The Panpositionable Hamiltonicity of $k$-ary $n$-cubes

Chia-Jung Tsai and Shin-Shin Kao

Abstract—The hypercube $Q_n$ is one of the most well-known and popular interconnection networks and the $k$-ary $n$-cube $Q^k_n$ is an enlarged family from $Q_n$, that keeps many pleasing properties from hypercubes. In this article, we study the panpositionable hamiltonicity of $Q^k_n$ for $k \geq 3$ and $n \geq 2$. Let $x, y \in V(Q^k_n)$ be two arbitrary vertices and $C$ be a hamiltonian cycle of $Q^k_n$. We use $d_C(x, y)$ to denote the distance between $x$ and $y$ on the hamiltonian cycle $C$. Define $l$ as an integer satisfying $d_C(x, y) \leq l \leq \frac{1}{2}|V(Q^k_n)|$. We prove the followings:

- When $k = 3$ and $n \geq 2$, there exists a hamiltonian cycle $C$ of $Q^k_n$ such that $d_C(x, y) = l$.
- When $k \geq 5$ is odd and $n \geq 2$, we request that $l \notin S$ where $S$ is a set of specific integers. Then there exists a hamiltonian cycle $C$ of $Q^k_n$ such that $d_C(x, y) = l$.
- When $k \geq 4$ is even and $n \geq 2$, we request $l - d_C(x, y)$ to be even. Then there exists a hamiltonian cycle $C$ of $Q^k_n$ such that $d_C(x, y) = l$.

The result is optimal since the restrictions on $l$ is due to the structure of $Q^k_n$ by definition.

Index Terms—Hamiltonian, panpositionable, bipanpositionable, $k$-ary $n$-cube.

I. INTRODUCTION

THE $n$-dimensional hypercube $Q_n$ is one of the most well-known and popular interconnection networks due to its excellent properties as the following: it is vertex-symmetric and edge-symmetric; it is hamiltonian; it allows cycle/path embedding when faults occur and so on. (See [1], [2] for these results and their references). Therefore, numerous studies have been devoted to the hypercube family [3]–[6], [11], [12].

The $k$-ary $n$-cube $Q^k_n$ is an enlarged family from $Q_n$ that keeps many pleasing properties from hypercubes. More precisely, each vertex of $Q^k_n$ is labeled by a $n$-bit finite sequence $(u_0, u_1, u_2, ..., u_{n-1})$, where $0 \leq u_i \leq k - 1$ for $0 \leq i \leq n - 1$, and every two vertices $u$ and $v$ are adjacent if and only if $|u_i - v_i| = 1$ for some $i$ and $u_j = v_j$ for any $0 \leq j \leq n - 1$ with $j \neq i$. It is obviously that $Q^k_n$ is indeed a subclass of the $k$-ary $n$-cube when $k = 2$. Some properties of $Q^k_n$ mentioned in [6] are listed here: it is known that $Q^k_n$ is vertex-symmetric and edge-symmetric [3]; it is hamiltonian [4], [5]; it has diameter $n \sqrt[k]{k}$ [4], [5]; it has a recursive structure; and it contains many important interconnection networks as cycles (of certain lengths) [3], meshes (of certain dimensions) [4], and even hypercubes.

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II. PRELIMINARIES

For the graph definitions and notations we follow [9].

$G = (V, E)$ is a graph if $V$ is a finite set and $E$ is a subset of $\{(u, v)\mid u, v \in V\}$ is an unordered pair of $V$. We say that $V$ is the vertex set and $E$ is the edge set of $G$. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E$. A path is represented by a finite sequence of vertices $(v_0, v_1, v_2, ..., v_n)$, where every two consecutive vertices are adjacent. If $P$ is a path represented by $(v_0, v_1, v_2, ..., v_n)$, then we define $\text{inv}(P) = (v_n, v_{n-1}, v_{n-2}, ..., v_0)$. The length of a path $P$ is the number of edges in $P$. We write the path $(v_0, v_1, ..., v_n)$ as $(v_0, v_1, ..., v_{n-1}, P_1, v_{n+1}, ..., v_j, P_2, v_{j+1}, ..., v_n)$, where $P_1 = (v_i, v_{i+1}, ..., v_j)$ and $P_2 = (v_{j+1}, v_j, ..., v_n)$. We use $d_G(u, v)$ to denote the distance between $u$ and $v$ in $G$, i.e., the length of the shortest path between $u$ and $v$ in $G$. A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of $G$ is a cycle that visits every vertex of $G$ exactly once. We use $d_G(u, v)$ to denote the distance between $u$ and $v$ in a cycle $C$ of $G$, i.e., the length of the shorter path between $u$ and $v$ in $C$. A hamiltonian graph is a graph with a hamiltonian cycle.

