Neighbors of Indefinite Binary Quadratic Forms

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Abstract—In this paper, we derive some algebraic identities on right and left neighbors \( R(F) \) and \( L(F) \) of an indefinite binary quadratic form \( F = F(x,y) = ax^2 + bxy + cy^2 \) of discriminant \( \Delta = b^2 - 4ac \). We prove that the proper cycle of \( F \) can be given by using its consecutive left neighbors. Also we construct a connection between right and left neighbors of \( F \).

Keywords—Quadratic form, indefinite form, cycle, proper cycle, right neighbor, left neighbor.

I. PRELIMINARIES.

A real binary quadratic form \( F \) is a polynomial in two variables \( x \) and \( y \) of the type

\[
F = F(x,y) = ax^2 + bxy + cy^2
\]

with real coefficients \( a, b, c \). We denote it by \( F = (a,b,c) \).

The discriminant of \( F \) is defined by the formula \( b^2 - 4ac \) and is denoted by \( \Delta = \Delta(F) \). \( F \) is an integral form if and only if \( a,b,c \in \mathbb{Z} \), and is called indefinite if and only if \( \Delta(F) > 0 \). An indefinite form \( F = (a,b,c) \) of discriminant \( \Delta \) is said to be reduced if

\[
\sqrt{\Delta} - 2|a| < b < \sqrt{\Delta}.
\]

Most properties of quadratic forms can be given by the aid of extended modular group \( \Gamma \) (see [5]). Gauss (1777-1855) defined the group action of \( \Gamma \) on the set of forms as follows:

\[
gF(x,y) = (ax^2 + brx + csx^2 + 2art + bru + bts + 2cstu) xy + (at^2 + btx + cu^2) y^2
\]

for \( g = \left( \begin{array}{cc} r & s \\ t & u \end{array} \right) \in \Gamma \). Hence two forms \( F \) and \( G \) are called equivalent if and only if there exists a \( g \in \Gamma \) such that \( gF = G \).

If \( det g = 1 \), then \( F \) and \( G \) are called properly equivalent, and if \( det g = -1 \), then \( F \) and \( G \) are called improperly equivalent.

If a form \( F \) is improperly equivalent to itself, then it called ambiguous.

Let \( \rho(F) \) denotes the normalization (it means that replacing \( F \) by its normalization) of \((c,-b,a)\). To be more explicit, we set

\[
\rho(F) = (c,-b+2cr_1,cr_1^2) (\text{mod } 2|\Delta|), \quad r_1 = r_1(F) = \begin{cases} \text{sign}(c) \left\lfloor \frac{b}{2|\Delta|} \right\rfloor & \text{for } |c| \geq \sqrt{\Delta} \\ \text{sign}(c) \left\lfloor \frac{b+\sqrt{\Delta}}{2|\Delta|} \right\rfloor & \text{for } |c| < \sqrt{\Delta} \end{cases}
\]

for \( i \geq 0 \). Then the number \( r_1 \) is called the reducing number and the form \( \rho'(F) \) is called the reduction of \( F \). Further, if \( F \) is reduced, then so is \( \rho'(F) \) by (2). In fact, \( \rho' \) is a permutation of the set of all reduced indefinite forms.

Now consider the following transformations

\[
\chi(F) = \chi(a,b,c) = (-c,b,-a) \\
\tau(F) = \tau(a,b,c) = (b,-a,c)
\]

If \( \chi(F) = F \), that is, \( F = (a,b,-a) \), then \( F \) is called symmetric. The cycle of \( F \) is the sequence \( \{\tau F\} \) for \( i \in \mathbb{Z} \), where \( G = (A,B,C) \) is a reduced form with \( A > 0 \) which is equivalent to \( F \). The cycle and proper cycle of \( F \) can be given by the following theorem.

**Theorem 1.1:** Let \( F = (a,b,c) \) be a reduced indefinite quadratic form of discriminant \( \Delta \). Then the cycle of \( F \) is a sequence \( F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1} \) of length \( l \), where

\[
s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor
\]

and

\[
F_{i+1} = \left( a_{i+1}, b_{i+1}, c_{i+1} \right) = \left( (c_i), -b_i+2s_i|c_i|, -(a_i+b_is_i+c_is_i^2) \right)
\]

for \( 1 \leq i \leq l-2 \). If \( l \) is odd, then the proper cycle of \( F \) is

\[
F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})
\]

of length \( 2l \) if \( l \) is even, then the proper cycle of \( F \) is

\[
F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1}) \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})
\]

of length \( l \). In this case the equivalence class of \( F \) is the disjoint union of the proper equivalence class of \( F \) and the proper equivalence class of \( \tau(F) \). [1], [4]

