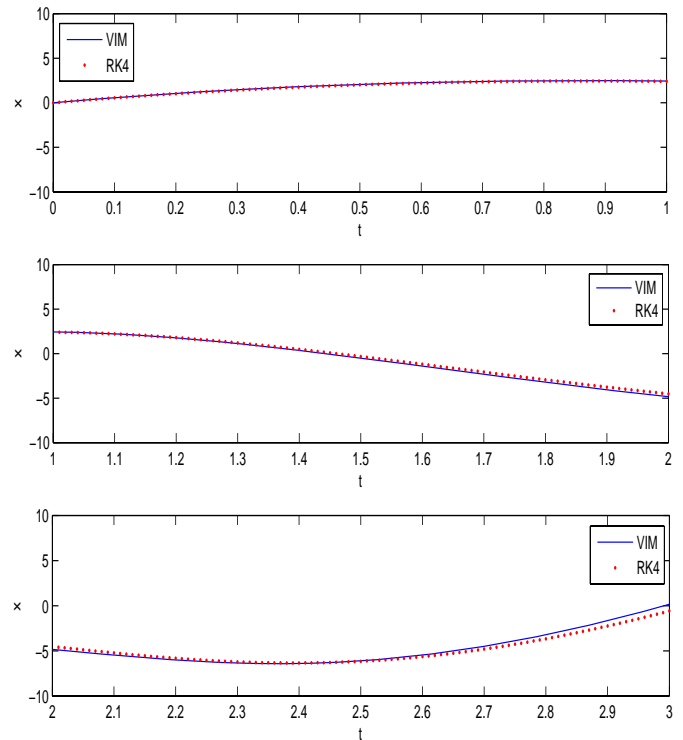


Fig. 2. Comparison of the VIM with the numerical RK4 for Example IV.2.

the numerical RK4 ( $h = 0.01$ ) shows good agreement between two methods.

**Example IV.3.** Consider a single-delay second order system as follows [20]:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.330 \end{bmatrix} x(t-1) + p(t), \\ x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & t \in [-1, 0] \end{cases}$$

where for each  $t \in [0, 2]$ ,

$$p(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

and

$$x(t) = [x_1(t) \ x_2(t)]^T.$$

Following transformations reformulate Example IV.3 into a non-delayed system on the interval  $[0, 1]$ :

$$\begin{cases} y_1(t) = x_1(t), & t \in [0, 1], \\ y_2(t) = x_1(t+1), & t \in [0, 1], \\ y_3(t) = x_2(t), & t \in [0, 1], \\ y_4(t) = x_2(t+1), & t \in [0, 1]. \end{cases}$$

The resulting non-delayed reformulation of Example IV.3 is as follows:

$$\dot{y}(t) = \begin{bmatrix} -1 & 0 & -3 & 0 \\ 1.66 & -1 & -0.697 & -3 \\ 2 & 0 & -5 & 0 \\ 0.93 & 2 & -0.330 & -5 \end{bmatrix} y(t) + \begin{bmatrix} 1.66 + \cos(t) \\ \cos(t+1) \\ 0.93 + \sin(t) \\ \sin(t+1) \end{bmatrix},$$

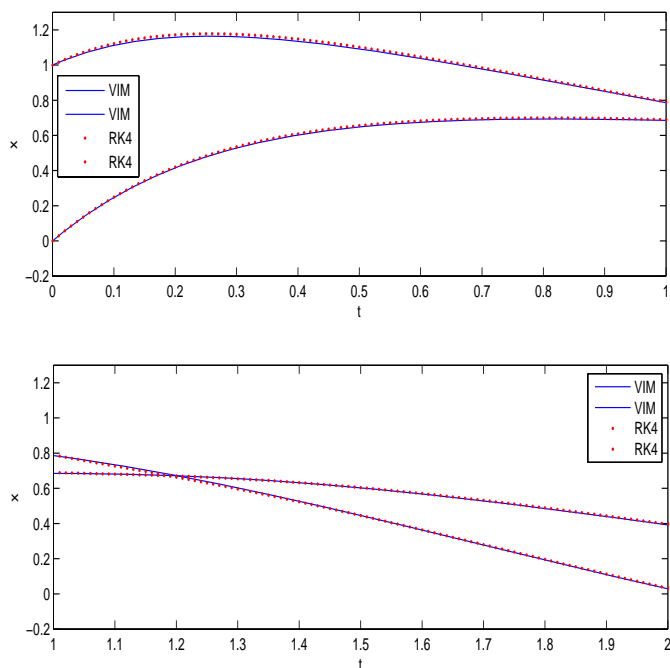


Fig. 3. Comparison of the VIM with the numerical RK4 for Example IV.3.

with initial value

$$y(0) = \begin{bmatrix} 1 \\ r_1 \\ 0 \\ r_3 \end{bmatrix},$$

where  $y(t) = [y_1(t) \ y_2(t) \ y_3(t) \ y_4(t)]^T$  and  $y(1) = [r_1 \ r_2 \ r_3 \ r_4]^T$ . Now the non-delayed initial value problem (7) can be solved on the interval  $[0, 1]$  by the iteration formula (8) with

$$C = \begin{bmatrix} -1 & 0 & -3 & 0 \\ 1.66 & -1 & -0.697 & -3 \\ 2 & 0 & -5 & 0 \\ 0.93 & 2 & -0.330 & -5 \end{bmatrix},$$

$$F(t) = \begin{bmatrix} \frac{1.66 + \cos(t)}{r_1} & 0 & 0 & 0 \\ 0 & \frac{\cos(t+1)}{r_2} & 0 & 0 \\ 0 & 0 & \frac{0.93 + \sin(t)}{r_3} & 0 \\ 0 & 0 & 0 & \frac{\sin(t+1)}{r_4} \end{bmatrix},$$

where  $r_1, r_2, r_3, r_4 \neq 0$ .

Figure 3 depicts the approximate analytical solution of Example IV.3 after using 8 iterations by the VIM. Comparison with the numerical RK4 ( $h = 0.01$ ) shows good agreement between two methods.

## V. CONCLUSION

In this paper, the VIM has been successfully applied to derive approximate analytical solutions for systems of linear

DDEs. We suggest the procedure which is simple and illustrative examples demonstrate that the method is valid and effective.

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