Abstract—In this paper, we investigate two parallel alternating methods for solving the system of linear equations $Ax = b$ and give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix. Furthermore, we give one example to show our results.

Keywords—nonsingular H-matrix, parallel alternating method, convergence.

I. INTRODUCTION

For the large system of linear equations

$$Ax = b,$$

where $A$ is a nonsingular square matrix of order $n$, $x, b \in \mathbb{R}^n$. Benzi and Szyld [1] analyzed the following alternating method:

Given an initial vector $x^{(0)}$, for $k = 0, 1, 2, \cdots$,

$$x^{(k+\frac{1}{2})} = M^{-1}N_x^{(k)} + M^{-1}b,$$

$$x^{(k+1)} = P^{-1}Qx^{(k+\frac{1}{2})} + P^{-1}b,$$

where $A = M - N = P - Q$ are two splittings of $A$. They proved its convergence under certain conditions when the coefficient matrix $A$ is a monotone matrix or a symmetric positive definite matrix.

In paper [2], Climent and Perea introduced two parallel alternating iterative methods.

Assume that

$$A = M_I - N_I = P_I - Q_I, \quad l = 1, 2, \cdots, p,$$

where $M_I$ and $P_I$ nonsingular matrices; $E_l$ satisfy $\sum_{l=1}^{p} E_l = I$ (I is an identity matrix), where $E_l$ are diagonal and $E_l \geq 0$.

Method 1: Let $x^{(0)}$ be a starting vector, $\varepsilon > 0$ is a given precision. For $k = 1, 2, \cdots$,

$$x^{(k+\frac{1}{2})}_l = (M_I^{-1}N_I)^{\nu(k,l)}x^{(k)} + \sum_{i=0}^{\nu(k,l)-1} (M_I^{-1}N_I)^iM_I^{-1}b,$$

$$x^{(k+1)}_l = (P_I^{-1}Q_I)^{\nu(k,l)}x^{(k+\frac{1}{2})}_l + \sum_{i=0}^{\nu(k,l)-1} (P_I^{-1}Q_I)^iP_I^{-1}b,$$

$$x^{(k+1)} = \sum_{l=1}^{p} E_lx^{(k+1)}_l.$$

If $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$, then quit.

It is easy to notice that the iterative matrix of Method 1 is

$$T = \sum_{l=1}^{p} E_l(M_I^{-1}N_I)^{\nu(k,l)}(M_I^{-1}N_I)^{\mu(k,l)}.$$

Method 2: Let $x^{(0)}$ be a starting vector, $\varepsilon > 0$ is a given precision. For $k = 1, 2, \cdots$,

$$x^{(k+\frac{1}{2})}_l = \sum_{l=1}^{p} E_l(M_I^{-1}N_I)^{\nu(k,l)}x^{(k)} + \frac{\nu(k,l)-1}{\mu(k,l)-1} \sum_{i=0}^{\nu(k,l)-1} (M_I^{-1}N_I)^iM_I^{-1}b,$$

$$x^{(k+1)}_l = \sum_{l=1}^{p} \frac{\nu(k,l)-1}{\mu(k,l)-1} (P_I^{-1}Q_I)^{\nu(k,l)}x^{(k+\frac{1}{2})}_l + \sum_{i=0}^{\nu(k,l)-1} (P_I^{-1}Q_I)^iP_I^{-1}b.$$

If $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$, then quit.

It is easy to notice that the iterative matrix of Method 2 is

$$S = \sum_{l=1}^{p} F_l(M_I^{-1}N_I)^{\nu(k,l)}(M_I^{-1}N_I)^{\mu(k,l)}.$$

In this paper, we give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix.

II. PRELIMINARIES

Let $A \in \mathbb{R}^{n \times n}$. We denote by $A \geq 0$ a nonnegative matrix, $|A|$ the absolute value of matrix $A$, and $\rho(A)$ the spectral radius of $A$.