A hamiltonian path in a graph $G$ is a path joining two distinct vertices $u$ and $v$ of $G$ that visits every vertex of $G$ exactly once. A graph $G$ is hamiltonian-connected if there is a hamiltonian path joining any two distinct vertices of $G$. Note that any (nontrivial) bipartite graph cannot be hamiltonian-connected, whereas a bipartite graph is hamiltonian laceable if there exists a hamiltonian path joining every two vertices which are in distinct partite [10].
The concept of hamiltonian panpositionability was first proposed by S. Kao et al. [7]. A hamiltonian graph $G$ is \textit{panpositionable} if for any two different vertices $u$ and $v$ of $G$ and any integer $l$ with $d_G(u, v) \leq l \leq \frac{V(G)}{2}$, there exists a hamiltonian cycle $C$ of $G$ with $d_G(u, v) = l$. A graph $G = (V_0 \cup V_1, E)$ is \textit{bipartite} if $V(G) = V_0 \cup V_1$ and $E(G)$ is a subset of $\{(u, v) | u \in V_0, v \in V_1\}$. A hamiltonian bipartite graph $G$ is \textit{bipanpositionable} if for any two different vertices $u$ and $v$ of $G$ and any integer $l$ with $d_G(u, v) \leq l \leq \frac{V(G)}{2}$ and $(l - d_G(u, v))$ is even, there exists a hamiltonian cycle $C$ of $G$ with $d_G(u, v) = l$.

The $k$-ary $n$-cube, $Q_k^n$, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass $Q_k^1$ is the well-studied hypercube family. The subclass $Q_k^n$ with $k \geq 3$ is defined as the cycle of length $k$. The $k$-ary $n$-cube, $Q_k^n$, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_k^n)$ be represented by $(u_0, u_1, \ldots, u_{n-1})$, where $0 \leq u_i \leq k - 1$. $u$ and $v$ are adjacent if and only if $|u_i - v_i| = 1$ or $k - 1$ for some $i$ and $u_i = v_i$ for any $0 \leq j \leq n - 1$ with $j \neq i$.

It is known that $Q_k^n$ is vertex-symmetric and edge-symmetric. Moreover, given any two distinct vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $Q_2^n$, there is an automorphism of $Q_2^n$ mapping $(u_1, u_2)$ and $(v_1, v_2)$ to $(m, 0)$ and $(0, n)$. Each vertex of $Q_k^n$ is represented by a $n$-bit tuple, and we will call the $d$th-bit the $d$th dimension. We can partition $Q_k^n$ over dimension $d$ by fixing the $d$th element of any vertex tuple at some value $a$, for every $a \in \{0, 1, \ldots, k - 1\}$. This results in $k$ copies $Q_{k,n-1}^{d,0}, Q_{k,n-1}^{d,1}, \ldots, Q_{k,n-1}^{d,k-1}$ of $Q_{k,n-1}$, with corresponding vertices in $Q_{k,n-1}^{d,0}, Q_{k,n-1}^{d,1}, \ldots, Q_{k,n-1}^{d,k-1}$ joined in a cycle of length $k$ (in dimension $d$) [6]. It is proven in [11], [12] that $Q_k^n$ is hamiltonian connected for odd $k$ and $Q_k^n$ is hamiltonian laceable for even $k$.

Let the length of a path between $u$ and $v$ in $Q_k^n$, where $k \geq 5$ is an odd integer, can not be arbitrary. For example, in $Q_3^n$, for any two vertices $u$ and $v$ and $d(u, v) = 1$, there exists no path $P$ between $u$ and $v$ with $|P| = 2$. In fact, we have the following observation. Given two vertices $u = (u_0, u_1, \ldots, u_{n-1})$ and $v = (v_0, v_1, \ldots, v_{n-1})$ of $Q_k^n$. Define the number $m_i = \min\{|u_i - v_i|, k - |u_i - v_i|\}$, where $0 \leq i \leq n - 1$. Let $s = \max\{m_0, \ldots, m_{n-1}\}$. Then there exists no path between $u$ and $v$ with length $r = d(u, v) = s + k - s - 2l = d(u, v) + k - 2s - 2l$, where $l$ is an integer and $1 \leq s \leq \frac{k}{2} - s$.

