N-Sun Decomposition of Complete Graphs and Complete Bipartite Graphs

R. Anitha, and R. S. Lekshmi

Abstract—Graph decompositions are vital in the study of combinatorial design theory. Given two graphs G and H, an H-decomposition of G is a partition of the edge set of G into disjoint isomorphic copies of H. An n-sun is a cycle C_n with an edge terminating in a vertex of degree one attached to each vertex. In this paper we have proved that the complete graph of order 2n, K_{2n}, can be decomposed into n-2 n-suns, a Hamilton cycle and a perfect matching, when n is even and for odd case, the decomposition is n-1 n-suns and a perfect matching. For an odd order complete graph K_{2n+1}, delete the star subgraph K_1, 2n  and the resultant graph K_{2n} is decomposed as in the case of even order. The method of building n-suns uses Walecki’s construction for the Hamilton decomposition of complete graphs. A spanning tree decomposition of even order complete graphs is also discussed using the labeling scheme of n-sun decomposition. A complete bipartite graph K_{n,n} can be decomposed into n/2 n-suns when n/2 is even. When n/2 is odd, K_{n,n} can be decomposed into (n-2)/2 n-suns and a Hamilton cycle.

Keywords—Hamilton cycle, n-sun decomposition, perfect matching, spanning tree.

I. INTRODUCTION

By a graph G = (V, E) we mean a finite undirected graph without loop unless otherwise stated. A cycle of length n is denoted by C_n. An n-sun is a cycle C_n with an edge terminating from each vertex of C_n [1]. Thus every n-sun graph contains exactly one cycle of length n and n pendant vertices. A decomposition of a graph is a collection of edge-disjoint subgraphs G_1, G_2, ..., G_m of G such that every edge of G belongs to exactly one G_i. Graph decompositions, known for its applications in combinatorial design theory, have been studied since the mid nineteenth century. Several decades after its introduction, Walecki has the credit of constructing Hamilton cycle decomposition of complete graphs [2]-[4]. Complete graphs are decomposed in many ways. In this paper we have decomposed complete graphs of even order, K_{2n}, into n-suns. The decomposition is based on Walecki’s construction of complete graphs into Hamilton cycles. The aim is to provide a systematic approach to the decomposition with a labeling scheme. By an orderly removal of edges from the cycles in n-suns, spanning tree decomposition is also possible in K_{2n}. Each of these spanning trees has the specialty of containing a perfect matching of K_{2n}. Complete bipartite graphs K_{n,n} are equally significant. Their n-sun decompositions are also given.

II. PRELIMINARIES

Let G be a graph with n vertices and m edges. A graph in which any two distinct points are adjacent is called a complete graph. A spanning cycle in a graph is called a Hamilton cycle. A partitioning of the edge set of G into Hamilton cycles if G is (2d+1)-regular [4]. The complete graph K_n has Hamilton decomposition for all n \geq 2. Any complete graph K_{2n} can be decomposed into (n-1)/2 Hamilton cycles if n is odd and (n-2)/2 Hamilton cycles if n is even [6].

In the decomposition of K_{2n} into n-suns we choose C_n to be the Hamilton cycles of its subgraph, K_{n,n}.

III. MAIN DEFINITIONS AND RESULTS

An n-sun decomposition of the complete graph K_{2n} is partitioning the edge set into n-suns and a perfect matching if n is odd and n-suns, a perfect matching and a Hamilton cycle if n is even.

A graph G is said to have total n-sun decomposition if every edge belongs to exactly one n-sun of the decomposition. An example graph is shown in Fig. 1. Note that regularity of graphs does not play a role in total n-sun decomposition.

Manuscript received March 24, 2007.

R. Anitha is with the Department of Mathematics and Computer Applications, PSG College of Technology, Coimbatore – 641 004, India (phone: 91-422-2572177; fax: 91-422-2573833; e-mail: anitha_nadarajan@mail.psgtech.ac.in).

