Parallel alternating two-stage methods for solving linear system

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Abstract—In this paper, we present parallel alternating two-stage methods for solving linear system $Ax = b$, where $A$ is a monotone matrix or an H-Matrix. And we give some convergence results of these methods for nonsingular linear system.

Keywords—parallel, alternating two-stage, convergence, linear system.

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I. INTRODUCTION

For the solution of the large linear system

$$Ax = b,$$

where $A$ is an $n \times n$ square matrix, and $x$ and $b$ are $n$-dimensional vectors, the basic iterative method is

$$Mx_{k+1} = Nx_k + b, \ k = 0, 1, \ldots$$

where $A = M - N$ and $M$ is nonsingular.

Alternating two-stage iterative methods[1] have been studied to approximate the linear system (2) by using an inner iteration. Let $M = P - Q = R - S$ be two splittings of the matrix $M$. In order to approximate (2), for each $k, k = 1, 2, \ldots$, we perform $s(k)$ inner iterations of the general class of iterative methods of the form

$$y_{j, \frac{k}{2}} = P^{-1}Q y_{j-1} + P^{-1}(Nx_{k-1} + b)$$

$$y_j = R^{-1}S y_{j, \frac{k}{2}} + R^{-1}(Nx_{k-1} + b), \ j = 1, 2, \ldots, s(k)$$

Thus, the resulting method is

$$x_k = (R^{-1}SP^{-1}Q)^s(k)x_{k-1} + \sum_{j=1}^{s(k)-1}(R^{-1}SP^{-1}Q)^jR^{-1}(SP^{-1} + I)(Nx_{k-1} + b), \ k = 1, 2, \ldots$$

On the other hand, with the development of parallel computation in recent years, the utilization of the parallel algorithms for the solution of large nonsingular linear system has become effective. Now we introduce the parallel alternating two-stage methods.

Given a parallel multisplitting of $A$, s.t.

(i) $A = M_l - N_l$,

(ii) $M_l = P_l - Q_l = R_l - S_l$,

(iii) $E_l \geq 0$ and $\sum_{l=1}^{\alpha} E_l = I$,

where $l = 1, 2, \ldots, \alpha$ and $I$ is the identity matrix.

Suppose that we have a multiprocessor with $\alpha$ processors connected to a host processor, that is, the same number of processors as splittings, and that all processors have the last update vector $x_{k-1}$. Then the $l$th processor only computes those entries of the vector $y_{i,j,\frac{k}{2}} = P_l^{-1}Q_l y_{i,j-1} + P_l^{-1}(N_l x_{k-1} + b)

y_{i,j} = R_l^{-1}S_l y_{i,j,\frac{k}{2}} + R_l^{-1}(N_l x_{k-1} + b), \ j = 1, 2, \ldots, s(k)$

with $y_{i,0} = x_{k-1}$, or equivalently

$$y_{i,j} = R_l^{-1}S_l P_l^{-1}Q_l y_{i,j-1} + R_l^{-1}(S_l P_l^{-1} + I)(N_l x_{k-1} + b), \ j = 1, 2, \ldots, s(k),$$

which correspond to the nonzero diagonal entries of $E_l$. The processor then scales these entries so as to be able to deliver the results to the host processor, performing the parallel multisplitting scheme

$$x_k = H(k)x_{k-1} + W(k)b, \ k = 1, 2, \ldots$$

where

$$H(k) = \sum_{l=1}^{\alpha} E_l [(R_l^{-1}S_l P_l^{-1}Q_l)^{(s(k))}]$$

$$+ \sum_{j=0}^{s(k)-1} (R_l^{-1}S_l P_l^{-1}Q_l)^j R_l^{-1}(S_l P_l^{-1} + I)N_l$$

and

$$W(k) = \sum_{l=1}^{\alpha} E_l \sum_{j=0}^{s(k)-1} (R_l^{-1}S_l P_l^{-1}Q_l)^j(R_l^{-1}S_l P_l^{-1} + R_l^{-1}).$$

Then, we can obtain the next algorithm:

Algorithm I (PATS):

for any given initial vector $x_0$

for $k = 1, 2, \ldots$ until convergent

for $l = 1, 2, \ldots, \alpha$

$y_{0,0} = x_{k-1}$

for $j = 1, 2, \ldots, s(l, k)$

$P_l y_{i,j,\frac{k}{2}} = Q_l y_{i,j-1} + (N_l x_{k-1} + b)$

$R_l y_{i,j} = S_l y_{i,j,\frac{k}{2}} + (N_l x_{k-1} + b)$

$x_k = \sum_{l=1}^{\alpha} E_l y_{i,l}$

Usually, we say that an parallel alternating two-stage method is stationary when $s(l, k) = s$, for all $l, k$, while an parallel alternating two-stage method is non-stationary if the number of inner iterations changes with the outer iteration $k$. In the following, we call them $SPATS$ method and $NSPATS$ method, respectively.