The right neighbor of \( F = (a,b,c) \) is denoted by \( R(F) \) is the form \( (A,B,C) \) determined by \( A = c, b+B \equiv 0 \pmod{2A}, \sqrt{\Delta} - 2|A| < B < \sqrt{\Delta} \) and \( B^2 - 4AC = \Delta \). It is clear from definition that

\[
R(F) = \left( \begin{array}{cc} 0 & -1 \\ 1 & -\delta \end{array} \right) (a,b,c),
\]

where \( b+B = 2c\delta \). The left neighbor is hence

\[
L(F) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) R(c,b,a) = \chi \tau(R(c,b,a)).
\]

So \( F \) is properly equivalent to its right and left neighbors (for further details on binary quadratic forms see [1], [2], [3], [4]).
II. Neighbors of Indefinite Quadratic Forms.

In this section, we will derive some properties of neighbors of indefinite quadratic forms. In [6], we proved the following theorem.

Theorem 2.1: Let \( F_0 \sim F_1 \sim \ldots \sim F_{l-1} \) be the cycle of \( F \) of length \( l \) and let \( R^i(F_0) \) be the consecutive right neighbors of \( F = F_0 \) for \( i \geq 0 \).

1) If \( l \) is odd, then the proper cycle of \( F \) is

\[ F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \ldots \sim R^{2l-3}(F_0) \sim R^{2l-1}(F_0) \]

of length \( 2l \).

2) If \( l \) is even, then the proper cycle of \( F \) is

\[ F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \ldots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0) \]

of length \( l \).

Also we proved that if \( l \) is odd, then \( R^{l-2}(F_0) \) and \( R^{2l-3}(F_0) \) are the symmetric right neighbors of \( F \). Further we proved the following corollary and two theorems in [6].

Corollary 2.2: Let \( F_0 \sim F_1 \sim \ldots \sim F_{l-1} \) be the cycle of \( F \) of length \( l \).

1) If \( l \) is odd, then

\[ R^i(F_0) = \begin{cases} F_i & \text{if } i \text{ is even} \\ \tau(F_i) & \text{if } i \text{ is odd} \end{cases} \]

for \( 1 \leq i \leq l-1 \) and

\[ R^l(F_0) = \begin{cases} F_{l-1} & \text{if } i \text{ is even} \\ \tau(F_{l-1}) & \text{if } i \text{ is odd} \end{cases} \]

for \( 1 \leq i \leq l-1 \).

2) If \( l \) is even, then

\[ R^i(F_0) = \begin{cases} F_i & \text{if } i \text{ is even} \\ \tau(F_i) & \text{if } i \text{ is odd} \end{cases} \]

for \( 1 \leq i \leq l-1 \).

Theorem 2.3: If \( l \) is odd, then \( F \) has \( 2l-1 \) right neighbors and if \( l \) is even, then \( F \) has \( l-1 \) right neighbors.

Theorem 2.4: If \( l \) is odd, then

1) \( R^i(F_0) = \tau(R^{2l-1-i}(F_0)) \) for \( 1 \leq i \leq 2l-2 \) and \( R^{2l-1}(F_0) = \tau(F_0) \).

2) \( R^i(F_0) = \tau(R^{i+l}(F_0)) \), \( R^l(F_0) = \tau(F_0) \) for \( l \leq i \leq l-1 \) and \( R^l(F_0) = \tau(R^{l-i}(F_0)) \) for \( l+1 \leq i \leq 2l-1 \).

In [7], we also derived some algebraic identities on proper cycles and right neighbors of \( F \). Now we can return our problem. Then we can give the following theorems.

Theorem 2.5: If \( l \) is odd, then in the proper cycle of \( F \), we have

1) \( \chi^l(F_0) = \tau(F_{l-i}) \) for \( 1 \leq i \leq 2l-1 \).

2) \( \chi^l(R^i(F_0)) = R^{2l-1-i}(F_0) \) for \( 0 \leq i \leq l-1 \).