R. S. Lekshmi is with the Department of Mathematics and Computer Applications, PSG College of Technology, Coimbatore – 641 004, India (phone: 91-422-2572177; fax: 91-422-2573833; e-mail: lekshmi_r_s@yahoo.com).
graphs have an n-sun decomposition which can be observed from Fig. 2.

**Lemma 3.1:** A necessary condition for the n-sun decomposition of $K_{2n}$ into $t$ isomorphic copies is that $|E|$ is divisible by $(1+2)n$ when $n$ is odd and $(3+2)n$ when $n$ is even.

**Proof:** When $n$ is odd, the total of (2n-1) edges in $K_{2n}$ are divided as follows. Since there are 2n edges in an n-sun, 2nt edges are in t isomorphic copies of n-suns and n edges in the perfect matching. Also $(2n-1)=2nt+n+2n$ implies $t=n-1$. The same line of reasoning is applicable for odd n in which $t=n-2$.

By the above lemma, $K_{2n}$ can be decomposed into n-1 n-suns and a perfect matching when n is odd. For n even, $K_{2n}$ can be decomposed into n-2 n-suns, a Hamilton cycle and a perfect matching. We use a labeling scheme to decompose $K_{2n}$ (n > 2) into n-suns. A proper labeling of G with k vertices is a bijection $f$: $V(G)$ → {1, 2, ..., k}.

**Theorem 3.2:** The complete graph $K_{2n}$ has n-sun decomposition for all odd $n > 2$.

**Proof:** Consider the complete graph $K_{2n}$ where n is odd. Split the vertex set $V = \{v_1, v_2, ..., v_{2n}\}$ of $K_{2n}$ into two such that $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{v_{n+1}, v_{n+2}, ..., v_{2n}\}$. Let U and W be the induced subgraph of $K_{2n}$ with vertex subset $V_1$ and $V_2$ respectively. Then U and W are complete graphs of odd order and have $n(n-1)/2$ edges each. The remaining $n^2$ edges of $K_{2n}$ form an edge cut (whose removal disconnects $K_{2n}$). To construct n-suns, we find Hamilton cycles of U and W using Walecki’s construction and to each Hamilton cycle, adjoin n edges from the edge cut. A similar functional notation of [2] is adopted in finding the Hamilton cycles of U and W.

Since n is odd, U can be decomposed into (n-1)/2 Hamilton cycles and a perfect matching and hence the maximum number of edge disjoint n-suns possible in U is (n-1)/2. In U, let C be the Hamilton cycle $v_1v_2v_3v_4v_5v_6...v_{n-1}v_nv_{n+1}v_{n+2}v_{n+3}v_{n+4}v_{n+5}v_{n+6}...v_{2n}v_1$ and $\alpha$ be the permutation $(v_1v_2v_3...v_n)$. Then $C, \alpha(C)\alpha^2(C), ..., \alpha^{n-1}(C)$ is a Hamilton cycle decomposition of U. For simplicity, let $\Phi_k$ denote the kth Hamilton cycle and $\Phi_k(v_i)$ denote vertex $v_i$ in that Hamilton cycle, where $k = 1, 2, ..., n-1/2$. Then $\Phi_k = \alpha^{k-1}(C)$.

Append the edges to the Hamilton cycles as

$$\Phi_k(v_i) = \left\lfloor \frac{n+k+i-1}{n} \right\rfloor \mod n \text{ if } k+i-1 \leq n \text{ and if } k+i-1 > n, \text{ then } i = 1, 2, ..., n.$$
change in the labels. Let the vertex set of $K_{2n}$ be split as $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{v_{n+1}, v_{n+2}, ..., v_{2n}\}$. Let $X$ and $Y$ be the induced subgraph of $K_{2n}$ with vertex subset $V_1$ and $V_2$ respectively. Here let $C$ be the Hamilton cycle $V_1V_2Y_3V_4Y_5V_6V_7V_8V_9V_{10}V_{11}V_{12}V_{13}V_{14}V_{15}V_{16}V_{17}V_{18}V_{19}V_{20}V_{21}V_{22}V_{23}V_{24}V_{25}V_{26}V_{27}V_{28}V_{29}V_{30}V_{31}V_{32}V_{33}V_{34} V_{35}V_{36}V_{37}V_{38}V_{39}V_{40}V_{41}V_{42}V_{43}V_{44}$ and $a$ be the permutation $(v_{45}, v_{46}, v_{47}, ..., v_{80})$ as in the odd case. Then $C, a(C), a^2(C), ..., a^{n-1}(C)$ is a Hamilton cycle decomposition of $X$.