Consequently, we modify the concept of panpositionability of $Q_k^n$ by saying that $Q_k^n$ is \textit{nearly-panpositionable} if for any two distinct vertices $x$ and $y$ of $Q_k^n$ and for any integer $l'$ with $d(xy) \leq l' \leq \frac{V(G)}{2}$ and $l' \notin \langle r : r = d(u, v) + k - 2s - 2l \leq 1 \leq \frac{k}{2} - s \rangle$, there exists a hamiltonian cycle $C$ of $Q_k^n$ with $d(xy) = l'$. Therefore, in this article, we want to prove that $Q_k^n$ is panpositionable if $k \geq 5$ is an odd integer, and is bipanpositionable if $k \geq 4$ is an even integer. First of all, we prove the following two lemmas.

\textbf{Lemma 1.} Let $k$ be an integer with $k \geq 3$. For any path $P$ with length $2$ in $Q_k^2$, there exists a hamiltonian cycle of $Q_k^2$ that contains $P$.

\textbf{Proof:} Let $c, r, i$ be nonzero integers, $\frac{c}{|r|} = s$, $\frac{c}{|r|} = t$, $\bar{a} = (a_1, a_2, \ldots, a_i)$ and $\bar{b} = (b_1, b_2, \ldots, b_i)$. If $c = 0$, then $s = 0$. Similarly, if $r = 0$, then $t = 0$. To construct the required hamiltonian cycles, we define some path patterns in the following.

Please see Fig. 1 and Fig. 2 for an illustration. Fig. 1 is examples of $f_{3,2}^3(1, 4)$ and $h_{3,2}^4(0, 5)$. Note that $f_{3,2}^3(1, 4) = \langle (1, 4), (2, 4), (3, 4), (3, 3), (3, 2), (3, 1) \rangle$ and $h_{3,2}^4(0, 5) = \langle (3, 2), (2, 3), (2, 2), (2, 1) \rangle$. Fig. 2 is an example of $H_{6,3}^4(1, 1)$, where $\bar{a} = (4, -2, -1)$ and $\bar{b} = (4, -3, 2)$. Note that $H_{6,3}^4(1, 1) = \langle (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 5), (3, 5), (4, 5), (5, 4), (5, 3), (5, 2), (4, 2), (3, 2), (3, 1), (2, 2) \rangle$. Let $P = \langle u, v, x \rangle$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $Q_k^2$. We have following cases.

\textbf{Case 1.} $k$ is odd.

\textbf{Case 1.1.} Either $u_1 = v_1$ or $u_2 = v_2$. W.L.O.G., let $u = (0, 0), v = (2, 0)$ and $P = \langle u, (1, 0), v \rangle$.

Let $s_i = (1 - i)(2 - k)$, for $i \leq k - 1$ and $a_k = 0$; $b = (0, -1, -1, \ldots, -1)$. There exists a hamiltonian cycle $C = \langle (0, 0), P, (2, 0), f_{k-1}^{-3}(2, 0), H_{6,3}^4(0, k - 1) \rangle$.
The hamiltonian cycle in Fig. 4 is a hamiltonian cycle for an illustration. The hamiltonian cycle in Fig. 3 is $C = \langle (0,0), P, (2,0), f^{-1}_6(2,0), H^6_{\overrightarrow{b},k}(0,6), (0,0) \rangle$ and $H^6_{\overrightarrow{b},k}(0,6) = \langle h^6_0(0,6), h^6_{-1}(5,6), h^6_{-1}(0,5), h^6_{-1}(5,4), h^6_{-1}(0,3), h^6_{-1}(5,2), h^6_{-1}(0,1) \rangle$.

**Case 1.2.** $u_1 \neq v_1, u_2 \neq v_2$. W.L.O.G., let $u = (0,0)$ and $v = (1,1)$.

The hamiltonian cycle is the same as in Case 1.1. Please see Fig. 3 for an illustration.