Proof: 1) Let \( F_0 = F = (a_0, b_0, c_0) \). Then applying (6), we get

\[ F_0 \sim (a_0, b_0, c_0) \]
\[ R^1(F_0) \sim (a_1, b_1, c_1) \]
\[ R^2(F_0) \sim (a_2, b_2, c_2) \]

\[ \vdots \]
\[ R^{l-2}(F_0) \sim \left( \frac{a_{l-2}, b_{l-2}, c_{l-2}}{l} \right) \]
\[ R^{l-1}(F_0) \sim \left( \frac{a_{l-1}, b_{l-1}, c_{l-1}}{l} \right) \]
\[ R^{l+1}(F_0) \sim \left( \frac{-c_{l-1}, a_{l-1}, b_{l-1}}{l} \right) \]
\[ \vdots \]
\[ R^{2l-3}(F_0) \sim \left( \frac{-c_{l-2}, a_{l-2}, b_{l-2}}{l} \right) \]
\[ R^{2l-2}(F_0) \sim \left( \frac{-c_{l-3}, a_{l-3}, b_{l-3}}{l} \right) \]
\[ R^{2l-1}(F_0) \sim \left( \frac{c_0, b_0, a_0}{l} \right) \]

Hence it is clear that

\[ R^1(F_0) = \tau(F_0) \]
\[ R^{l+1}(F_0) = \tau(F_1) \]
\[ R^{l+2}(F_0) = \tau(F_2) \]
\[ \vdots \]
\[ R^{l-2}(F_0) = \tau(F_{l-3}) \]
\[ R^{l-1}(F_0) = \tau(F_{l-2}) \]
\[ R^{l+1}(F_0) = \tau(F_{l-1}) \]

So \( R^i(F_0) = \tau(F_{l-i}) \) for \( 1 \leq i \leq 2l-1 \).

2) Similarly we find that

\[ \chi^l(F_0) = R^{2l-1}(F_0) \]
\[ \chi^l(R^1(F_0)) = R^{2l-2}(F_0) \]
\[ \chi^l(R^2(F_0)) = R^{2l-3}(F_0) \]

\[ \vdots \]
\[ \chi^l(R^{2l-2}(F_0)) = R^{2l-4}(F_0) \]
\[ \chi \tau(R_{i+1}^i(F_0)) = R^{2i-1}_i(F_0) \]
\[ \chi \tau(R_{i+2}^i(F_0)) = R^{2i+1}_i(F_0) \]
\[ \cdots \]
\[ \chi \tau(R_{i-3}^i(F_0)) = R^{i+2}_i(F_0) \]
\[ \chi \tau(R_{i-2}^i(F_0)) = R^{i+1}_i(F_0) \]
\[ \chi \tau(R_{i-1}^i(F_0)) = R^i_0(F_0). \]

So \( \chi \tau(R^i_0(F_0)) = R^{2i-1-i}_i(F_0) \) for \( 0 \leq i \leq l-1 \).

Now we consider the left neighbors of \( F \). Recall that the left neighbor of \( F \) is defined to be

\[ L(F) = L(a, b, c) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} R(c, b, a). \]

Then we can give the following theorem.

**Theorem 2.6:** Let \( F_0 \sim F_1 \sim \cdots \sim F_{l-1} \) denote the cycle of \( F \). If \( l \) is odd, then

1) \[ L^i(F_0) = \begin{cases} \tau(F_{l-i}) & i \text{ is odd} \\
F_{l-i} & i \text{ is even} \end{cases} \]
for \( 1 \leq i \leq l \)

2) \[ L^i(F_0) = \begin{cases} \tau(F_{2l-i}) & i \text{ is odd} \\
F_{2l-i} & i \text{ is even} \end{cases} \]
for \( l + 1 \leq i \leq 2l \).

3) \[ \chi(L^i(F_0)) = \begin{cases} \tau(F_{l-i-1}) & i \text{ is odd} \\
F_{l-i-1} & i \text{ is even} \end{cases} \]
for \( 1 \leq i \leq l \)

4) \[ \chi(L^i(F_0)) = \begin{cases} \tau(F_{l-i-1}) & i \text{ is odd} \\
F_{l-i-1} & i \text{ is even} \end{cases} \]
for \( l + 1 \leq i \leq 2l \).

**Proof:** 1) Applying (7), we get

\[ L^1(F_0) = (c_0, b_0, a_0) = \tau(F_{l-1}) \]
\[ L^2(F_0) = (-c_1, b_1, -a_1) = F_{l-2} \]
\[ L^3(F_0) = (c_2, b_2, a_2) = \tau(F_{l-3}) \]
\[ \cdots \]
\[ L^i(F_0) = (-a_0, b_0, -c_0) = \tau(F_0) \]
\[ L^{i+1}(F_0) = (-c_0, b_0, -a_0) = F_{l-2} \]
\[ \cdots \]
\[ L^{2l-1}(F_0) = (-a_0, b_0, -c_0) = \tau(F_0) \]
\[ L^{2l}(F_0) = (a_0, b_0, c_0) = F_0. \]

So the result is clear. The others can be proved similarly.

Note that we proved in Theorem 2.1 that the proper cycle of \( F \) can be given by using its consecutive right neighbors. Similarly we can give the following theorem.