Let $\Phi_k(v_i)$ denote vertex $v_i$ in the $k$th Hamilton cycle of $X$, where $k = 1, 2, ..., \frac{n-2}{2}$. Then $\Phi_k = a^{k-1}(C)$. Append the edges to the Hamilton cycles of $X$ as $\Phi_k(v_i) = \begin{cases} \begin{array}{l} n + k + i - 1 \text{ if } k + i - 1 \leq n \\ \{[k + i - 1] \text{mod } n\} + n \text{ if } k + i - 1 > n' \end{array} \end{cases}$ and $i = 1, 2, ..., \frac{n-2}{2}$.

Similarly in $Y$, $\beta(C), \beta^2(C), ..., \beta^{\frac{n-4}{2}}(C)$ is a Hamilton cycle decomposition where $C'$ is the cycle $v_{n+1}v_{n+2}v_{n+3}v_{n+4}v_{n+5}v_{n+6}v_{n+7}v_{n+8}v_{n+9}v_{n+10}v_{n+11}v_{n+12}v_{n+13}v_{n+14}v_{n+15}v_{n+16}v_{n+17}v_{n+18}v_{n+19}v_{n+20}v_{n+21}v_{n+22}v_{n+23}v_{n+24}v_{n+25}v_{n+26}v_{n+27}v_{n+28}v_{n+29}v_{n+30}$ and $\beta$ is the permutation $(v_{n+1})(v_{n+2})(v_{n+3})...v_{2n}$.

Let $\Phi_k'(v_i)$ denote vertex $v_i$ in the $k$th Hamilton cycle $\Phi_k'(Y)$ where $k = 1, 2, ..., \frac{n-2}{2}$. Append the pendant edges using the labeling scheme $\Phi_k'(v_{n+i}) = \begin{cases} \begin{array}{l} k + i \text{ if } k + i \leq n \\ \{k + i \text{mod } n\} + k \text{ if } k + i > n' \end{array} \end{cases}$, $i = 1, 2, ..., n$. The perfect matching of $X$: $\{(v_i, v_{n+i})\}$ and that of $Y$: $\{(v_{n+i}, v_{n+i+1})\}$ forms a perfect matching of $K_{2n}$ where $i = 1, 2, ..., \frac{n}{2}$ and $j = \begin{cases} \begin{array}{l} \frac{n}{2} + 1, \text{ if } i = 1 \\ \frac{n}{2} + 2, \text{ if } i > 1 \end{array} \end{cases}$.

The remaining edges form a Hamilton cycle whose labeled structure is shown in Fig. 3.

A decomposition of $K_{12}$ into four 6-suns, a perfect matching and a Hamilton cycle is shown in Fig. 5 of Appendix.

**Corollary 3.6:** Let $K_{2n} - I$ denote the subgraph of $K_{2n}$ with a perfect matching removed. Then $(K_{2n} - I) + 2C_n$ can be decomposed into $n$-suns.

Add two sets of $n$ multi edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), ..., (v_{n-1}, v_n)$. The multi edges form two $n$-cycles. Since a Hamilton cycle is the union of two perfect matchings, append one matching each to the two $n$-cycles to obtain the required decomposition.