Definition 2.1 Let $A = B - C$ be a splitting of $A$. If $B^{\varepsilon} \geq 0, B^{\varepsilon}C \geq 0, C^{\varepsilon} \geq 0$, then $A = B - C$ is a weak regular splitting[3]. If $B^{\varepsilon} \geq 0, C^{\varepsilon} \geq 0$, then $A = B - C$ is a regular splitting[4]. If $B$ is an M-matrix and $C \geq 0$, then $A = B - C$ is an M-splitting[5].

In paper [2], a weak regular splitting is also called a weak nonnegative splitting of the first type.

It’s obvious that an M-splitting is a regular splitting and a regular splitting is a weak regular splitting.

Definition 2.2[6] Let $A \in \mathbb{R}^{n \times n}$. $A = M - N(M, N \in \mathbb{R}^{n \times n})$ is called an H-splitting if $M > -|N|$ is an M-matrix. If $M > -|N|$, then $A = M - N$ is called an H-compatible splitting.

III. CONVERGENCE THEOREMS

In this section, we give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix.

Some results on parallel alternating methods

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Lemma 3.1[2] Let $A \in R^{n \times n}$ and $A^{-1} \geq 0$. If $A = M_l - N_l = P_l - Q_l \ (l = 1, 2, \cdots, p)$ are all weak nonnegative splittings of the first type, then

$$\rho(T) < 1,$$

where

$$T = \sum_{l=1}^{p} E_l(P_l^{-1}Q_l)^{(k,l)}(M_l^{-1}N_l)^{\mu(k,l)}.$$

Lemma 3.2[2] Let $A \in R^{n \times n}$ and $A^{-1} \geq 0$. If $A = M_l - N_l = P_l - Q_l \ (l = 1, 2, \cdots, p)$ are all weak nonnegative splittings of the first type, then

$$\rho(S) < 1,$$

where

$$S = \sum_{l=1}^{p} E_l(P_l^{-1}Q_l)^{(k,l)}\sum_{l=1}^{p} E_l(M_l^{-1}N_l)^{\mu(k,l)}.$$

Lemma 3.3[3] If $A \in R^{n \times n}$ is a nonsingular H-matrix, then $|A^{-1}| \leq A > -1$.

Theorem 3.1 Let $A \in R^{n \times n}$ be a nonsingular H-matrix, $A = M_l - N_l = P_l - Q_l \ (l = 1, 2, \cdots, p)$ are H-splittings while $B \in R^{n \times n}$ be a nonsingular M-matrix, $B = (\langle B_{ij} \rangle)_{1 \leq i, j \leq n}$ are M-splittings of $B$. Thus we obtain

$$|T| \geq \sum_{l=1}^{p} E_l(P_l^{-1}Q_l)^{(k,l)}(M_l^{-1}N_l)^{\mu(k,l)}$$

$$\sum_{l=1}^{p} E_l(P_l^{-1}Q_l)^{(k,l)}(M_l^{-1}N_l)^{\mu(k,l)} = T.$$
It's easy to test that
\[
\begin{bmatrix}
10 & -3 & -8 \\
-6 & 10 & 0 \\
-7 & -7 & 12
\end{bmatrix}
\]
is a nonsingular M-matrix, but
\[
\begin{bmatrix}
10 & -3 & -8 \\
-6 & 10 & 0 \\
-7 & -7 & 12
\end{bmatrix}
\neq A,
\]
so
\[
A = M_l - N_l = P_l - Q_l \quad (l = 1, 2)
\]
are H-splittings.

Case 1: We choose
\[
E_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\quad
E_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
then
\[
T = \begin{bmatrix}
48 & -89 & 237 \\
31 & 12518 & 40 & 9861 \\
-52 & 212 & 150 & 949
\end{bmatrix}
\]
\[
\rho(T) = \frac{103}{3963} < 1.
\]

Case 2: We choose
\[
E_1 = I/3, \quad E_2 = 2I/3, \quad F_1 = 3I/4, \quad F_2 = I/4, \quad l = 1, 2,
\]
then
\[
S = \begin{bmatrix}
191 & 119 & 231 \\
-64 & 19 & 25 \\
-30 & 75 & 133
\end{bmatrix}
\]
\[
\rho(S) = \frac{163}{6917} < 1.
\]

REFERENCES