**Case 1.2.2.** $P = \langle u, (0,1), v \rangle$, where $k = 3$.

The hamiltonian cycle is $C = \langle (0,0), P(1,0), f^{-1}_3(1,0), f^{-1}_2(2,0), (0,0) \rangle$.

**Case 1.2.3.** $P = \langle u, (0,1), v \rangle$, where $k \geq 5$.

Let $a_i = (-1)^i(k - 2)$, for $1 \leq i < k$, $a_{k-1} = 4 - k$ and $a_k = k - 3$; $b = (0, -1, -1, ..., -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (1,0), (0,1), \rangle$. Please see Fig. 4 for an illustration. The hamiltonian cycle in Fig. 4 is $C = \langle (0,0), P(1,0), f^{-1}_3(6,1), f^{-1}_2(6,0), (0,0) \rangle$ and $H^6_{\overrightarrow{b},k}(0,6) = \langle h^6_0(0,6), h^6_{-1}(5,6), h^6_{-1}(0,5), h^6_{-1}(5,4), h^6_{-1}(0,3), h^6_{-1}(5,2), h^6_{-1}(2,1) \rangle$.

**Case 2.** $k$ is even.

**Case 2.1.** Either $u_1 = v_1$ or $u_2 = v_2$. W.L.O.G., let $u = (0,0)$ and $v = (2,0)$ and $P = \langle u, (0,1), v \rangle$.

Let $a_i = (-1)^i(2 - k)$, for $1 \leq i \leq k$, $a_1 = k - 3$, $a_2 = 1 - k$ and $a_{k+1} = 0$; $b = (0, k - 1, -1, -1, ..., -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (2,0), H^6_{\overrightarrow{b},k+1}(2,0), (0,0) \rangle$. Please see Fig. 5 for an illustration. The hamiltonian cycle in Fig. 5 is $C = \langle (0,0), P, (2,0), H^6_{\overrightarrow{b},k+2}(2,0), (0,0) \rangle$ and $H^6_{\overrightarrow{b},k+2}(2,0) = \langle h^6_0(2,0), h^6_{-1}(5,0), h^6_{-1}(4,0), h^6_{-1}(0,3), h^6_{-1}(4,2), h^6_{-1}(0,1) \rangle$.

**Case 2.2.** $u_1 \neq v_1, u_2 \neq v_2$. W.L.O.G., let $u = (0,0)$ and $v = (1,1)$.

Case 2.2.1. $P = \langle u, (0,1), v \rangle$.

The hamiltonian cycle is the same as in Case 2.1. Please see Fig. 5 for an illustration.

**Case 2.2.2.** $P = \langle u, (0,1), v \rangle$.

Let $a_i = (-1)^i(k - 2)$, for $2 \leq i \leq k - 2$, $a_1 = 1 - k$, $a_{k-1} = 4 - k$ and $a_k = k - 3$; $b = (k - 2, -1, -1, ..., -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (1,0), (0,1), H^6_{\overrightarrow{b},k}(k - 1,1), (0,0) \rangle$. Please see Fig. 6 for an illustration. The hamiltonian cycle in Fig. 6 is $C = \langle (0,0), P, (1,0), (0,1), H^6_{\overrightarrow{b},k}(5,1), (0,0) \rangle$ and $H^6_{\overrightarrow{b},k}(5,1) = \langle h^6_{-1}(5,1), h^6_{-1}(0,5), h^6_{-1}(4,4), h^6_{-1}(0,3), h^6_{-1}(4,2), h^6_{-1}(2,1) \rangle$.

The lemma is proved.

To facilitate our derivation in the following, we define some path patterns. We shall use $x_{i0}, x_{i1}, x_{i2}, ..., x_{i-k-1}$ to denote the $k^{n-1}$ vertices of $Q^{k,i}_{d,n-1}$ for some $d$. For simplicity, denote $Q^{k,i}_{d,n-1}$ as $Q^{k,i}_{n-1}$. Let the path $p(x_{i0}, x_{i1}) = \langle x_{i0}, x_{i1}, x_{i2}, ..., x_{i-k-1} \rangle$ and $a_i = (a + i \mod k^{n-1})$. For example, if $k_{n-1} = 64$, then $p(x_{60}, x_{63}) = \langle x_{60}, x_{61}, x_{62}, x_{63}, x_{1}, x_{2} \rangle$. It is known that there exists a hamiltonian cycle in $Q^{k,i}_{n-1}$ [4]. Thus $x_{i0}$ and $x_{i-k-1}$ are adjacent and so are $x_{i1}$ and $x_{i-k}$.