**Theorem 2.7:** Let \( L^i(F) \) denote the consecutive left neighbors of \( F \).

1) If \( l \) is odd, then the proper cycle of \( F = F_0 \) is

\[ F_0 \sim L^{2l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0) \]
of length \( 2l \).

2) If \( l \) is even, then the proper cycle of \( F = F_0 \) is

\[ F_0 \sim L^{l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0) \]
of length \( l \).

**Proof:** 1) Let \( l \) be odd. Then by Theorem 1.1 the proper cycle of \( F \) is

\[ F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim F_0 \]
of length \( 2l \). We also see Theorem 2.6 that

\[ L^1(F_0) = \begin{cases} \tau(F_{l-i}) & i \text{ is odd} \\
F_{l-i} & i \text{ is even} \end{cases} \]
for \( 1 \leq i \leq l \)

\[ L^i(F_0) = \begin{cases} \tau(F_{2l-i}) & i \text{ is odd} \\
F_{2l-i} & i \text{ is even} \end{cases} \]
for \( l + 1 \leq i \leq 2l \).

2) For \( l \) is even, then the proper cycle of \( F = F_0 \) is

\[ F_0 \sim L^{l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0) \]
of length \( l \).

**Example 2.1:** 1) The cycle of \( F = (1, 5, -4) \) is \( F_0 = (1, 5, -4) \sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1) \) of length 5. So its proper cycle is hence

\[ F_0 = (1, 5, -4) \sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1) \]
of length 10. The consecutive left neighbors of \( F \) are

\[ L^1(F) = (-4, 5, 1), L^2(F) = (2, 3, -4), \]
\[ L^3(F) = (-2, 5, 2), L^4(F) = (4, 3, -2), \]
\[ L^5(F) = (-4, 5, 1), L^6(F) = (2, 3, -4), \]
\[ L^7(F) = (-2, 5, 2), L^8(F) = (4, 3, -2), \]
\[ L^9(F) = (-4, 5, 1), L^{10}(F) = F. \]

So it is easily seen that the proper cycle of \( F \) is

\[ F \sim L^3(F) \sim L^6(F) \sim L^9(F) \sim L^5(F) \sim L^8(F) \sim L^5(F) \sim L^8(F) \sim L^1(F). \]
2) The cycle of \( F = (1, 8, -5) \) is \( F_0 = (1, 8, -5) \sim F_1 = (5, 2, -4) \sim F_2 = (4, 6, -3) \sim F_3 = (3, 6, -4) \sim F_4 = (4, 2, -5) \sim F_5 = (5, 8, -1) \) of length 6. So its proper cycle is
\[
F_0 = (1, 8, -5) \sim F_1 = (-5, 2, 4) \sim F_2 = (4, 6, -3) \sim F_3 = (-3, 6, 4) \sim F_4 = (4, 2, -5) \sim F_5 = (-5, 8, 1).
\]
The left neighbors of \( F \) are
\[
L^1(F) = (-5, 8, 1), \quad L^2(F) = (4, 2, -5),
\]
\[
L^3(F) = (-3, 6, 4), \quad L^4(F) = (4, 6, -3),
\]
\[
L^5(F) = (-5, 2, 4), \quad L^6(F) = F.
\]
So its proper cycle is \( F \sim L^5(F) \sim L^4(F) \sim L^3(F) \sim L^2(F) \sim L^1(F). \)

From above theorem, we can give the following result.

**Theorem 2.8:** If \( l \) is odd, then \( F \) has \( 2l - 1 \) left neighbors and if \( l \) is even it has \( l - 1 \) left neighbors.

**Proof:** Let \( l \) be odd. Then we get
\[
F_0 = (a_0, b_0, c_0),
\]
\[
F_1 = (a_1, b_1, c_1),
\]
\[
F_2 = (a_2, b_2, c_2),
\]
\[
F_3 = (a_3, b_3, c_3),
\]
\[
\ldots
\]
\[
F_{l-3} = (-c_2, b_2, -a_2),
\]
\[
F_{l-2} = (-c_1, b_1, -a_1),
\]
\[
F_{l-1} = (-c_0, b_0, -a_0).
\]
The first left neighbor of \( F = F_0 \) is
\[
L^1(F_0) = (a_1, b_1, c_1),
\]
\[
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
\[
R(c_0, b_0, a_0)
\]
\[
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
\[
(a_0, -b_0 + 2a_0b_0, c_0 - \delta b_0 + a_0d_0)
\]
\[
= (c_0 - \delta b_0 + a_0d_0, -b_0 + 2a_0b_0, a_0)
\]
\[
= (c_0, b_0, a_0)
\]
Similarly we obtain
\[
L^2(F_0) = (-c_1, b_1, -a_1)
\]
\[
L^3(F_0) = (c_2, b_2, a_2)
\]
\[
L^4(F_0) = (-c_3, b_3, -a_3)
\]
\[
\ldots
\]
\[
L^{l-1}(F_0) = (-c_0, b_0, -a_0)
\]
\[
L^l(F_0) = (a_0, b_0, c_0)
\]
\[
\ldots
\]
\[
L^{2l-1}(F_0) = (-a_1, b_1, -c_1)
\]
\[
L^{2l}(F_0) = (a_0, b_0, c_0) = F_0.
\]
So \( F \) has \( 2l - 1 \) left neighbors. Similarly it can be shown that \( F \) has \( l - 1 \) left neighbors if \( l \) is even.