In this paper, our study concentrates on the parallel alternating two-stage method. With this aim, in the next section, we introduce the notation and preliminaries needed in this paper. In section 3, we present convergence conditions of these methods for nonsingular...
linear systems, when the matrix $A$ of the linear system is monotone or H-matrix. In section 4, we also give two relaxation parallel alternating two-stage methods, $RPATS\,I$ and $RPATS\,II$, and analyze their convergence conditions.

II. NOTATION AND PRELIMINARIES

We need the following definitions and results.

We say that a vector $x$ is nonnegative, denoted $x \geq 0$, if all of its entries are nonnegative. A nonsingular matrix $A \in \mathbb{R}^{n \times n}$ is an M-matrix if and only if $A$ is a monotone matrix ($A^{-1} \geq 0$). For any matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, we define its comparison matrix $< A > = (b_{ij})$ by $b_{ij} = |a_{ij}|$, $b_{ij} = -|a_{ij}|$, $i \neq j$. A nonsingular matrix $A$ is said to be an H-matrix if $< A >$ is an M-matrix.

**Lemma 1**: ([2]). Let $A, B \in \mathbb{R}^{n \times n}$,

1. If $A$ is an H-matrix, then $|A^{-1}| \leq |A|^{-1}$, and
2. If $|A| \leq B$, then $|A| \leq |B|$.

**Lemma 2**: ([3, 4]). Let $A \in \mathbb{R}^{n \times n}$. A splitting $A = M - N$ is called

1. regular if $M^{-1} \geq 0$ and $N \geq 0$,
2. weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$,
3. H-splitting if $< M > [2]$ is a nonsingular M-matrix, and
4. H-compatible splitting if $< A > = (M, N)$.

**Lemma 3**: ([3, 4]). Let $A = M - N$ be a splitting.

1. If the splitting is weak regular, then $\rho (M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$,
2. If the splitting is an H-splitting, then $A$ and $M$ are H-matrices and $\rho (M^{-1}N) < |A|^{-1}$ if $< A > = (M, N) < 1$,
3. If the splitting is an H-compatible splitting and $A$ is an H-matrix, then it is an H-splitting and thus convergent.

**Lemma 4**: ([5]). Let $T_1, T_2, \ldots, T_k \cdots$ be a sequence of nonnegative matrices in $\mathbb{R}^{n \times n}$. If all of them are $H$-matrices, then there exists $\theta > 0$, for $j = 1, 2, \cdots$, such that $\rho (H_k) \leq \theta^k < 1$, where $H_k = T_kT_{k-1} \cdots T_2T_1$, and therefore $\lim_{k \to \infty} H_k = 0$.

III. CONVERGENCE THEOREMS

Firstly, we deal with the convergence of the $PATS$ method when $A$ is a monotone matrix.

**Theorem 1**: Let $A^{-1} \geq 0$. If the outer splitting $A = M_l - N_l$ is regular splitting and the inner splitting $M_l = P_l - Q_l = R_l - S_l$ are weak regular splitting, then the $PATS$ method converges to $x = A^{-1}b$ for any initial vector $x_0$ and for any sequence of inner iteration numbers $s(k) \geq 1, k = 1, 2, \cdots$.

**Proof**: Let $x_k = A^{-1}b$ and $e_k = x_k - x_0$ be the error vector at the $k$th outer iteration of the $PATS$ method. It is easy to prove that $x_k$ is a fixed point of (3). Therefore,

$$e_k = H(k)H(k-1) \cdots H(1)e_0, \quad k = 1, 2, \cdots$$

We suppose that $H(k) = \sum_{i=1}^{n} e_i T_i(k)$, where

$$T_i(k) = (R_i^{-1}S_iP_i^{-1}Q_i)^{i(k)} + \sum_{j=0}^{i(k)} (R_i^{-1}S_iP_i^{-1}Q_i)^{j(k)}R_i^{-1}(S_iP_i^{-1} + I)N_i,$$

we can easily have

$$T_i(k) \geq 0 \text{ and } H(k) \geq 0.$$

Now we consider the vector $e = (1, 1, \cdots, 1)^T$, and suppose $x = A^{-1}e$, since $A^{-1} \geq 0$, and $A^{-1}$ have no rows that all entries are zero, then $x > 0$. Similarly, $R_i^{-1}(S_iP_i^{-1} + I) > 0$. From

$$(I - M_l^{-1}N_l)x = M_l^{-1}Ax = M_l^{-1}e,$$

it follows that

$$T_i(k)x = [(R_i^{-1}S_iP_i^{-1}Q_i)^{i(k)} + \sum_{j=0}^{i(k)} (R_i^{-1}S_iP_i^{-1}Q_i)^{j(k)}R_i^{-1}(S_iP_i^{-1} + I)N_i]x,$$

$$= [(I - (R_i^{-1}S_iP_i^{-1}Q_i)^{(l(k)}) + M_l^{-1}N_l]x,$$

$$= x - \sum_{j=0}^{i(k)} (R_i^{-1}S_iP_i^{-1}Q_i)^{j(k)}(S_iP_i^{-1} + I)(P_l - Q_l)M_l^{-1}e,$$