**Lemma 2.** Let $k$ be an integer with $k \geq 3$. For any path $P$
with length 2 in $Q^h_k$, there exists a hamiltonian cycle of $Q^n_k$ that contains $P$.

Proof: The lemma will be proved by mathematical induction. By Lemma 1, the statement holds for $Q^0_k$. Using the induction hypothesis, we assume that the statement holds for $Q^{k-1}_{n-1}$, where $n \geq 3$. Now we want to prove that the lemma is true for $Q^n_k$. There are three cases.

**Case 1.** $P$ is in $Q^0_{n-1}$. W.L.O.G., let $i = 0$.

By the induction hypothesis, there exists a hamiltonian cycle $C^0$ of $Q^{0}_{n-1}$ that contains $P$. Let $P = (x^0_0, x^0_1, x^0_2)$ and $C^0 = (x^0_0, P, x^0_2, x^0_3, ..., x^0_{k-1}, x^0_0)$. Since $Q^{k-1}_{n-1}$ is hamiltonian [4], let the hamiltonian cycles in $Q^{k-1}_{n-1}$ be $C^k = (x^0_0, x^0_1, x^0_2, x^0_3, ..., x^0_{k-1}, x^0_0)$.

1) $k$ is odd. Then the hamiltonian cycle is

$$C = (x^0_0, P, x^0_1, x^0_2, x^0_3, x^0_4, ..., x^0_{k-1}, p(x^0_{k-1}, x^0_{k-1}),$$

$$\text{inv}(p(x^0_{k-1}, x^0_{k-1})), p(x^0_{k-1}, x^0_{k-3}), \text{inv}(p(x^0_{k-1}, x^0_{k-3})))$$

$$, ..., p(x^0_2, x^0_2), \text{inv}(p(x^0_1, x^0_1)), p(x^0_1, x^0_0), x^0_0).$$

2) $k$ is even. Then the hamiltonian cycle is

$$C = (x^0_0, P, x^0_1, x^0_2, x^0_3, x^0_4, ..., x^0_{k-1}, \text{inv}(p(x^0_{k-1}, x^0_{k-1})),$$

$$p(x^0_{k-1}, x^0_{k-3}), \text{inv}(p(x^0_{k-1}, x^0_{k-3})), p(x^0_{k-1}, x^0_2), ...,$$

$$p(x^0_2, x^0_2), \text{inv}(p(x^0_1, x^0_1)), p(x^0_1, x^0_0), x^0_0).$$

Please see Fig. 7 for an illustration, where the hamiltonian cycle in Fig. 7 is $C = (x^0_0, P, x^0_1, x^0_2, x^0_3, x^0_4, ..., x^0_{k-1}, p(x^0_{k-1}, x^0_{k-1}),$  \(\text{inv}(p(x^0_{k-1}, x^0_{k-1})), p(x^0_{k-1}, x^0_{k-3}), \text{inv}(p(x^0_{k-1}, x^0_{k-3}))))$... $p(x^0_2, x^0_2), \text{inv}(p(x^0_1, x^0_1)), p(x^0_1, x^0_0), x^0_0).

**Case 2.** $P$ passes through two $Q^{k-1}_{n-1}$. W.L.O.G., those two are $Q^{k-1}_{n-1}$ and $Q^{k-1}_{n-1}$.
Let $P = (x^0_0, x^0_1, x^0_2)$. In [11], [12], it has been shown that there exists a hamiltonian path $(x^0_1, p(x^0_1, x^0_0), x^0_0)$ in $Q^{k-1}_{n-1}$.