**Theorem 2.9:** Let \( F_0 \sim F_1 \sim \ldots \sim F_{l-1} \) be the cycle of \( F \) of length \( l \). If \( l \) is odd, then
1) \( L(F_i) = \tau(F_{i-1}) \) for \( 1 \leq i \leq l - 1 \) and \( L(F_0) = \tau(F_{l-1}). \)
2) \( L(F_i) = \chi \tau(F_{i-1}) \) for \( 1 \leq i \leq l - 1 \) and \( L(F_0) = \chi \tau(F_0). \)

**Proof:** 1) Let \( F = F_0 = (a_0, b_0, c_0). \) Then
\[
F_1 = (a_1, b_1, c_1)
\]
\[
= ((c_0, -b_0 + 2s_0|c_0|, -(a_0 + b_0s_0 + c_0s_0^2))
\]
\[
= (-c_0, -b_0 - 2s_0c_0, -a_0 + b_0s_0 - c_0s_0^2).
\]

Now we try to determine the first left neighbor of \( F_1 \). Applying its definition, we get
\[
L(F_1) = L(-c_0, -b_0 - 2s_0c_0, -a_0 + b_0s_0 - c_0s_0^2)
\]
\[
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
\[
R(-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0).
\]

So we have to find out the right neighbor of \((-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0).\) To get this we make the change of variables \( x \rightarrow y \) and \( y \rightarrow x - \delta y \). Then we get
\[
R(-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0)
\]
\[
= ((a_0, -b_0 - 2s_0c_0, -a_0 + b_0s_0 - c_0s_0^2))
\]
\[
+ (0, 1, 0)
\]
\[
R(-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0)
\]
\[
= (-c_0, -b_0 - 2s_0c_0, -a_0 + b_0s_0 - c_0s_0^2).
\]

Also for \( i = 0 \), we get \( s_0 = -\delta_0 = 0 \). So (10) becomes
\[
R(-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0)
\]
\[
= (-c_0x^2 + (b_0 + 2c_0s_0 - 2c_0\delta_0)xy
\]
\[
+ (-a_0 - b_0s_0 - c_0s_0^2 + b_0s_0 + 2s_0c_0\delta_0 - c_0s_0^2)xy^2).
\]

Since \( s_0 = -\delta_0 = 0 \), (11) becomes
\[
R(-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0)
\]
\[
= (-c_0x^2 + (b_0)s_0 - a_0g^2).
\]

So applying (9) and (12), we get
\[
L(F_1) = L(-c_0, -b_0 - 2s_0c_0, -a_0 - b_0s_0 - c_0s_0^2)
\]
\[
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
\[
R(-a_0 - b_0s_0 - c_0s_0^2, -b_0 - 2s_0c_0, -c_0)
\]
\[
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
\[
(-c_0, b_0, -a_0)
\]
\[
= \tau(F_0).
\]

Similarly we find that \( L(F_2) = \tau(F_1) \), \( L(F_3) = \tau(F_2) \), \ldots, \( L(F_{l-1}) = \tau(F_{l-2}) \) and \( L(F_0) = \tau(F_{l-1}). \) The other case can be proved similarly.
Example 2.2: The cycle of $F = (1, 7, -6)$ is

$$F_0 = (1, 7, -6) \sim F_1 = (6, 5, -2) \sim F_2 = (2, 7, -3) \sim F_3 = (3, 5, -4) \sim F_4 = (4, 3, -4) \sim F_5 = (4, 5, -3) \sim$$