**Corollary 3.7:** An odd order complete graph can be decomposed into $n$-suns, a Hamilton cycle, a perfect matching and a star graph when $n$ is even. For odd $n$, $n$-suns, a perfect matching and a star partition the edges of $K_{2n+1}$.

**Proof:** In the case of odd order complete graphs $K_{2n+1}$, $n$-sun decomposition is not possible, since the maximum matching is $2n$. By the removal of a star graph $K_{1, 2n}$, the resultant is a complete graph of even order which has $n$-sun decomposition.

In the next section we discuss about the $n$-sun decomposition of complete bipartite graphs $K_{m, m}$, $m = 2n$. We split $K_{m, m}$ into two complete bipartite subgraphs $K_{n, n}$ and the remaining edges form an edge cut of $K_{m, m}$. We find Hamilton cycles in each subgraph $K_{n, n}$ and append edges from the edge cut for the pendants of the $n$-sun. Since the minimum cycle length in bipartite graphs is four and to append pendants, $m \geq 4$.

In [7] decomposition of $r$-partite graphs into edge-disjoint Hamilton cycles is discussed. We shall now the procedure to find edge-disjoint Hamilton cycles for bipartite graphs which uses consecutive perfect matchings (1-factors).

Let the vertices of $K_{n, n}$ be partitioned as $U = \{u_1, u_2, ..., u_n\}$ and $V = \{v_1, v_2, ..., v_n\}$. Let the set of perfect matching be $\Phi = \{F_j = j_i\}$, $j = 1, 2, ..., n$ and the suffices of $v$ are taken modulo $n$. Then $H_1 = F_{2n-1} \cup F_{2n}$, $k = 1, 2, ..., n/2$ is a set of edge disjoint Hamilton cycles of $K_{n, n}$. When $n$ is even, $K_{n, n}$ can be decomposed into $n/2$ edge-disjoint Hamilton cycles. For odd $n$, the decomposition is $(n-1)/2$ Hamilton cycles plus a perfect matching. When $n$ is odd the edges of the perfect matching is $\{(u_i, v_i)\}$ where $i = 1, 2, ..., n$.

An $n$-sun decomposition of the complete bipartite graph $K_{n, n}$ is partitioning the edge set into $n$-suns and $n$ copies of $C_4$ if $n/2$ is odd and $n$-suns if $n/2$ is even.

**Theorem 3.8:** The complete bipartite graph $K_{n, n}$ has $n$-sun decomposition for all $n>2$ even.

**Proof:** To simplify the labeling scheme, let the vertices of $K_{n, n}$ be partitioned as $U$ and $V$,

$U = \{u_1, u_2, ..., u_n\}$ and $V = \{v_1, v_2, ..., v_n\}$.

Let us split $K_{n, n}$ into $K_{n, n}^{1,2, n/2}$ and $K_{n, n}^{n/2, n/2}, n/2$ and an edge-cut with $n/2$ edges where $K_{n, n}^{1,2, n/2}$ and $K_{n, n}^{n/2, n/2}$ are the induced subgraphs formed by $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and $\{u'_1, u'_2, ..., u'_n, v'_1, v'_2, ..., v'_n\}$ respectively.

The maximum number of $n$-suns possible in the decomposition of $K_{n, n}$ is $n/2$ since the maximum number of Hamilton cycles in $K_{n/2, n/2}$ and $K_{n/2, n/2}$ put together is $n/2$. 