$$= x - R_i^{-1}(S_iP_i^{-1} + I)e$$

$$= x - R_i^{-1}(S_iP_i^{-1} + I)e,$$

therefore,

$$H(k)x = x - \sum_{i=1}^{\infty} E_i R_i^{-1}(S_iP_i^{-1} + I)e$$

$$= x - \sum_{i=1}^{\infty} E_i R_i^{-1}(S_iP_i^{-1} + I)e,$$

$$< x - \sum_{i=1}^{\infty} E_i R_i^{-1}(S_iP_i^{-1} + I)e.$$

Since $T_i(k)x \geq 0$, we have $x - \sum_{i=1}^{\infty} E_i R_i^{-1}(S_iP_i^{-1} + I)e < x$. Therefore, there exists $0 \leq \theta < 1$, such that

$$x - \sum_{i=1}^{\infty} E_i R_i^{-1}(S_iP_i^{-1} + I)e \leq \theta x,$$

then

$$H(k)x < \theta x, \quad k = 1, 2, \cdots,$$

and from Lemma 4 it follows that

$$\lim_{k \to \infty} H(k) = 0,$$

where $H(k) = H(k)H(k-1) \cdots H(1)$. So we have

$$\lim_{k \to \infty} e(k) = 0,$$

and the proof is complete.

Now, we study the convergence of the $NSPATS$ method when $A$ is an H-matrix, and therefore not necessarily a monotone matrix.

**Theorem 2**: Let $A$ is an H-matrix. If the outer splitting $A = M_l - N_l$ is H-splitting and the inner splitting $M_l = P_l - Q_l = R_l - S_l$ are H-compatible splitting. Then the $PATS$ method converges to $x = A^{-1}b$ for any initial vector $x_0$ and for any sequence of inner iteration numbers $s(k) \geq 1, k = 1, 2, \cdots$.

**Proof**: By Lemma 3(2), the matrices $P_l$ and $R_l$ are H-matrices.
and using Lemma 1(1) we can obtain,

\[ |H(k)| = \sum_{t=1}^{\alpha} E_t T_t(k) / \sum_{t=1}^{\alpha} E_t T_t(k) \leq \sum_{l=1}^{\alpha} H_l(k) \]

\[ \leq \sum_{l=1}^{\alpha} E_l[(|R_l^{-1}||S_l||P_l^{-1}||Q_l|)]^{t(l,k)} \]

\[ + \sum_{l=1}^{\alpha} E_l[(|R_l^{-1}||S_l||P_l^{-1}||Q_l|)]^{t(l,k)} \]

\[ \leq \sum_{l=1}^{\alpha} \sum_{j=0}^{s(l,k)-1} (|R_l^{-1}||S_l||P_l^{-1}||Q_l|)]^{t(l,k)} \]

\[ \times \sum_{j=0}^{s(l,k)-1} (|R_l^{-1}||S_l||P_l^{-1}||Q_l|)]^{t(l,k)} \]

\[ = H(k). \]

Then we have

\[ |H(k)||H(k-1)| \cdots |H(1)| \leq |H(k)||H(k-1)| \cdots |H(1)|. \]

Moreover, \( H(k) \) is the iteration matrix of the NASTSM method for the method \( (M_l > -r_1) \) with the regular splittings \( M_l > r_1 \) and \( -M_l > r_1 \). From the definitions of H-splitting, H-compatible splitting and Theorem 1, we can obtain

\[ \lim_{k \to \infty} E(k) = 0, \]

and the proof is complete.

IV. RELAXATION ITERATION METHODS

Now we introduce the relaxation factor to the parallel alternating two-stage methods. Then, we can obtain the following two algorithms:

**Algorithm 2 (RPATS I):**

for any given initial vector \( x_0 \) and \( \omega \in (0,1) \)
for \( k = 1, 2, \ldots \) until convergent
\[ y_{0,j} = x_{k-1} \]
for \( j = 1, 2, \ldots, s(l,k) \)
\[ y_{k,j} = \omega y_{k,j-1} + (N x_{k-1} + b) \]
\[ R_1 y_{k,j} = S_1 y_{k,j-1} + (N_1 x_{k-1} + b) \]
\[ x_k = \omega \sum_{i=1}^{s} E_i x_{k-1} + (1 - \omega) x_{k-1} \]

**Algorithm 3 (RPATS II):**

for any given initial vector \( x_0 \) and \( \omega \in (0,1) \)
for \( k = 1, 2, \ldots \) until convergent
\[ y_{0,j} = x_{k-1} \]
for \( j = 1, 2, \ldots, s(l,k) \)
\[ y_{k,j} = \omega (Q_l y_{k,j-1} + (N x_{k-1} + b)) + (1 - \omega) y_{k,j-1} \]
\[ R_1 y_{k,j} = \omega (S_1 y_{k,j-1} + N_1 x_{k-1} + b) \]
\[ x_k = \sum_{i=1}^{s} E_i y_{i,j} \]