$$F_6 = (3, 7, -2) \sim F_7 = (2, 5, -6) \sim F_8 = (6, 7, -1).$$

Then

$$L(F_0) = L(1, 7, -6) = (-6, 7, 1) = \tau(F_8) = \chi(F_0)$$

$$L(F_1) = L(6, 5, -2) = (-1, 7, 6) = \tau(F_0) = \chi(F_8)$$

$$L(F_2) = L(2, 7, -3) = (-6, 5, 2) = \tau(F_1) = \chi(F_7)$$

$$L(F_3) = L(3, 5, -4) = (-2, 7, 3) = \tau(F_2) = \chi(F_6)$$

$$L(F_4) = L(4, 3, -4) = (-3, 5, 4) = \tau(F_3) = \chi(F_5)$$

$$L(F_5) = L(4, 5, -3) = (-4, 3, 4) = \tau(F_4) = \chi(F_4)$$

$$L(F_6) = L(3, 7, -2) = (-4, 5, 3) = \tau(F_5) = \chi(F_3)$$

$$L(F_7) = L(2, 5, -6) = (-3, 7, 2) = \tau(F_6) = \chi(F_2)$$

$$L(F_8) = L(6, 7, -1) = (-2, 6, 5) = \tau(F_7) = \chi(F_1)$$

as we wanted.

From above theorem, we can give the following corollary.

Corollary 2.10: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of $F$ of length $l$. If $l$ is odd, then

1) $\tau(L^i(F_0)) = L^{i+1}(F_0)$ for $1 \leq i \leq l$.

2) $\chi(L^i(F_0)) = L^{i+1-1}(F_0)$ for $1 \leq i \leq l$ and $\chi(L^1(F_0)) = L^{3l+1-1}(F_0)$ for $l + 1 \leq i \leq 2l$.

Theorem 2.11: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of $F$ of length $l$. If $l$ is odd, then $L^{i+1}(F_0)$ and $L^{3l+1}(F_0)$ are the symmetric left neighbors of $F$.

Proof: We know that $F$ has $2l - 1$ left neighbors when $l$ is odd. Also

$$L^1(F_0) = (c_0, b_0, a_0)$$

$$L^2(F_0) = (-c_1, b_1, -a_1)$$

$$L^3(F_0) = (c_2, b_2, a_2)$$

$$L^{i+1}(F_0) = (-a_{i+1}, b_{i+1}, -c_{i+1})$$

So $L^{i+1}(F_0)$ and $L^{3l+1}(F_0)$ are symmetric left neighbors.

**Theorem 2.12:** If $l$ is odd, then in the proper cycle of $F$, we have

1) $L^i(F_0) = F_{2i-1}$ for $1 \leq i \leq 2l$.

2) $L^l(F_0) = \tau(F_{2i-1})$ for $1 \leq i \leq l$ and $L^l(F_0) = \tau(F_{3l-2})$ for $l + 1 \leq i \leq 2l$.

3) $L^l(F_0) = \chi(F_{l-1})$ for $1 \leq i \leq l$ and $L^l(F_0) = \chi(F_{2l-1})$ for $1 \leq i \leq 2l$.

**Proof:** 1) Before starting our proof, we try to determine the cycle and proper cycle of $F$. To get this let $F = F_0 = (a_0, b_0, c_0)$. Then the cycle of $F$ is $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-2} \sim F_{l-1}$, where

$$F_0 = (a_0, b_0, c_0)$$

$$F_1 = (a_1, b_1, c_1)$$

$$F_2 = (a_2, b_2, c_2)$$

$$F_3 = (a_3, b_3, c_3)$$

$$\cdots$$

Now we determine the left neighbors of $F = F_0$. Then applying (7), we get

$$L^1(F_0) = (c_0, b_0, a_0) = F_{2l-1}$$

$$L^2(F_0) = (-c_1, b_1, -a_1) = F_{2l-2}$$

$$\cdots$$

$$L^{l}(F_0) = (c_{l-1}, b_{l-1}, a_{l-1}) = F_{l-1}$$

$$L^{l+1}(F_0) = (-a_0, b_0, -c_0) = F_1$$

$$L^{l+1}(F_0) = (a_0, b_0, a_0) = F_0$$

$$F_{l-1} = (a_0, b_0, -c_0) = F_{l-1}$$
\[ L^{2l-1}(F_0) = (-a_1, b_1, -c_1) = F_l \]
\[ L^{2l}(F_0) = (a_0, b_0, a_0) = F_0. \]

So \( L^i(F_0) = F_{2i-1} \) for \( 1 \leq i \leq 2l \).