Fig. 3: The Hamilton cycle structure in the $n$-sun decomposition of $K_{n, n}$ even.
Let \( \{F_j = \bigcup_{i=1}^{n/2} u_i v_{i+j}, j = 1, 2, \ldots, n/2\} \) be the perfect matchings of \( K_{n/2,n/2} \), the suffix of \( v \) being taken modulo \( n/2 \). Let \( H_k = F_{2k-1} \cup F_{2k}, k = 1, 2, \ldots, n/4 \) be a Hamilton cycle decomposition of \( K_{n/2,n/2} \). Append edges from the edge cut as follows: \( H_k(u_i) = v_{p}, H_k(v_i) = u_{p}', p = (k+i-1 \mod (n/2)), i = 1, 2, \ldots, n/2 \) and \( k = 1, 2, \ldots, n/4 \). Similarly let \( \{F_j' = \bigcup_{i=1}^{n/2} u_i v_{i+j}', j = 1, 2, \ldots, n/2\} \) be the perfect matchings of \( K_{n/2,n/2} \), the suffix of \( v \) being taken modulo \( n/2 \). Let \( H_k' = F_{2k-1} \cup F_{2k}', k = 1, 2, \ldots, n/4 \) be a Hamilton cycle decomposition of \( K_{n/2,n/2} \). Append edges for the pendants of \( n \)-sun as \( H_k(u_i) = v_{k+i} \) and \( H_k'(v_i) = u_{k+i}, i = 1, 2, \ldots, n/2, k = 1, 2, \ldots, n/4, \) the suffixes \( k+i \) are taken modulo \( n/2 \). The Hamilton cycles with the appended edges form \( n \)-suns. Since every edge is in exactly one \( n \)-sun, \( K_{n,n} \) has a total \( n \)-sun decomposition.

**Theorem 3.9:** The complete bipartite graph \( K_{n,n} \) has \( n \)-sun decomposition for all \( n/2 \) odd.

**Proof:** Let the notations \( U, V, F_k \) and \( H_k \) be as in Theorem 3.8 where \( n/2 \) is odd, \( k = 1, 2, \ldots, (n-2)/4 \) and the maximum number of \( n \)-suns possible is \((n-1)/2\). By the construction, the perfect matching left out in \( K_{n/2,n/2} \) and \( K_{n/2,n/2} \) after the \( n \)-suns are constructed are \( F_{n/2} \) and \( F_{n/2} \) respectively. These edges along with the edge set \( \{u_i v_{q}' \} \cup \{u'_i v_q\}, i = 1, 2, \ldots, n/2 \) and \( q = i+(n-1)/2 \) is taken modulo \( n/2 \), forms a Hamilton cycle. In fact \( \{u_i v_{q}' \} \cup \{u'_i v_q\} \) is a perfect matching of \( K_{n,n} \).

Examples of \( n \)-sun decompositions of \( K_{4,4} \) and \( K_{6,6} \) are shown in Fig. 6 and Fig. 7 of Appendix.

**IV. CONCLUSION**

The aim of this communication has been to present a new kind of decomposition of \( K_{2n} \) and \( K_{n,n} \). It is hoped that this decomposition may stimulate further studies on \( n \)-sun decompositions. The \( n \)-suns and the Hamilton cycles of \( K_{2n} \) have close association since both are spanning subgraphs with exactly one cycle. When \( n \) is odd, the total number of \( n \)-suns in the decomposition of \( K_{2n} \) is \( n-1 \) which is exactly the same number as in the decomposition of \( K_{2n} \) into Hamilton cycles, whereas for even \( n \), the number is \( n-2 \) which is one less than the number of Hamilton cycles in the decomposition. Also the deletion of any one edge of the cycle in the \( n \)-sun or Hamilton cycle results in a spanning tree where the tree contains a perfect matching. Since the maximum degree of a vertex in the \( n \)-sun is three, the spanning trees obtained from \( n \)-suns also have the same maximum degree. The important feature of these spanning trees is that the diameter is \((n/2)+1\) for all \( n \) where as in Hamilton paths the diameter is \( n-1 \). An \( n \)-sun decomposition of \( K_{n,n} \) is also discussed. Finding a strong sufficient condition for the existence of \( n \)-sun and total \( n \)-sun decomposition of \( K_{2n} \) and \( K_{n,n} \) will be well appreciated. Tree decomposition for \( K_{n,n} \) using \( n \)-suns can be studied.
Fig. 6 A 4-sun decomposition of $K_{4,4}$

Fig. 7 A 6-sun decomposition of $K_{6,6}$

REFERENCES