1) $k$ is odd. Then the hamiltonian cycle is

$$C = (x^0_0, P, x^0_1, x^0_2, x^0_3, x^0_4, ..., x^0_{k-1}, p(x^0_{k-1}, x^0_{k-1}),$$

$$\text{inv}(p(x^0_{k-1}, x^0_{k-1})), p(x^0_{k-1}, x^0_{k-3}), \text{inv}(p(x^0_{k-1}, x^0_{k-3})))$$

$$, ..., p(x^0_2, x^0_2), \text{inv}(p(x^0_1, x^0_1)), p(x^0_1, x^0_0), x^0_0).$$

2) $k$ is even. Then the hamiltonian cycle is

$$C = (x^0_0, P, x^0_1, x^0_2, x^0_3, x^0_4, ..., x^0_{k-1}, \text{inv}(p(x^0_{k-1}, x^0_{k-1})),$$

$$p(x^0_{k-1}, x^0_{k-3}), \text{inv}(p(x^0_{k-1}, x^0_{k-3})), p(x^0_{k-1}, x^0_2), ...,$$

$$p(x^0_2, x^0_2), \text{inv}(p(x^0_1, x^0_1)), p(x^0_1, x^0_0), x^0_0).$$

Please see Fig. 8 for an illustration, where the hamiltonian cycle in Fig. 8 is $C = (x^0_0, P, x^0_1, x^0_2, x^0_3, x^0_4, ..., x^0_{k-1}, p(x^0_{k-1}, x^0_{k-1}),$  \(\text{inv}(p(x^0_{k-1}, x^0_{k-1})), p(x^0_{k-1}, x^0_{k-3}), \text{inv}(p(x^0_{k-1}, x^0_{k-3}))))$... $p(x^0_2, x^0_2), \text{inv}(p(x^0_1, x^0_1)), p(x^0_1, x^0_0), x^0_0).

**Case 3.** $P$ passes through three $Q^{k-1}_{n-1}$.
It is known that we can partition $Q^n_k$ over dimension $d$ by fixing the $d$th element of any vertex tuple at some value $a$, for every $a \in \{0, 1, ..., k-1\}$. In this case, $P = (u, x, v)$ passes through three $Q^{k-1}_{n-1}$, i.e., $u$, $x$, and $v$ have the same value in at least one element of vertex tuple. Hence this case is equivalent to Case 1.

By the mathematical induction, the lemma is proved.

**III. THE PANPOSITIONABILITY OF $Q^n_k$, WHERE $k \geq 3$ IS AN ODD INTEGER AND $n \geq 2$ IS AN INTEGER.**

**Lemma 3.** $Q^n_k$ is a panpositionable hamiltonian graph.

Proof: There are two cases: Case 1. $u = (0, 0)$ and $v = (1, 0)$; Case 2. $u = (1, 0)$ and $v = (0, 1)$. By brute force, we construct the required hamiltonian cycles. Please see Fig. 9.

**Theorem 1.** $Q^n_k$ is a panpositionable hamiltonian graph.

Proof: The theorem is proved by mathematical induction using Lemma 3 as base case. The detailed derivation is skipped.

**Lemma 4.** Let $k$ be an odd integer with $k \geq 5$. Then $Q^2_k$ is nearly-panpositionable.

Proof: The proof is by brute force and hence is skipped.

**Theorem 2.** Let $k$ be an odd integer with $k \geq 5$. $Q^n_k$ is nearly-panpositionable hamiltonian.

Proof: We will prove the theorem using the mathematical induction. By Lemma 4, $Q^5_k$ is nearly-panpositionable hamiltonian. With the induction hypothesis, we assume that $Q^{k-1}_{n-1}$ is nearly-panpositionable hamiltonian for some $n \geq 3$. We need to show that $Q^n_k$ is nearly-panpositionable hamiltonian. Let $u, v \in Q^n_k$ and $l$ be an integer with
By the induction hypothesis, for any two vertices $H_k, i$, Lemma 2, for any path with length $n$, there exists a corresponding vertex $x_{0,H_k, i}$. For example, let $x_{0,H_k, i}$ and $x_{0,H_k, j}$ be vertices. Then we have the hamiltonian cycle

$$C = (x_{0,1}^0, x_{0,2}^0, x_{0,3}^0, ..., x_{0,n-1}^0, x_{0,1}^0), \text{inv}(p(x_{1,1}^0, x_{1,2}^0, x_{1,3}^0, ..., x_{1,n-1}^0)), p(x_{1,1}^1, x_{1,2}^1, x_{1,3}^1, ..., x_{1,n-1}^1), ..., p(x_{1,n-1}^1, x_{1,1}^2, x_{1,2}^2, ..., x_{1,n-1}^2), ..., p(x_{1,1}^n, x_{1,2}^n, x_{1,3}^n, ..., x_{1,n-1}^n)),$$

$$p(v_1^0, v_2^0, v_3^0, ..., v_{n-1}^0), p(v_1^1, v_2^1, v_3^1, ..., v_{n-1}^1), ..., p(v_1^n, v_2^n, v_3^n, ..., v_{n-1}^n).$$