2) Similarly we obtain
\[
\begin{align*}
L^1(F_0) &= (a_0, b_0, a_0) = \tau(F_{l-1}) \\
L^2(F_0) &= (-a_1, b_1, -a_1) = \tau(F_{l-2}) \\
&\vdots \\
L^{l-2}(F_0) &= (-a_2, b_2, -c_2) = \tau(F_2) \\
L^{l-1}(F_0) &= (a_1, b_1, c_1) = \tau(F_1) \\
L^l(F_0) &= (-a_0, b_0, -c_0) = \tau(F_0) \\
L^{l+1}(F_0) &= (-c_0, b_0, -a_0) = \tau(F_{2l-1}) \\
L^{l+2}(F_0) &= (c_1, b_1, a_1) = \tau(F_{2l-2}) \\
&\vdots \\
L^{2l-1}(F_0) &= (-a_1, b_1, -c_1) = \tau(F_{l+1}) \\
L^{2l}(F_0) &= (a_0, b_0, a_0) = \tau(F_l).
\end{align*}
\]

So \( L^i(F_0) = \tau(F_{l-i}) \) for \( l \leq i \leq l \) and \( L^i(F_0) = \tau(F_{2l-i}) \) for \( l+1 \leq i \leq 2l \).

The others are proved similarly.

**Example 2.3:** The cycle of \( F = (1, 7, -6) \) is
\[
\begin{align*}
F_0 &= (1, 7, -6) \sim F_1 = (6, 5, -2) \sim F_2 = (2, 7, -3) \sim F_3 = (3, 5, -4) \sim F_4 = (4, 3, -4) \sim F_5 = (4, 5, -3) \sim F_6 = (3, 7, -2) \sim F_7 = (2, 5, -6) \sim F_8 = (6, 7, -1)
\end{align*}
\]

and hence the proper cycle of is
\[
\begin{align*}
F_0 &= (1, 7, -6) \sim F_1 = (6, 5, 2) \sim F_2 = (2, 7, 3) \sim F_3 = (3, 5, 4) \sim F_4 = (4, 3, 4) \sim F_5 = (4, 5, 3) \sim F_6 = (3, 7, 2) \sim F_7 = (2, 5, 6) \sim F_8 = (6, 7, 1)
\end{align*}
\]

Now we give the connection between right and left neighbors of \( F \). To get this we can give the following theorem.

**Theorem 2.13:** Let \( R^i(F_0) \) and \( L^i(F_0) \) be denote the right and left neighbors of \( F \), respectively.

1) If \( l \) is odd, then \( L^i(F_0) = R^{2l-i}(F_0) \) for \( 1 \leq i \leq 2l - 1 \).

2) If \( l \) is even, then \( L^i(F_0) = R^{l-i}(F_0) \) for \( 1 \leq i \leq l - 1 \).

**Proof:** 1) Let \( l \) be odd. Then the proper cycle of \( F \) can be given by using its consecutive right neighbors, that is, \( F_0 \sim R^1(F_0) \sim R^3(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0) \) by Theorem 2.1. Also by considering the proper cycle \( F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim \tau(F_{l-2}) \sim \tau(F_{l-1}) \) of \( F \), we get
\[
R^i(F_0) = \begin{cases} 
F_i & \text{if } i \text{ is even} \\
\tau(F_i) & \text{if } i \text{ is odd}
\end{cases}
\]
for \( 1 \leq i \leq l - 1 \) and
\[
R^i(F_0) = \begin{cases} 
F_{l-i} & \text{if } i \text{ is even} \\
\tau(F_{l-i}) & \text{if } i \text{ is odd}
\end{cases}
\]
for \( l \leq i \leq 2l - 1 \) by Corollary 2.2. Also
\[
L^i(F_0) = \begin{cases} 
\tau(F_{l-i}) & \text{if } i \text{ is odd} \\
F_{l-i} & \text{if } i \text{ is even}
\end{cases}
\]
for \( 1 \leq i \leq l \) and
\[
L^i(F_0) = \begin{cases} 
\tau(F_{2l-i}) & \text{if } i \text{ is odd} \\
F_{2l-i} & \text{if } i \text{ is even}
\end{cases}
\]
for \( l+1 \leq i \leq 2l \). On the other hand, since the proper cycle of \( F \) is \( L^{2l}(F_0) \sim L^{2l-1}(F_0) \sim \cdots \sim L^3(F_0) \sim L^1(F_0) \), we conclude that \( L^i(F_0) = R^{2l-i}(F_0) \) for \( 1 \leq i \leq 2l - 1 \).

Similarly if \( l \) is even, then \( L^i(F_0) = R^{l-i}(F_0) \) for \( 1 \leq i \leq l \).

**Example 2.4:** 1) The cycle of \( F = (1, 5, -4) \) is \( F_0 = (1, 5, -4) \sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1) \). The consecutive left and right neighbors of \( F \) are
\[
L^i(F) = (-4, 5, 1) = R^8(F)
\]
\[
L^i(F) = (2, 3, -4) = R^6(F)
\]
\[
L^i(F) = (-2, 5, 2) = R^7(F)
\]
\[
L^i(F) = (4, 3, -2) = R^6(F)
\]
\[
L^i(F) = (-1, 5, 4) = R^4(F)
\]
\[
L^i(F) = (4, 5, -1) = R^4(F)
\]
\[
L^i(F) = (2, 3, -4) = R^5(F)
\]
\[
L^i(F) = (2, 5, -2) = R^6(F)
\]
\[
L^i(F) = (-4, 3, 2) = R^5(F).
\]

2) The cycle of \( F = (1, 8, -5) \) is \( F_0 = (1, 8, -5) \sim F_1 = (5, 2, -4) \sim F_2 = (4, 6, -3) \sim F_3 = (3, 6, -4) \sim F_4 = (4, 2, -5) \sim F_5 = (5, 8, -1) \). The consecutive left and right neighbors of \( F \) are
\[
L^i(F) = (-4, 8, 5) = R^8(F)
\]
\[
L^i(F) = (5, 2, -4) = R^6(F)
\]
\[
L^i(F) = (-2, 8, 5) = R^7(F)
\]
\[
L^i(F) = (4, 6, -3) = R^6(F)
\]
\[
L^i(F) = (-3, 8, 5) = R^4(F)
\]
\[
L^i(F) = (4, 2, -5) = R^4(F).
\]
neighbors of $F$ are

\[
L^1(F) = (-5, 8, 1) = R^3(F),
\]
\[
L^2(F) = (4, 2, -5) = R^3(F),
\]
\[
L^3(F) = (-3, 6, 4) = R^3(F),
\]
\[
L^4(F) = (4, 6, -3) = R^2(F),
\]
\[
L^5(F) = (-5, 2, 4) = R^3(F).
\]

From above theorem, we can give the following result.

**Corollary 2.14:** Let $R^i(F_0)$ and $L^i(F_0)$ denote the right and left neighbors of $F_0$, respectively. If $l$ is odd, then

1. $L^i(F_0) = \tau(R^{i-1}(F_0))$ for $1 \leq i \leq l$ and $L^l(F_0) = \tau(R^{l-i}(F_0))$ for $1 \leq i \leq 2l$.
2. $L^i(F_0) = \chi(R^{i-1}(F_0))$ for $1 \leq i \leq l$ and $L^l(F_0) = \chi(R^{l-i-1}(F_0))$ for $1 \leq i \leq 2l$.

If $l$ is even, then $L^i(F_0) = \chi(R^{l-1}(F_0))$ for $1 \leq i \leq l-1$.

Finally, we can give the following theorem.

**Theorem 2.15:** $R(F_0)$ and $L(F_0)$ denote the right and left neighbors of $F_0$, respectively. Then

\[
R(L(F_0)) = L(R(F_0)) = F_0.
\]

**Proof:** Recall that the right neighbor of $F = (a, b, c)$ is the form $R(F) = (A, B, C)$, where $A = c, b + B \equiv 0 \pmod{2A}, \sqrt{\Delta} - 2A < B < \sqrt{\Delta}$ and $B^2 - 4AC = \Delta$. Also $R(F) = [0; -1, 1; -\delta](a, b, c)$ for $b + B = 2c\delta$ and $L(F) = \chi \tau(R(c, b, a))$. For $F = F_0 = (a_0, b_0, c_0)$, we get

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
c_0 \\
b_0
\end{pmatrix} = R(c_0, b_0, a_0).
\]

(13)

Now we try to find $R(c_0, b_0, a_0)$. It is easily seen that

\[
R(c_0, b_0, a_0) = (a_0, -b_0 + 2a_0\delta_0, c_0 - b_0\delta_0 + a_0\delta_0^2).
\]

So (13) becomes

\[
L(F_0) = (c_0 - b_0\delta_0 + a_0\delta_0^2, -b_0 + 2a_0\delta_0, a_0).
\]

Note that $-b_0 + 2a_0\delta_0 \equiv -b_0 \pmod{2a_0}$. Also $\sqrt{\Delta} - 2a_0 < -b_0 + 2a_0\delta_0 < \sqrt{\Delta}$. So if we take the right neighbor of $L(F_0)$, then we get

\[
R(L(F_0)) = R(c_0 - b_0\delta_0 + a_0\delta_0^2, -b_0 + 2a_0\delta_0, a_0)
\]
\[
= (a_0, b_0, c_0)
\]
\[
= F_0.
\]

Similarly it can be proved that $L(R(F_0)) = F_0$.

**REFERENCES**