Please see Fig. 11 for an illustration, where $m = 0, r = 2, t' = 8$ and the hamiltonian cycle is

$$C = (x_{0,1}^0, x_{0,2}^0, x_{0,3}^0, L_3^1, p(x_{1,1}^1, x_{1,2}^1, x_{1,3}^1), p(x_{1,4}^1, x_{1,5}^1), p(x_{1,6}^1, x_{1,7}^1), p(x_{1,8}^1, x_{1,9}^1), x_{0,1}^0).$$

If $r$ is odd, let $t_1 = t_2 = t_3 = v_1$, $x_{1+r}^0 = z_{1+r}^0$ and $x_{1+r}^1 = z_{1+r}^1$. There is a hamiltonian cycle

$$C = (x_{0,1}^0, x_{0,2}^0, x_{0,3}^0, L_3^1, p(x_{1,1}^1, x_{1,2}^1, x_{1,3}^1), p(x_{1,4}^1, x_{1,5}^1), p(x_{1,6}^1, x_{1,7}^1), p(x_{1,8}^1, x_{1,9}^1), x_{0,1}^0).$$

Let $m$ and $r$ be integers and $0 \leq r \leq k^{n-1} - 1$ such that $k^{n-1} - 1 = m + r \cdot k^{n-1} + l + 1 = l$. W.L.O.G., let $u = x_{0,1}^0$ and $v = x_{0,1}^0$. For simplicity, denote $x_{1}^{k^{n-1} - 1 - m}$ as $v_1^0, x_{1}^{k^{n-1} - 1 - m} - a$ as $v_1^0, x_{1}^{k^{n-1} - 1 - m} - v_1^0$ as $v_1^0$. If $r$ is even, let $t_1 = t_2 = v_1, z_{1+r} = x_{0,1}^0 + t$ and $z_{1+r} = x_{0,1}^0$. There is
Let 0 ≤ r ≤ \frac{k-1}{2} be an integer, d + 2(t - d') + r \cdot k^{n-1} = l, d' \leq k^{n-1} - 1 and e = k - 1 - r.

Let r be an odd integer. We have the hamiltonian cycle

\[ C = (x_0^r, x_1^r, \ldots, x_{t-r-1}^r), \quad \text{where} \quad x_{n-1}^r = x_0^r. \]

The panpositionability of any k-ary n-cube \( Q^k_n \) is nearly-papositionable for any odd integer \( k \geq 5 \). Moreover, we prove that \( Q^k_n \) is bipanpositionable hamiltonian for any even integer \( k \geq 4 \). It is known that the hypercube, \( Q^2_n \), is bipanpositionable [7]. Thus \( Q^k_n \) is bipanpositionable for all even integers \( k \geq 2 \).

The panpositionability of any k-ary n-cube has been completely studied and the result is optimal in the sense that given any two vertices \( u \) and \( v \), there exists no more hamiltonian cycle on which \( d_C(u, v) \) equals any of the numbers we miss in the nearly-papositionable \( Q^k_n \) when \( k \) is odd.

V. CONCLUSIONS

In this paper, we prove that the k-ary n-cube \( Q^k_n \) is panpositionable hamiltonian and \( Q^k_n \) is nearly-papositionable for any odd integer \( k \geq 5 \). Moreover, we prove that \( Q^k_n \) is bipanpositionable hamiltonian for any even integer \( k \geq 4 \). It is known that the hypercube, \( Q^2_n \), is bipanpositionable [7]. Thus \( Q^k_n \) is bipanpositionable for all even integers \( k \geq 2 \).

The panpositionability of any k-ary n-cube has been completely studied and the result is optimal in the sense that given any two vertices \( u \) and \( v \), there exists no more hamiltonian cycle on which \( d_C(u, v) \) equals any of the numbers we miss in the nearly-papositionable \( Q^k_n \) when \( k \) is odd.

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